Consensus of multiple autonomous underwater vehicles with double independent Markovian switching topologies and timevarying delays*

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A new method in which the consensus algorithm is used to solve the coordinate control problems of leaderless multiple autonomous underwater vehicles (multi-AUVs) with double independent Markovian switching communication topologies and time-varying delays among the underwater sensors is investigated. This is accomplished by first dividing the communication topology into two different switching parts, i.e., velocity and position, to reduce the data capacity per data package sent between the multi-AUVs in the ocean. Then, the state feedback linearization is used to simplify and rewrite the complex nonlinear and coupled mathematical model of the AUVs into a double-integrator dynamic model. Consequently, coordinate control of the multi-AUVs is regarded as an approximating consensus problem with various time-varying delays and velocity and position topologies. Considering these factors, sufficient conditions of consensus control are proposed and analyzed and the stability of the multi-AUVs is proven by Lyapunov–Krasovskii theorem. Finally, simulation results that validate the theoretical results are presented.

Keywords: multiple autonomous underwater vehicles, consensus control, Markovian switching topology, time-varying delay

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1. Introduction

Coordination of multiple autonomous underwater vehicles (multi-AUVs) has attracted much attention as it has broad applications in both civilian and military underwater tasks.[1–3] When single AUV is used in complex missions, it has to be self-sufficient and thus requires sufficient power, sensors, and other devices. However, a single AUV is sometimes unable to complete the required tasks safely and stably by itself in severe ocean environments.[4] In contrast, multiple AUVs can be adapted to offsetting the weaknesses of a single AUV. The multi-AUVs can take more sensors and provide higher efficiency and applicability than single AUV. This means that the multi-AUVs can receive and process the information obtained from the sensors, and deal with the communication problems caused by the underwater sensors. In a practical AUV system, the data package sent by sonar consists of state information and flag bits, such as starting flag bit, checking flag bit and ending flag bit. If the receiver neither obtains nor mismatches flag bits, the data packet is ineffective. So the large data packet means more time delays and more data-packet dropouts. The method of dividing the whole data packet into several smaller ones can reduce the influence.

In the AUV research, stability is a common target in both single AUV and multi-AUVs. In the literature, numerous methods have been proposed to solve this problem, such as nonlinear control theory,[5] sliding mode control,[6,7] fuzzy control,[8] and backstepping control.[9,10] However, methods used in single-AUV scenarios cannot completely resolve control problem of multi-AUVs, as each member has to complete the mission while coordinate with the others based on common information. Further, the communication topology used for sharing such information among the members usually contains time delays or data packet dropouts. In addition, underwater communication, positioning, and navigation all depend heavily on acoustics, which limits the communication bandwidth, communication distance, and communication volume. The dynamic behaviours of AUVs are nonlinear and coupled,[11] and the stable control of multi-AUVs is even more difficult when practical and complex dynamics and communication constraints are involved. Qi,[12] Cui *et al.*,[13] and Yang and Gu[14] all proposed the methods in which various nonlinear approaches used in single AUV with good performance are extended to the control of underactuated AUVs. Their proposed methods navigate all the members in a formation to track a reference path or maintain the formation by communicating with a leader or a common target. In their methods, only the position of the leader is required to ensure that his motion follows the AUVs. However, in all the proposed methods, the motion information of the following AUVs is ignored.

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Consequently, the proposed methods lack coordination and autonomous ability.

Various methods used in multi-agents systems have recently been adopted and applied to coordinate control or group stability.[15,16] With the aid of these methods, autonomous ability and cooperation have been strengthened. Li and Wang[17] applied the consensus method to multi-AUVs with using a finite-time position consensus algorithm. Their algorithm enables the AUVs to avoid obstacles in the path and change configuration to avoid collisions while maintaining connectivity. However, communication constraints exist when the AUVs share information about underwater. The stability of multi-agent systems with delays can be analyzed and discussed by Lyapunov–Razumikhin and Lyapunov–Krasovskii theories.[18–20] Shang[21] and Xie and Cheng[22] proposed new methods in which the Lyapunov theory is used to solve the consensus problem under Markovian switching topology with time delays. Switching topologies are significant as they could simulate the communication constraints in ocean environments better.

Motivated by the considerations above, we focus on consensus of multi-AUVs under limited communication by using double independent Markovian switching topologies. The contributions of this paper consist of three parts. Firstly, the nonlinear mathematical model of AUV is transformed into a linear dynamic model via state feedback linearization with minimal inaccuracy. This process is accomplished by the coordinate transformation instead of the method in which the nonlinear parameter presented in Ref. [23] is ignored. Secondly, just as done in Ref. [24], the communication topology is divided into two parts, i.e., position and velocity, which are respectively used to deliver position and velocity information. This method reduces the capacity of per data packet. Subsequently, the same time-varying delays in both topologies are used to verify the feasibility of two independent position and velocity topologies. Thirdly, as the same delays in double topologies are difficult to realize in real ocean environment, two different time-varying delays are analyzed and discussed in this paper. This means that the multi-AUVs operate in the case of different time-varying delays with double independent Markovian switching topologies. Furthermore, in both delay cases, the sufficient conditions for uniformity are ultimately obtained via Lyapunov–Krasovskii theory.

The remainder of this paper is organized as follows. In Section 2, we introduce preliminary graph theory notions and transform the single AUV nonlinear mathematical model into a double-integrator dynamic model via state feedback linearization. In Section 3, we present two main results respectively for the cases of same and different time delays in double independent Markovian switching topologies based on the simplified mathematical model. In Section 4 we give the simulation results that validate the theoretical results. Finally, in Section 5, we draw some conclusions from the study in this paper.

2. Problem formulation

2.1. Graph theory

Let $G=(V,\varepsilon,A)$ represent a directed graph, where$V=(v_1,v_2,\ldots)$ is a set of nodes (or AUVs), and $\varepsilon\subseteq V\times V$ is a set of edges. The adjacency matrix $A=[a_{ij}]\in\mathbb{R}^{n\times n}$ is defined by $a_{ij}\neq 0$ if $(v_i,v_j)\in\varepsilon$, which indicates that node (AUV) j can receive information from node (AUV) i, and $a_{ij}=0$ otherwise. The neighbor set of node (AUV) i is denoted as $N_i=\{j\in V|\,(i,j)\in\varepsilon\}$. Define the in-degree matrix as a diagonal matrix $D=\mathrm{diag}\left(d_1^{\mathrm{in}},d_2^{\mathrm{in}},\ldots,\right)$, where d_i^{in} and d_i^{out} are the in-degree and out-degree of the node (or AUV) $d_i^{\mathrm{in}}=\sum\limits_{j=1}^{n}a_{ij}$ and $d_i^{\mathrm{out}}=\sum\limits_{j=1}^{n}a_{ji}$. The G is said to be balanced if $d_i^{\mathrm{in}}=d_i^{\mathrm{out}}$ for all $i=1,2,\ldots,n$.[25] Define the Laplacian matrix as $L=[l_{ij}]=D-A$, which has all row sums equal to zero.

A directed tree is a graph in which each node has exactly one parent, except for one node, called the root, which has no parent and a directed path to each node. A directed spanning tree of a graph is a directed tree formed by graph edges that connect all the nodes of the graph. A graph has a directed spanning tree if there exists at least one node with a directed path to all the other nodes.[26] According to the lemma in Ref. [27], L has exactly one zero eigenvalue if and only if the directed graph associated with G has a spanning tree. It is obvious that $\lambda_2(L)$ is the smallest non-zero eigenvalue.

In this paper, the communication topology is divided into two parts: position topology and velocity topology. Two topologies are respectively used to send position and velocity information. In what follows, G_{p} and G_{v} represent the position and velocity topologies, respectively. Similarly, $L_{\mathrm{p}},L_{\mathrm{v}},N_i^{\mathrm{p}},N_i^{\mathrm{v}}$, respectively represent the Laplacian matrices of G_{p} and G_{v} and the neighbor sets of the i-th AUV. For a given positive integer N, the finite topology graphs can be represented as $G_{\mathrm{p}1},G_{\mathrm{p}2},\ldots,G_{\mathrm{p}N}$ and $G_{\mathrm{v}1},G_{\mathrm{v}2},\ldots,G_{\mathrm{v}N}$ associated with $G_{\cdot i}=(V_i,\varepsilon_i,A_i)$ and $\{\cdot\}$ signifies p or v. Let G_{p}^u denote the union of position graphs and G_{v}^u represent the union of velocity graphs. Then, $G_{\mathrm{p}}^u\triangleq G_{\mathrm{p}1}\cup\cdots\cup G_{\mathrm{p}N}\triangleq\left\{V_{\mathrm{p}1}\cup\cdots\cup V_{\mathrm{p}N},\varepsilon_{\mathrm{p}1}\cup\cdots\cup\varepsilon_{\mathrm{p}N},\sum\limits_{i=1}^{N}A_i\right\}$ where $V_{\mathrm{p}1}\cup\cdots\cup V_{\mathrm{p}N}$ and $\varepsilon_{\mathrm{p}1}\cup\cdots\cup\varepsilon_{\mathrm{p}N}$ denote the unions of nodes and edges. Similarly, $G_{\mathrm{v}}^u\triangleq G_{\mathrm{v}1}\cup\cdots\cup G_{\mathrm{v}N}\triangleq\left\{V_{\mathrm{v}1}\cup\cdots\cup V_{\mathrm{v}N},\varepsilon_{\mathrm{v}1}\cup\cdots\cup\varepsilon_{\mathrm{v}N},\sum\limits_{i=1}^{N}A_i\right\}$. The L_{p}^u and L_{v}^u denote the unions of Laplacian matrices with $L_{\mathrm{p}}^u\triangleq L_{\mathrm{p}1}\cup\cdots\cup L_{\mathrm{p}N}$ and $L_{\mathrm{v}}^u\triangleq L_{\mathrm{v}1}\cup\cdots\cup L_{\mathrm{v}N}$, respectively. The $L_{\mathrm{p}}(t)$ and $L_{\mathrm{v}}(t)$ represent the Laplacian matrices of position and velocity topologies at time t, and $L_{\mathrm{p}}(t)\in\{L_{\mathrm{p}1},\ldots,L_{\mathrm{p}N}\}$ and

$L_{\rm v}(t) \in \{L_{\rm v1}, \ldots, L_{\rm vN}\}$. Then, the union disoriented Laplacian matrices are designed as $\hat{L}_{\rm p}^{u} = \left(L_{\rm p}^{u} + L_{\rm p}^{u\rm T}\right)/2$ and $\hat{L}_{\rm v}^{u} = \left(L_{\rm v}^{u} + L_{\rm v}^{u\rm T}\right)/2$.

2.2. The AUV model and state feedback linearization

The dynamic model of AUV can typically be described by a 6-DOF model according to the body-fixed and earth-fixed coordinate systems as indicated in the following equations. As rolling has little influence on translational motion, the roll speed is ignored in this paper.[28] Thus, the nonlinear and coupled model can be presented as follows:

$$\begin{aligned} &\dot{\eta}_i = J(\eta_i)v_i, \\ &M_i\dot{v}_i + C_i(v_i)v_i + D_i(v_i)v_i + g(\eta_i) = T_i, \end{aligned} \tag{1}$$

where $\eta_i = [x_i, y_i, z_i, \theta_i, \varphi_i]^{\rm T} \in \mathbb{R}^5$ describes the states of the position/Euler angles, and $v_i = [u_i, v_i, w_i, q_i, r_i]^{\rm T} \in \mathbb{R}^5$ describes the states of velocities for the i-th AUV. M_i, $C_i(v_i)$, and $D_i(v_i)$ denote inertia, Coriolis, and damping, respectively, while $g(\eta_i)$ is the magnitude of a vector of generalized gravitational and buoyancy forces. T_i is the control inputs. $J(\eta_i) = \begin{bmatrix} J_1(\eta_i) & 0 \\ 0 & J_2(\eta_i) \end{bmatrix}$ is the Jacobian matrix from body-fixed frame to earth-fixed frame.

$$J_1(\eta_i) = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi & \cos\psi\sin\theta \\ \sin\psi\cos\theta & \cos\psi & \sin\psi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$J_2(\eta_i) = \begin{bmatrix} 1 & 0 \\ 0 & 1/\cos\theta \end{bmatrix}.$$

In this paper, an AUV is assumed to be symmetrical in plane and height and have a torpedo-like shape that can provide good hydrodynamic performance. This shape also means that some parameters can be ignored or simplified. It also means that $g(\eta)$ is equal to 0. Noting that both M_i and $C_i(v)$ contain the body-rigid part and hydrodynamic part for added mass, and are expressed as

$$M_i = M_i^{\rm RB} + M_i^{\rm Add}, \quad C_i(v) = C_i^{\rm RB}(v) + C_i^{\rm Add}(v),$$

$$D_i(v) = -\,{\rm diag}\{X_u, Y_{\rm v}, Z_w, M_q, N_r\},$$

$$M_i^{\rm RB} = {\rm diag}\{m, m, m, I_y, I_z\},$$

$$C_i^{\rm RB} = \begin{bmatrix} 0 & 0 & 0 & mw & -mv \\ 0 & 0 & 0 & 0 & mu \\ 0 & 0 & 0 & -mu & 0 \\ -mw & 0 & mu & 0 & 0 \\ mv & -mu & 0 & 0 & 0 \end{bmatrix},$$

$$M_i^{\rm Add} = -\begin{bmatrix} X_{\dot{u}} & 0 & 0 & 0 & 0 \\ 0 & Y_{\dot{v}} & 0 & 0 & Y_{\dot{r}} \\ 0 & 0 & Z_{\dot{w}} & Z_{\dot{q}} & 0 \\ 0 & 0 & M_{\dot{w}} & M_{\dot{q}} & 0 \\ 0 & N_{\dot{v}} & 0 & 0 & N_{\dot{r}} \end{bmatrix},$$

$$C_i^{\rm Add}(v) = \begin{bmatrix} 0 & 0 & 0 & Z_{\dot{w}}w & -Y_{\dot{v}}v \\ 0 & 0 & 0 & 0 & X_{\dot{u}}u \\ 0 & 0 & 0 & -X_{\dot{u}}u & 0 \\ -Z_{\dot{w}}w & 0 & X_{\dot{u}}u & 0 & 0 \\ Y_{\dot{v}}v & -X_{\dot{u}}u & 0 & 0 & 0 \end{bmatrix},$$

where $M_i^{\rm RB}$ is the rigid-body mass matrix, $C_i^{\rm RB}(v)$ is the rigid-body Coriolis and centripetal matrix. $M_i^{\rm Add}$ is the inertia matrix of added mass terms, and $C_i^{\rm Add}(v)$ is the hydrodynamic Coriolis and Centripetal matrix of added mass terms. The hydrodynamic parameters $X_{(\cdot)}$, $Y_{(\cdot)}$, $Z_{(\cdot)}$, $M_{(\cdot)}$, $N_{(\cdot)}$ are presented in Refs. [28] and [29].

Consider that

$$\underbrace{M_i\dot{v}_i}_{M_i\dot{v}_i} + \underbrace{C_i(v_i)v_i + D_i(v_i)v_i}_{N_i(v_i)v_i} = \underbrace{\tau_i}_{g'(\zeta)\hat{u}_i}, \tag{2}$$

where $\hat{u}_i = [T_u, T_{\rm v}, T_w, \delta_s, \delta_r]^{\rm T}$ represents the forces and rudder angles, $g'(\zeta)$ can be described as

$$g'(\zeta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & g'_{25} \\ 0 & 0 & 1 & g'_{34} & 0 \\ 0 & 0 & 0 & g'_{44} & 0 \\ 0 & 0 & 0 & 0 & g'_{55} \end{bmatrix},$$

g'_{ij} is the hydrodynamic coefficients caused by rudders.

According to Eqs. (1) and (2), we can derive the following equation,

$$\begin{bmatrix} \dot{\eta}_i \\ \dot{v}_i \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -M_i^{-1} \end{bmatrix} \begin{bmatrix} J(\eta_i)v_i \\ N_i(v_i)v_i \end{bmatrix} + \begin{bmatrix} 0 \\ M_i^{-1}g'(\zeta) \end{bmatrix} \hat{u}_i. \tag{3}$$

Then we take mathematical model of single AUV for example. The standard nonlinearization function can be described as

$$\begin{aligned} &\dot{\zeta} = f(\zeta) + g(\zeta)\hat{u}, \\ &\mu = h(\zeta), \end{aligned} \tag{4}$$

where $\zeta = \left[\eta^{\rm T}, v^{\rm T}\right]^{\rm T}$, $h(\zeta) = \eta$, $f(\zeta) = \begin{bmatrix} I & 0 \\ 0 & -M_i^{-1} \end{bmatrix} \begin{bmatrix} J(\eta_i)v_i \\ N_i(v_i)v_i \end{bmatrix}$, and $g(\zeta) = \begin{bmatrix} 0 \\ M_i^{-1}g'(\zeta) \end{bmatrix}$.

Furthermore, for the first five elements of $h(\zeta)$, the Lie derivative of $h_i(\zeta)$ can be defined as

$$L_f h_i(\zeta) = f_i(\zeta), \quad 1 \le i \le 5, \tag{5}$$

and the second Lie derivative of $h_i(\zeta)$ is

$$\begin{aligned} L_f^2 h_1(\zeta) = {} & (w\cos\psi\cos\theta - u\cos\psi\sin\theta)f_4(\zeta) \\ & + (-u\sin\psi\cos\theta - v\cos\psi)f_5(\zeta) \\ & - w\sin\psi\sin\theta f_5(\zeta) + \cos\psi\cos\theta f_6(\zeta) \\ & - \sin\psi f_7(\zeta) + \cos\psi\sin\theta f_8(\zeta), \\ L_f^2 h_2(\zeta) = {} & (w\sin\psi\cos\theta - u\sin\psi\sin\theta)f_4(\zeta) \\ & + (u\cos\psi\cos\theta - v\cos\psi)f_5(\zeta) \\ & + w\cos\psi\sin\theta f_5(\zeta) + \sin\psi\cos\theta f_6(\zeta) \\ & - \cos\psi f_7(\zeta) + \sin\psi\sin\theta f_8(\zeta), \end{aligned}$$

$$L_f^2 h_3(\zeta) = (-u\cos\theta - w\sin\theta) f_4(\zeta) - \sin\theta f_6(\zeta) + \cos\theta f_8(\zeta),$$
$$L_f^2 h_4(\zeta) = f_9(\zeta),$$
$$L_f^2 h_5(\zeta) = \frac{r\sin\theta}{\cos^2\theta}, \quad \text{and} \quad f_4(\zeta) + \frac{f_{10}(\zeta)}{\cos\theta}.$$

Then, we can design two new vectors $x \in \mathbb{R}^5$ and $v \in \mathbb{R}^5$ as follows:

$$x = [h_i(\zeta)]^{\mathrm{T}}, \ v = \left[L_f h_i(\zeta)\right]^{\mathrm{T}}, \ i = 1,2,\ldots,5, \tag{6}$$
$$\Gamma(\zeta) = [\Gamma_{ij}(\zeta)] \in \mathbb{R}^{5\times 5}, \ B(\zeta) = \left[L_f^2 h_i(\zeta)\right]^{\mathrm{T}}, \tag{7}$$

where $\Gamma_{ij}(\zeta) = L_{g_j} L_f h_i(\zeta)$, $1 \le i \le p$, $1 \le j \le p$.

The new control input, which is used in the linearization system, can be defined as

$$u = B(\zeta) + \Gamma(\zeta)\hat{u}, \tag{8}$$

where $\hat{u}$ is the input of the nonlinear system in Eq. (2).

Thus, the mathematical model of the i-th AUV can be modified into a standard double integrator dynamic model by state feedback linearization as follows:

$$\dot{x}_i = v_i, \dot{v}_i = u_i, \tag{9}$$

where $x_i \in \mathbb{R}^5$, $v_i \in \mathbb{R}^5$, $u_i \in \mathbb{R}^5$, $i = 1,2,\ldots,n$.

2.3. Basic lemmas and notations

In this section, various lemmas that play important roles in the subsequent analysis are introduced.

Lemma 1:[30] For a given symmetric matrix $S \in \mathbb{R}^{n\times n}$ with the form

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{10}$$

where $S_{11} \in \mathbb{R}^{r\times r}$, $S_{12} \in \mathbb{R}^{r\times(n-r)}$, $S_{21} \in \mathbb{R}^{(n-r)\times r}$, $S_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, $S < 0$ if and only if $S_{11} < 0$, $S_{22} - S_{21}S_{11}^{-1}S_{12} < 0$, or $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{21} < 0$.

Lemma 2[31] Suppose that $f(t)$ is F_t-measurable and assume that $E[f(t)1_{\{\theta=i\}}]$ exists, then, for any $i \in n$,

$$E\left[f(x)d(1_{\{\theta=i\}})\right] = \sum_{j=1}^{n} \gamma_{ji} E\left[f(x)1_{\{\theta=i\}}\right]\mathrm{d}t + o(\mathrm{d}t), \tag{11}$$

Lemma 3[32] For any constant matrix $M \in \mathbb{R}^{n\times n}$, $M = M^{\mathrm{T}} > 0$, scalar γ, vector function $x : [0,\gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined as

$$-\gamma\int_0^\gamma x^{\mathrm{T}}(s)Mx(s) \le -\left(\int_0^\gamma x(s)\mathrm{d}s\right)^{\mathrm{T}} M \left(\int_0^\gamma x(s)\mathrm{d}s\right). \tag{12}$$

Notation 1 In this paper, $\mathbb{R}$ represents the real number set, thus $\mathbb{R}^m$ and $\mathbb{R}^{n\times m}$ are the $m\times 1$ column vector and the $n\times m$ matrix with real numbers. I_n denotes the identity matrix with n dimensions, and $\mathbf{1}_n$ (sometimes, $\mathbf{1}$, for short) denotes a column vector with all entries equal to one. A^{T} denotes the transpose of A, and $A > 0$ means that A is positive definite. $\otimes$ represents the Kronecker product of matrices. Let $\|\cdot\|$ denote the Euclidean norm.

3. Main result

In this work, the multi-AUVs operate in the form of a stochastically switching topology. The switching process is assumed to be driven by a finite ergodic Markov chain with a topology set $\{G_1, G_2, \ldots, G_N\}$ and transition probability matrix $\Pi = [\gamma_{ij}]_{N\times N} \in \mathbb{R}^{N\times N}$. Note that Π is a transition rate matrix whose row summation is zero and all off-diagonal elements are nonnegative.[33]

Given a probability space (U, F, P), where U is the sample space, F is the algebra of events, and P is the probability measure defined on F, then

$$P(\theta(t+\Delta t) = j|\theta(t) = i) = \begin{cases} \gamma_{ij}\Delta t + o(\Delta t), & \text{if} \quad i \ne j, \\ 1 + \gamma_{ii}\Delta t + o(\Delta t), & \text{if} \quad i = j, \end{cases} \tag{13}$$

where $\Delta t > 0$, γ_{ij} is the transition rate from i to j if $i \ne j$, otherwise, $\gamma_{ii} = -\sum_{j=1}^{j\ne i} \gamma_{ij}$, $o(\Delta t)$ denotes an infinitesimal of a higher order than Δt, which means $\lim_{t\to\infty}\left[o(\Delta t)/\Delta t\right] = 0$.

Next, the stability conditions for multi-AUVs based on the simplified model defined by Eq. (9) under protocol (14) are discussed:

$$\begin{aligned} u_i(t) \ = \ & -K_{\mathrm{p}} \sum_{j\in N_i^x} a_{ij}(t)(x_i(t) - x_j(t)) \\ & -K_{\mathrm{v}} \sum_{j\in N_i^v} b_{ij}(t)(v_i(t) - v_j(t)), \end{aligned} \tag{14}$$

where $K_{\mathrm{p}} \in \mathbb{R}^{5\times 5}$ and $K_{\mathrm{v}} \in \mathbb{R}^{5\times 5}$ are the common consensus gain matrices for position and velocity topologies, respectively, and $a_{ij}(t)$ and $b_{ij}(t)$ are the (i,j) entries of the adjacency matrices A_{p} and A_{v}.

Thus, $\varepsilon_i(t) = \left[x_i^{\mathrm{T}}(t) \ v_i^{\mathrm{T}}(t)\right] \in \mathbb{R}^{10}$ is defined. Combination of protocols (14) and (9) can be changed into

$$\begin{aligned} \dot{\varepsilon}_i(t) \ &= A\varepsilon_i(t) + Bu_i(t) \\ &= A\varepsilon_i(t) + BK_{\mathrm{p}} \sum_{j\in N_i^x} a_{ij}(t)(x_j(t) - x_i(t)) \\ &\quad + BK_{\mathrm{v}} \sum_{j\in N_i^v} b_{ij}(t)(v_j(t) - v_i(t)), \end{aligned} \tag{15}$$

where

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{10\times 10}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathbb{R}^{10\times 5},$$

$I \in \mathbb{R}^{5\times 5}$, and $K_{\mathrm{p}}, K_{\mathrm{v}} \in \mathbb{R}^{5\times 5}$.

For $i = 1,2,\ldots,n$, the error vector is defined as $\xi_i(t) = \varepsilon_i(t) - \bar{\varepsilon}(t)$, where $\bar{\varepsilon}(t) = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i(t)$ is the average state vector. Thus, equation (15) can be rewritten as

$$\dot{\xi}_i(t) \ = A\xi_i(t) + BK_{\mathrm{p}} \sum_{j\in N_i^x} l_{ij}^{\mathrm{p}}(t) Q^{\mathrm{p}} \xi_j(t)$$

$$+BK_{\mathrm{v}}\sum_{j\in N_i^{\mathrm{v}}} l_{ij}^{\mathrm{v}}(t)Q^{\mathrm{v}}\xi_j(t), \tag{16}$$

where $Q^{\mathrm{p}}=\begin{bmatrix} I & 0\end{bmatrix}\in\mathbb{R}^{5\times 10}$ and $Q^{\mathrm{v}}=\begin{bmatrix} 0 & I\end{bmatrix}\in\mathbb{R}^{5\times 10}$ are defined to filter position and velocity states in $\xi_i(t)$.

Let $\xi(t)=\left[\xi_1^{\mathrm{T}}(t),\ldots,\xi_n^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, then it will follow that

$$\begin{aligned}\dot{\xi}(t) \;=\; & (I_n\otimes A)\,\xi(t)-\left(L_{\mathrm{p}}(t)\otimes BK_{\mathrm{p}}Q^{\mathrm{p}}\right)\xi(t)\\ & -\left(L_{\mathrm{v}}(t)\otimes BK_{\mathrm{v}}Q^{\mathrm{v}}\right)\xi(t).\end{aligned} \tag{17}$$

Note that (A,B) is stabilizable if there is a feedback gain matrix K_{AB} such that $A+BK_{AB}$ is of Hurwitz.[34] Based on the matrix Riccati equation,[35] there exists $\bar{P}>0$ that satisfies

$$\bar{P}A+A^{\mathrm{T}}\bar{P}-2\bar{P}BB^{\mathrm{T}}\bar{P}<0. \tag{18}$$

Thus, the gain matrices K_{P} and K_{v} are designed as

$$K_{\mathrm{p}}=\alpha\bar{P}B,\;\; K_{\mathrm{v}}=\beta\bar{P}B, \tag{19}$$

where $\alpha=k_\alpha\cdot\begin{bmatrix}0 & I\end{bmatrix}^{\mathrm{T}}\in\mathbb{R}^{5\times 10}$, $\beta=k_\beta\cdot\begin{bmatrix}0 & I\end{bmatrix}^{\mathrm{T}}\in\mathbb{R}^{5\times 10}$, and k_α and k_β are positive constants.

Definition 1 The multi-AUVs system with simplified mathematical model (9) under control protocol (14) is said to reach consensus if there exists a consensus gain K in any case of $x_i(0)\in\mathbb{R}^5$, $v_i(0)\in\mathbb{R}^5$, and initial distribution of $\theta(0)$:

$$\begin{aligned}&\lim_{t\to\infty}E\left(\left\|x_i(t)-x_j(t)\right\|^2\right)=0,\;\text{for any}\;\; i,j\in V,\\ &\lim_{t\to\infty}E\left(\left\|v_i(t)-v_j(t)\right\|^2\right)=0,\;\text{for any}\;\; i,j\in V,\end{aligned} \tag{20}$$

where the expectation $E(\cdot)$ is taken under the measure P.

Assumption 1 The Markov process $\theta(t)$ is ergodic and the communication graphs $G_1,G_2,\ldots,G_N$ are balanced.

3.1. Consensus of multi-AUVs with the same delays in double Markovian switching topologies

In this section, the time delays in switching topologies are assumed to be the same. Thus, the control protocol $u_i(t)$ is described as

$$\begin{aligned}u_i(t) \;=\; & -K_{\mathrm{p}}\sum_{j\in N_i^x} a_{ij}(t)(x_i(t-\tau)-x_j(t-\tau))\\ & -K_{\mathrm{v}}\sum_{j\in N_i^{\mathrm{v}}} b_{ij}(t)(v_i(t-\tau)-v_j(t-\tau)).\end{aligned} \tag{21}$$

Assumption 2 The time delay satisfies $0<\tau(t)<h$ and $\dot{\tau}(t)<\kappa<1$ for $t\geq 0$, where $h,\kappa\geq 0$.

Thus, the multi-AUVs system with the same delays can be summarized as

$$\dot{\xi}(t)=(I_n\otimes A)\,\xi(t)-\left(L_{\mathrm{p}}(t)\otimes BK_{\mathrm{p}}Q^{\mathrm{p}}+L_{\mathrm{v}}(t)\otimes BK_{\mathrm{v}}Q^{\mathrm{v}}\right)\xi(t-\tau). \tag{22}$$

For short, $\dot{\xi}(t)=\tilde{A}\xi(t)+\tilde{B}\xi(t-\tau)$, where $\tilde{A}=I_n\otimes A$, $\tilde{B}=-L_{\mathrm{p}}(t)\otimes BK_{\mathrm{p}}Q^{\mathrm{p}}-L_{\mathrm{v}}(t)\otimes BK_{\mathrm{v}}Q^{\mathrm{v}}$.

As $K_{\mathrm{P}}=\alpha\bar{P}B$, $K_{\mathrm{V}}=\beta\bar{P}B$, we can derive that

$$\tilde{B}_{\mathrm{L}}=-L_{\mathrm{p}}^u\otimes BK_{\mathrm{p}}Q^{\mathrm{p}}-L_{\mathrm{v}}^u\otimes BK_{\mathrm{v}}Q^{\mathrm{v}}. \tag{23}$$

For matrix representation simplicity, $\varsigma_i\in\mathbb{R}^{nN\times 3nN}$ are defined as block entry matrices, such as $\varsigma_1=\begin{bmatrix}I_{nN} & 0 & 0\end{bmatrix}$. The notations of several matrices are defined as

$$\phi(t)=\left[\xi^{\mathrm{T}}(t),\xi^{\mathrm{T}}(t-\tau),\xi^{\mathrm{T}}(t-h-\tau)\right]^{\mathrm{T}}, \tag{24}$$

$$\begin{aligned}\Theta=\; & \varsigma_1^{\mathrm{T}}\left(\tilde{A}^{\mathrm{T}}P+P\tilde{A}+Q_1-\tau^{-1}R_1+\tilde{A}^{\mathrm{T}}\Omega\tilde{A}\right)\varsigma_1\\ & +\varsigma_2^{\mathrm{T}}\left[(1-\kappa)(Q_2-Q_1)-\tau^{-1}R_1-h^{-1}R_1+\tilde{B}_{\mathrm{L}}^{\mathrm{T}}\Omega\tilde{B}_{\mathrm{L}}\right]\varsigma_2\\ & -\varsigma_3^{\mathrm{T}}\left[(1-\kappa)Q_2-h^{-1}R_1\right]\varsigma_3\\ & +\varsigma_1^{\mathrm{T}}\left(P\tilde{B}_{\mathrm{L}}+\tau^{-1}R_1+\tilde{A}^{\mathrm{T}}\Omega\tilde{B}_{\mathrm{L}}\right)\varsigma_2\\ & +\varsigma_2^{\mathrm{T}}\left(\tilde{B}_{\mathrm{L}}^{\mathrm{T}}P+\tau^{-1}R_1+\tilde{B}_{\mathrm{L}}^{\mathrm{T}}\Omega\tilde{A}\right)\varsigma_1\\ & +\varsigma_2^{\mathrm{T}}\left(h^{-1}R_1\right)\varsigma_3+\varsigma_3^{\mathrm{T}}\left(h^{-1}R_1\right)\varsigma_2,\end{aligned} \tag{25}$$

where $\Omega=\tau R_1+hR_2$.

Theorem 1 For the multi-AUVs system simplified by state feedback linearization, suppose that Assumptions 1 and 2 hold, and assume that G_{p}^u and G_{v}^u contain spanning trees. The consensus gains are chosen as $K_{\mathrm{p}}=\alpha\bar{P}B$ and $K_{\mathrm{v}}=\beta\bar{P}B$ with $\bar{P}$ given from Eq. (18),

$$k_\alpha\geq\lambda_2(\hat{L}_{\mathrm{p}}^u)\cdot\min_{i\in N}\pi_i,\;\;\text{and}\;\; k_\beta\geq\lambda_2(\hat{L}_{\mathrm{v}}^u)\cdot\min_{i\in N}\pi_i. \tag{26}$$

The AUVs in system (9) converge to the central state stochastically if there exist real matrices $Q_1\geq 0$, $Q_2\geq 0$, $R_1\geq 0$, $R_2\geq 0$ with appropriate dimensions satisfying the following LMIs:

$$\begin{bmatrix}\psi_{11} & P\tilde{B}_{\mathrm{L}}+\tau^{-1}R_1 & 0 & \tilde{A}^{\mathrm{T}}\\ \tilde{B}_{\mathrm{L}}^{\mathrm{T}}P+\tau^{-1}R_1 & \psi_{22} & h^{-1}R_1 & \tilde{B}_{\mathrm{L}}^{\mathrm{T}}\\ 0 & h^{-1}R_1 & \psi_{33} & 0\\ \tilde{A} & \tilde{B}_{\mathrm{L}} & 0 & \psi_{44}\end{bmatrix}<0, \tag{27}$$

where $\tilde{A}=I_n\otimes A$, $\psi_{11}=\tilde{A}^{\mathrm{T}}P+P\tilde{A}+Q_1-\tau^{-1}R_1$, $\psi_{22}=(1-\kappa)(Q_2-Q_1)-\left(\tau^{-1}+h^{-1}\right)R_1$, $\psi_{33}=h^{-1}R_1-(1-\kappa)Q_2$, $\psi_{44}=-(\tau R_1+hR_2)$, and $\tilde{B}_{\mathrm{L}}=-L_{\mathrm{p}}^u\otimes BK_{\mathrm{p}}Q^{\mathrm{p}}-L_{\mathrm{v}}^u\otimes BK_{\mathrm{v}}Q^{\mathrm{v}}$.

Proof For each $k\in N$, choosing the following Lyapunov–Krasovskii functional candidate as

$$V(\xi(t),k)=\sum_{i=1}^{3}V_i(\xi(t),k), \tag{28}$$

where

$$\begin{aligned}&V_1=\xi^{\mathrm{T}}P\xi,\quad V_2=\int_{t-\tau}^{t}\xi^{\mathrm{T}}Q_1\xi\,\mathrm{d}s+\int_{t-h-\tau}^{t-\tau}\xi^{\mathrm{T}}Q_2\xi\,\mathrm{d}s,\\ &V_3=\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\xi}^{\mathrm{T}}R_1\dot{\xi}\,\mathrm{d}s\mathrm{d}\theta+\int_{-h-\tau}^{-\tau}\int_{t+\theta}^{t}\dot{\xi}^{\mathrm{T}}R_2\dot{\xi}\,\mathrm{d}s\mathrm{d}\theta,\\ &P=I_n\otimes\bar{P},\;\; Q_1>0,\;\; Q_2>0,\;\; R_1>0,\;\;\text{and}\;\; R_2>0.\end{aligned}$$

In what follows, V, V_i, and ξ are, respectively, short for $V(\xi(t),k)$, $V_i(\xi(t),k)$, and $\xi(t)$.

Let $\bar{V}$ represent the expectation of the Lyapunov function,

$$\bar{V}=E(V)=E\left(\sum_{i=1}^{3}V_i\right), \tag{29}$$

and for the i-th topology at time t, $V^i = \sum_{j=1}^{3} V_j 1_{\{\theta(t)=i\}}$.

Note that $V^i,\ i \in N$ represents the Lyapunov function for the i-th topology and $V_i,\ i \in \{1,2,3\}$ represents the Lyapunov function for the i-th part in Eq. (28). Thus, the Lyapunov function can be changed into

$$\bar{V}^i = E\left(\sum_{j=1}^{3} V_j 1_{\{\theta(t)=i\}}\right), \tag{30}$$

and the derivatives of the Lyapunov function can be rewrite as

$$\mathrm{d}\bar{V}^i = E\left(\sum_{j=1}^{3} \mathrm{d}\left(V_j 1_{\{\theta(t)=i\}}\right)\right). \tag{31}$$

The derivatives of these parts are

$$\begin{aligned}
&\mathrm{d}\left(V_1 1_{\{\theta(t)=i\}}\right)\\
&= \mathrm{d}\xi^{\mathrm{T}} P\xi 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} P \mathrm{d}\xi 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} P\xi \,\mathrm{d}1_{\{\theta(t)=i\}} + o(\mathrm{d}t),\\
&\mathrm{d}\left(V_2 1_{\{\theta(t)=i\}}\right)\\
&= \mathrm{d}\left(\int_{t-\tau}^{t} \xi^{\mathrm{T}} Q_1 \xi \,\mathrm{d}s + \int_{t-h-\tau}^{t-\tau} \xi^{\mathrm{T}} Q_2 \xi \,\mathrm{d}s\right) 1_{\{\theta(t)=i\}}\\
&\quad + \left(\int_{t-\tau}^{t} \xi^{\mathrm{T}} Q_1 \xi \,\mathrm{d}s + \int_{t-h-\tau}^{t-\tau} \xi^{\mathrm{T}} Q_2 \xi \,\mathrm{d}s\right) \mathrm{d}1_{\{\theta(t)=i\}} + o(\mathrm{d}t),\\
&\mathrm{d}\left(V_3 1_{\{\theta(t)=i\}}\right)\\
&= \mathrm{d}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} \,\mathrm{d}s\mathrm{d}\theta + \int_{-h-\tau}^{-\tau}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} \,\mathrm{d}s\mathrm{d}\theta\right) 1_{\{\theta(t)=i\}}\\
&\quad + \left(\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} \,\mathrm{d}s\mathrm{d}\theta\right) \mathrm{d}1_{\{\theta(t)=i\}}\\
&\quad + \left(\int_{-h-\tau}^{-\tau}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} \,\mathrm{d}s\mathrm{d}\theta\right) \mathrm{d}1_{\{\theta(t)=i\}} + o(\mathrm{d}t).
\end{aligned}$$

According to Lemma 2, we can obtain that

$$\begin{aligned}
\mathrm{d}\bar{V}^i = E\Big\{ &\mathrm{d}\xi^{\mathrm{T}} P\xi 1_{\{\theta(t)=i\}} + \mathrm{d}\left(\int_{t-\tau}^{t} \xi^{\mathrm{T}} Q_1 \xi \,\mathrm{d}s\right) 1_{\{\theta(t)=i\}}\\
&+ \xi^{\mathrm{T}} P \mathrm{d}\xi 1_{\{\theta(t)=i\}} + \mathrm{d}\left(\int_{t-h-\tau}^{t-\tau} \xi^{\mathrm{T}} Q_2 \xi \,\mathrm{d}s\right) 1_{\{\theta(t)=i\}}\\
&+ \mathrm{d}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} \,\mathrm{d}s\mathrm{d}\theta\right) 1_{\{\theta(t)=i\}}\\
&+ \mathrm{d}\left(\int_{-h-\tau}^{-\tau}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} \,\mathrm{d}s\mathrm{d}\theta\right) 1_{\{\theta(t)=i\}}\Big\}\\
&+ \sum_{j=1}^{N} \gamma_{ji} \bar{V}^j \mathrm{d}t + o(\mathrm{d}t).
\end{aligned} \tag{32}$$

Then the integrations of $\xi(\cdot)$ and $\dot{\xi}(\cdot)$ can be transformed based on the characteristics of derivatives. Moreover, equation (31) can be changed into the following equation:

$$\begin{aligned}
\dot{\bar{V}}^i \le E\Big\{ &\dot{\xi}^{\mathrm{T}} P\xi 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} P\dot{\xi} 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} Q_1 \xi 1_{\{\theta(t)=i\}}\\
&+ (1-\kappa)\left(\xi_\tau^{\mathrm{T}} Q_2 \xi_\tau - \xi_{h\cdot\tau}^{\mathrm{T}} Q_2 \xi_{h\cdot\tau} - \xi_\tau^{\mathrm{T}} Q_1 \xi_\tau\right) 1_{\{\theta(t)=i\}}\\
&+ \tau\dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} 1_{\{\theta(t)=i\}} - \int_{t-\tau}^{t} \dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} \,\mathrm{d}s 1_{\{\theta(t)=i\}}\\
&+ h\dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} 1_{\{\theta(t)=i\}} - \int_{t-h-\tau}^{t-\tau} \dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} \,\mathrm{d}s 1_{\{\theta(t)=i\}}\Big\}\\
&+ \sum_{j=1}^{N} \gamma_{ji} \bar{V}^j,
\end{aligned} \tag{33}$$

where ξ_τ and $\xi_{h\cdot\tau}$ are the abbreviations of $\xi(t-\tau)$ and $\xi(t-h-\tau)$ respectively.

By Lemma 3, equation (33) can be rewritten as

$$\begin{aligned}
\dot{\bar{V}}^i \le E\Big\{ &\dot{\xi}^{\mathrm{T}} P\xi 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} P\dot{\xi} 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} Q_1 \xi 1_{\{\theta(t)=i\}}\\
&+ (1-\kappa)\left(\xi_\tau^{\mathrm{T}} Q_2 \xi_\tau - \xi_{h\cdot\tau}^{\mathrm{T}} Q_2 \xi_{h\cdot\tau} - \xi_\tau^{\mathrm{T}} Q_1 \xi_\tau\right) 1_{\{\theta(t)=i\}}\\
&+ \tau\dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} 1_{\{\theta(t)=i\}} - \tau^{-1}(\xi-\xi_\tau)^{\mathrm{T}} R_1 (\xi-\xi_\tau) 1_{\{\theta(t)=i\}}\\
&+ h\dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} 1_{\{\theta(t)=i\}}\\
&- h^{-1}(\xi_\tau-\xi_{h\cdot\tau})^{\mathrm{T}} R_1 (\xi_\tau-\xi_{h\cdot\tau}) 1_{\{\theta(t)=i\}}\Big\}\\
&+ \sum_{j=1}^{N} \gamma_{ji} \bar{V}^j.
\end{aligned} \tag{34}$$

Note that $\dot{\bar{V}} = \sum_{i=1}^{N} \dot{\bar{V}}^i(\cdot)$ and $\sum_{i=1}^{N} \gamma_{ji} = 0,\ i \in N,\ \dot{\xi} = \tilde{A}\xi + \tilde{B}\xi_\tau$. For the union of switching topologies, $\dot{\bar{V}}$ can be summarized as follows:

$$\begin{aligned}
\dot{\bar{V}} \le E\Big\{ &\xi^{\mathrm{T}}\left(\tilde{A}^{\mathrm{T}}P + P\tilde{A} + Q_1\right)\xi + \xi_\tau^{\mathrm{T}} \tilde{B}_{\mathrm{L}}^{\mathrm{T}} P\xi + \xi^{\mathrm{T}} P\tilde{B}_{\mathrm{L}}\xi_\tau\\
&+ \xi_\tau^{\mathrm{T}}\left[(1-\kappa)(Q_2-Q_1)\right]\xi_\tau - \xi_{h\cdot\tau}^{\mathrm{T}}(1-\kappa)Q_2\xi_{h\cdot\tau}\\
&+ \xi^{\mathrm{T}}\tilde{A}^{\mathrm{T}}(\tau R_1 + hR_2)\tilde{A}\xi + \xi_\tau^{\mathrm{T}}\tilde{B}_{\mathrm{L}}^{\mathrm{T}}(\tau R_1 + hR_2)\tilde{B}_{\mathrm{L}}\xi_\tau\\
&+ \xi^{\mathrm{T}}\tilde{A}^{\mathrm{T}}(\tau R_1 + hR_2)\tilde{B}_{\mathrm{L}}\xi_\tau + \xi_\tau^{\mathrm{T}}\tilde{B}_{\mathrm{L}}^{\mathrm{T}}(\tau R_1 + hR_2)\tilde{A}\xi\\
&- \tau^{-1}\left(\xi^{\mathrm{T}} R_1 \xi - \xi^{\mathrm{T}} R_1 \xi_\tau - \xi_\tau^{\mathrm{T}} R_1 \xi + \xi_\tau^{\mathrm{T}} R_1 \xi_\tau\right)\\
&- h^{-1}\left(\xi_\tau^{\mathrm{T}} R_1 \xi_\tau - \xi_\tau^{\mathrm{T}} R_1 \xi_{h\cdot\tau} - \xi_{h\cdot\tau}^{\mathrm{T}} R_1 \xi_\tau + \xi_{h\cdot\tau}^{\mathrm{T}} R_1 \xi_{h\cdot\tau}\right)\Big\},
\end{aligned} \tag{35}$$

where $\tilde{B}_{\mathrm{L}} = -L_{\mathrm{p}}^u \otimes BK_{\mathrm{p}}Q^{\mathrm{p}} - L_{\mathrm{v}}^u \otimes BK_{\mathrm{v}}Q^{\mathrm{v}}$, $K_{\mathrm{p}} = \alpha\bar{P}B$, and $K_{\mathrm{v}} = \beta\bar{P}B$.

Then $\dot{\bar{V}}$ can be rewrite as

$$\begin{aligned}
\dot{\bar{V}} &\le E\Big\{ \phi^{\mathrm{T}}\varsigma_1^{\mathrm{T}}\left(\tilde{A}^{\mathrm{T}}P + P\tilde{A} + Q_1\right)\varsigma_1\phi + \phi^{\mathrm{T}}\varsigma_2^{\mathrm{T}}\tilde{B}_{\mathrm{L}}^{\mathrm{T}} P\varsigma_1\phi\\
&\quad + \phi^{\mathrm{T}}\varsigma_1^{\mathrm{T}} P\tilde{B}_{\mathrm{L}}\varsigma_2\phi + \phi^{\mathrm{T}}\varsigma_2^{\mathrm{T}}\left[(1-\kappa)(Q_2-Q_1)\right]\varsigma_2\phi\\
&\quad - \phi^{\mathrm{T}}\varsigma_3^{\mathrm{T}}(1-\kappa)Q_2\varsigma_3\phi + \phi^{\mathrm{T}}\varsigma_1^{\mathrm{T}}\tilde{A}^{\mathrm{T}}\Omega\tilde{A}\varsigma_1\phi\\
&\quad + \phi^{\mathrm{T}}\varsigma_2^{\mathrm{T}}\tilde{B}_{\mathrm{L}}^{\mathrm{T}}\Omega\tilde{B}_{\mathrm{L}}\varsigma_2\phi + \phi^{\mathrm{T}}\varsigma_1^{\mathrm{T}}\tilde{A}^{\mathrm{T}}\Omega\tilde{B}_{\mathrm{L}}\varsigma_2\phi + \phi^{\mathrm{T}}\varsigma_2^{\mathrm{T}}\tilde{B}_{\mathrm{L}}^{\mathrm{T}}\Omega\tilde{A}\varsigma_1\phi\\
&\quad - \tau^{-1}\phi^{\mathrm{T}}\left(\varsigma_1^{\mathrm{T}} R_1\varsigma_1 - \varsigma_1^{\mathrm{T}} R_1\varsigma_2 - \varsigma_2^{\mathrm{T}} R_1\varsigma_1 + \varsigma_2^{\mathrm{T}} R_1\varsigma_2\right)\phi\\
&\quad - h^{-1}\phi^{\mathrm{T}}\left(\varsigma_2^{\mathrm{T}} R_1\varsigma_2 - \varsigma_2^{\mathrm{T}} R_1\varsigma_3 - \varsigma_3^{\mathrm{T}} R_1\varsigma_2 + \varsigma_3^{\mathrm{T}} R_1\varsigma_3\right)\phi\Big\}\\
&= E\left\{\phi^{\mathrm{T}}\Theta\phi\right\},
\end{aligned} \tag{36}$$

where $\Omega = \tau R_1 + hR_2$, and ϕ and Θ are defined in Eqs. (24) and (25).

With $\varsigma(t) = [\varsigma_1, \varsigma_2, \varsigma_3]^{\mathrm{T}}$,

$$\dot{\bar{V}}(\xi(t), k) \le E\left\{\phi^{\mathrm{T}}(t)\varsigma^{\mathrm{T}}(t)\Psi\varsigma(t)\phi(t)\right\}, \tag{37}$$

and

$$\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} & 0\\ \psi_{21} & \psi_{22} & h^{-1}R_1\\ 0 & h^{-1}R_1 & \psi_{33} \end{bmatrix} + \begin{bmatrix} A^{\mathrm{T}}\\ B_{\mathrm{L}}^{\mathrm{T}}\\ 0 \end{bmatrix} \Omega \begin{bmatrix} A & B_{\mathrm{L}} & 0 \end{bmatrix}, \tag{38}$$

where $\psi_{11} = \tilde{A}^{\mathrm{T}}P + P\tilde{A} + Q_1 - \tau^{-1}R_1$, $\psi_{12} = \psi_{21}^{\mathrm{T}} = P\tilde{B}_{\mathrm{L}} + \tau^{-1}R_1$, $\psi_{22} = (1-\kappa)(Q_2 - Q_1) - \left(\tau^{-1} + h^{-1}\right)R_1$, and $\psi_{33} = h^{-1}R_1 - (1-\kappa)Q_2$.

According to Lemma 1, and when condition (27) in Theorem 1 holds, it can be derived that

$$\Psi < 0, \tag{39}$$

Thus, it is easy to obtain $\dot{V}(\xi(t),k) < 0$ for $t > 0$. Moreover, there exists a constant $\rho > 0$, satisfying

$$\dot{V}(\xi(t),k) < -\rho \|\xi(t)\|^2. \tag{40}$$

This means that system (9) is asymptotically stable in the case of the same time-varying delays.

Remark 1 From Theorem 1, it obviously follows that $\xi \to 0$ as $t \to \infty$, which means that equation (20) holds. Thus, all AUVs will unite and hold the same motion states. However, in real ocean missions, a fixed formation consisting of predesigned relevant distances and angles is required. As the relevant distances and angles are invariable when the multi-AUVs are operated, a scenario in which the AUVs gather at a central point and keep the common motion states can represent a scenario in which they hold a fixed formation. Consequently, the conditions in Theorem 1 can solve and guarantee the stability of multi-AUVs with the same delays.

Remark 2 In this section discussed mainly is the stable phase involving the same point and motion states. Because a formation has no leader or reference path, all the members will maintain the same random motion state when the AUVs arrive at the common point. This state also causes them to have random positions. Thus, in the phase after all members reach a common point, characteristics such as randomness, convergence and unity will appear.

3.2. Consensus of multi-AUVs with different delays in double Markovian switching topologies

In this subsection, the time delays in double switching topologies are assumed to be different. The following assumption, which is similar to Assumption 2, is needed.

Assumption 3 In this case, with different time-varying delays in communication topologies, the delays are assumed to satisfy $0 < \tau_1(t) < h_1, 0 < \tau_2(t) < h_2$ and for $t \geq 0$, $\dot{\tau}_1(t) < \kappa_1 < 1, \dot{\tau}_2(t) < \kappa_2 < 1$, where $h_1, \kappa_1, h_2, \kappa_2 \geq 0$.

The control protocol $u_i(t)$ is given as

$$\begin{aligned} u_i(t) \;=\; & -K_{\mathrm{p}} \sum_{j \in N_i^x} a_{ij}(t)(x_i(t-\tau_1) - x_j(t-\tau_1)) \\ & - K_{\mathrm{v}} \sum_{j \in N_i^{\mathrm{v}}} b_{ij}(t)(v_i(t-\tau_2) - v_j(t-\tau_2)). \end{aligned} \tag{41}$$

Thus, the system model can be summarized as

$$\begin{aligned} \dot{\xi}(t) \;=\; & (I_n \otimes A)\xi(t) + \left(-L^{\mathrm{p}}(t) \otimes BK_{\mathrm{p}}Q^{\mathrm{p}}\right)\xi(t-\tau_1) \\ & + \left(-L^{\mathrm{v}}(t) \otimes BK_{\mathrm{v}}Q^{\mathrm{v}}\right)\xi(t-\tau_2). \end{aligned} \tag{42}$$

Let $\dot{\xi}(t) = \tilde{A}\xi(t) + \tilde{B}_1\xi(t-\tau_1) + \tilde{B}_2\xi(t-\tau_2)$, where $\tilde{B}_1 = -L_{\mathrm{p}}(t) \otimes BK_{\mathrm{p}}Q^{\mathrm{p}}$, $\tilde{B}_2 = -L_{\mathrm{v}}(t) \otimes BK_{\mathrm{v}}Q^{\mathrm{v}}$ and $\tilde{A} = I_n \otimes A$.

The control gains can be designed as $K_{\mathrm{p}} = \alpha\bar{P}B$, $K_{\mathrm{v}} = \beta\bar{P}B$ and we can derive the union matrices $\tilde{B}_{1\mathrm{L}}$ and $\tilde{B}_{2\mathrm{L}}$ as

$$\tilde{B}_{1\mathrm{L}} = \left(-L_{\mathrm{p}}^u \otimes BK_{\mathrm{p}}Q^{\mathrm{p}}\right), \quad \tilde{B}_{2\mathrm{L}} = \left(-L_{\mathrm{v}}^u \otimes BK_{\mathrm{v}}Q^{\mathrm{v}}\right). \tag{43}$$

For matrix representation simplicity in the following, $\varsigma_i \in \mathbb{R}^{nN \times 5nN}$ are defined as the block entry matrices, such as $\varsigma_1 = \begin{bmatrix} I & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{nN \times 5nN}$. The notations of several matrices are defined as

$$\begin{aligned} \phi(t) = & \left[\xi^{\mathrm{T}}(t), \xi^{\mathrm{T}}(t-\tau_1), \xi^{\mathrm{T}}(t-\tau_2), \xi^{\mathrm{T}}(t-h_1-\tau_1), \right. \\ & \left. \xi^{\mathrm{T}}(t-h_2-\tau_2)\right]^{\mathrm{T}}, \end{aligned} \tag{44}$$

$$\begin{aligned} \Theta = & \varsigma_1^{\mathrm{T}}\left(\Upsilon + Q_1 + Q_3 + \tau_1^{-1}R_1 + \tau_2^{-1}R_3 + \tilde{A}^{\mathrm{T}}\Omega_2\tilde{A}\right)\varsigma_1 \\ & + \varsigma_2^{\mathrm{T}}\left(\tilde{B}_{1\mathrm{L}}^{\mathrm{T}}\Omega_2\tilde{B}_{1\mathrm{L}} - (1-\kappa_1)Q_1 - h_1^{-1}R_2 - \tau_1^{-1}R_1\right)\varsigma_2 \\ & + \varsigma_3^{\mathrm{T}}\left(\tilde{B}_{2\mathrm{L}}^{\mathrm{T}}\Omega_2\tilde{B}_{2\mathrm{L}} - (1-\kappa_2)Q_3 - \tau_2^{-1}R_3 - h_2^{-1}R_4\right)\varsigma_3 \\ & + \varsigma_3^{\mathrm{T}}\tilde{B}_{2\mathrm{L}}^{\mathrm{T}}\Omega_2\tilde{B}_{1\mathrm{L}}\varsigma_2 + \varsigma_2^{\mathrm{T}}\left(h_1^{-1}R_2\right)\varsigma_4 + \varsigma_4^{\mathrm{T}}\left(h_1^{-1}R_2\right)\varsigma_2 \\ & + \varsigma_2^{\mathrm{T}}\tilde{B}_{1\mathrm{L}}^{\mathrm{T}}\Omega_2\tilde{B}_{2\mathrm{L}}\varsigma_3 - \varsigma_4^{\mathrm{T}}\left(h_1^{-1}R_2\right)\varsigma_4 + \varsigma_3^{\mathrm{T}}\left(h_2^{-1}R_4\right)\varsigma_5 \\ & + \varsigma_2^{\mathrm{T}}\left(\tilde{B}_{1\mathrm{L}}^{\mathrm{T}}P + \tau_1^{-1}R_1 + \tilde{B}_{1\mathrm{L}}^{\mathrm{T}}\Omega_2\tilde{A}\right)\varsigma_1 \\ & + \varsigma_3^{\mathrm{T}}\left(\tilde{B}_{2\mathrm{L}}^{\mathrm{T}}P + \tau_2^{-1}R_3 + \tilde{B}_{2\mathrm{L}}^{\mathrm{T}}\Omega_2\tilde{A}\right)\varsigma_1 \\ & + \varsigma_1^{\mathrm{T}}\left(P\tilde{B}_{1\mathrm{L}} + \tau_1^{-1}R_1 + \tilde{A}^{\mathrm{T}}\Omega_2\tilde{B}_{1\mathrm{L}}\right)\varsigma_2 \\ & + \varsigma_1^{\mathrm{T}}\left(P\tilde{B}_{2\mathrm{L}} + \tau_2^{-1}R_3 + \tilde{A}^{\mathrm{T}}\Omega_2\tilde{B}_{2\mathrm{L}}\right)\varsigma_3 \\ & + \varsigma_5^{\mathrm{T}}\left(h_2^{-1}R_4\right)\varsigma_3 - \varsigma_5^{\mathrm{T}}\left(h_2^{-1}R_4\right)\varsigma_5, \end{aligned} \tag{45}$$

where $\Upsilon = \tilde{A}^{\mathrm{T}}P + P\tilde{A}$ and $\Omega_2 = \tau_1R_1 + \tau_2R_3 + h_1R_2 + h_2R_4$.

Theorem 2 For the system of multi-AUVs simplified by state feedback linearization, assume that Assumptions 1 and 3 hold, and further, assume that G_{p}^u and G_{v}^u contain spanning trees. The consensus gains are chosen as $K_{\mathrm{p}} = \alpha\bar{P}B$ and $K_{\mathrm{v}} = \beta\bar{P}B$ with $\bar{P}$ given by Eq. (18), and

$$\alpha \geq \lambda_2(\hat{L}_{\mathrm{p}}^u) \cdot \min_{i \in N} \pi_i \;\text{ and }\; \beta \geq \lambda_2(\hat{L}_{\mathrm{v}}^u) \cdot \min_{i \in N} \pi_i. \tag{46}$$

The AUVs in system (9) converge to the central state stochastically if there exist real matrices $Q_1 \geq 0$, $Q_2 \geq 0$, $R_1 \geq 0$, $R_2 \geq 0$ of appropriate dimensions satisfying the following LMIs:

$$\begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & 0 & 0 & \tilde{A}^{\mathrm{T}} \\ \varphi_{21} & \varphi_{22} & 0 & h_1^{-1}R_2 & 0 & \tilde{B}_{1\mathrm{L}}^{\mathrm{T}} \\ \varphi_{31} & 0 & \varphi_{33} & 0 & h_2^{-1}R_4 & \tilde{B}_{2\mathrm{L}}^{\mathrm{T}} \\ 0 & h_1^{-1}R_2 & 0 & -h_1^{-1}R_2 & 0 & 0 \\ 0 & 0 & h_2^{-1}R_4 & 0 & -h_2^{-1}R_4 & 0 \\ \tilde{A} & \tilde{B}_{1\mathrm{L}} & \tilde{B}_{2\mathrm{L}} & 0 & 0 & -\varphi_{66} \end{bmatrix} < 0, \tag{47}$$

where $\varphi_{11} = \tilde{A}^{\mathrm{T}}P + P\tilde{A} + Q_1 + Q_3 - \tau_1^{-1}R_1 - \tau_2^{-1}R_3$, $\varphi_{22} = -(1-\kappa_1)Q_1 - h_1^{-1}R_2 - \tau_1^{-1}R_1$, $\varphi_{12} = \varphi_{21}^{\mathrm{T}} = P\tilde{B}_1 + \tau_1^{-1}R_1$, $\varphi_{13} = \varphi_{31}^{\mathrm{T}} = P\tilde{B}_{2\mathrm{L}} + \tau_2^{-1}R_3$, $\varphi_{33} = -(1-\kappa_2)Q_3 - \tau_2^{-1}R_3 - h_2^{-1}R_4$, $\varphi_{66} = \tau_1R_1 + \tau_2R_3 + h_1R_2 + h_2R_4$, $\tilde{B}_{1\mathrm{L}} = -L_{\mathrm{p}}^u \otimes BK_{\mathrm{p}}Q^{\mathrm{p}}$, $\tilde{A} = I_n \otimes A$, and $\tilde{B}_{2\mathrm{L}} = -L_{\mathrm{v}}^u \otimes BK_{\mathrm{v}}Q^{\mathrm{v}}$.

Proof For each $k \in N$, we also choose the following Lyapunov–Krasovskii functional candidate,

$$V(\xi(t),k) = \sum_{i=1}^{3} V_i(\xi(t),k), \tag{48}$$

and

$$\begin{aligned}
V_1 &= \xi^{\mathrm{T}} P \xi,\\
V_2 &= \int_{t-\tau_1}^{t} \xi^{\mathrm{T}} Q_1 \xi \,\mathrm{d}s + \int_{t-\tau_2}^{t} \xi^{\mathrm{T}} Q_3 \xi \,\mathrm{d}s,\\
V_3 &= \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} \,\mathrm{d}s\,\mathrm{d}\theta + \int_{-h-\tau_1}^{-\tau_1}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} \,\mathrm{d}s\,\mathrm{d}\theta\\
&\quad + \int_{-\tau_2}^{0}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_3 \dot{\xi} \,\mathrm{d}s\,\mathrm{d}\theta + \int_{-h-\tau_2}^{-\tau_2}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_4 \dot{\xi} \,\mathrm{d}s\,\mathrm{d}\theta,
\end{aligned} \tag{49}$$

where $P = I_n \otimes \bar{P}$, $Q_1 > 0$, $Q_3 > 0$, $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, and $R_4 > 0$.

Just as indicated in Subsection 3.1, ξ, ξ_{τ_1}, and ξ_{τ_2} are abbreviations of $\xi(t)$, $\xi(t-\tau_1)$, and $\xi(t-\tau_2)$ respectively; the definitions of $\bar{V}$, V^i, V_i, $\bar{V}^i$, and $\mathrm{d}\bar{V}^i$ are also the same as those in the above section.

$$\begin{aligned}
\mathrm{d}\bar{V}^i = E\big(&\mathrm{d}\left(V_1 1_{\{\theta(t)=i\}}\right) + \mathrm{d}\left(V_2 1_{\{\theta(t)=i\}}\right)\\
&+ \mathrm{d}\left(V_3 1_{\{\theta(t)=i\}}\right)\big).
\end{aligned} \tag{50}$$

The derivatives of three parts are

$$\begin{aligned}
\mathrm{d}\left(V_1 1_{\{\theta(t)=i\}}\right) &= \mathrm{d}\xi^{\mathrm{T}} P \xi 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} P \mathrm{d}\xi 1_{\{\theta(t)=i\}}\\
&\quad + \xi^{\mathrm{T}} P \xi \,\mathrm{d}1_{\{\theta(t)=i\}} + o(\mathrm{d}t),\\
\mathrm{d}\left(V_2 1_{\{\theta(t)=i\}}\right) &= \mathrm{d}\left(V_2\right) 1_{\{\theta(t)=i\}} + \left(V_2\right) \mathrm{d}1_{\{\theta(t)=i\}} + o(\mathrm{d}t),\\
\mathrm{d}\left(V_3 1_{\{\theta(t)=i\}}\right) &= \mathrm{d}\left(V_3\right) 1_{\{\theta(t)=i\}} + \left(V_3\right) \mathrm{d}1_{\{\theta(t)=i\}} + o(\mathrm{d}t).
\end{aligned}$$

The elements of $\{\cdot\}1_{\{\theta(t)=i\}}$ can be transformed in accordance with Lemma 2,

$$\begin{aligned}
\mathrm{d}\bar{V}^i = E\Big\{&\mathrm{d}\xi^{\mathrm{T}} P \xi 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} P \mathrm{d}\xi 1_{\{\theta(t)=i\}}\\
&+ \mathrm{d}\left(\int_{t-\tau_1}^{t} \xi^{\mathrm{T}} Q_1 \xi \,\mathrm{d}s + \int_{t-\tau_2}^{t} \xi^{\mathrm{T}} Q_3 \xi \,\mathrm{d}s\right) 1_{\{\theta(t)=i\}}\\
&+ \mathrm{d}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} \,\mathrm{d}s\,\mathrm{d}\theta + \int_{-h-\tau}^{-\tau}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} \,\mathrm{d}s\,\mathrm{d}\theta\right.\\
&\left.+ \int_{-\tau_2}^{0}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_3 \dot{\xi} \,\mathrm{d}s\,\mathrm{d}\theta + \int_{-h-\tau_2}^{-\tau_2}\int_{t+\theta}^{t} \dot{\xi}^{\mathrm{T}} R_4 \dot{\xi} \,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\cdot 1_{\{\theta(t)=i\}}\Big\} + \sum_{j=1}^{N} \gamma_{ji} \bar{V}^j(\xi(t),k)\,\mathrm{d}t + o(\mathrm{d}t).
\end{aligned} \tag{51}$$

For convenient calculation, V_E' is determined to represent $\mathrm{d}\left(V_3\right) 1_{\{\theta(t)=i\}}$ in the following processing stage. Then,

$$\begin{aligned}
\dot{\bar{V}}^i \le E\Big\{&\dot{\xi}^{\mathrm{T}} P \xi 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} P \dot{\xi} 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} Q_1 \xi 1_{\{\theta(t)=i\}}\\
&- (1-\kappa_1)\xi_{\tau_1}^{\mathrm{T}} Q_1 \xi_{\tau_1} 1_{\{\theta(t)=i\}} + \xi^{\mathrm{T}} Q_3 \xi 1_{\{\theta(t)=i\}}\\
&- (1-\kappa_2)\xi_{\tau_2}^{\mathrm{T}} Q_3 \xi_{\tau_2} 1_{\{\theta(t)=i\}} + V_E'\Big\} + \sum_{j=1}^{N} \gamma_{ji} \bar{V}^j \,\mathrm{d}t,
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
V_E' = {}& \tau_1 \dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} 1_{\{\theta(t)=i\}} - \int_{t-\tau_1}^{t} \dot{\xi}^{\mathrm{T}} R_1 \dot{\xi} \,\mathrm{d}s 1_{\{\theta(t)=i\}}\\
&+ h_1 \dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} 1_{\{\theta(t)=i\}} - \int_{t-h_1-\tau_1}^{t-\tau_1} \dot{\xi}^{\mathrm{T}} R_2 \dot{\xi} \,\mathrm{d}s 1_{\{\theta(t)=i\}}\\
&+ \tau_2 \dot{\xi}^{\mathrm{T}} R_3 \dot{\xi} 1_{\{\theta(t)=i\}} - \int_{t-\tau_2}^{t} \dot{\xi}^{\mathrm{T}} R_3 \dot{\xi} \,\mathrm{d}s 1_{\{\theta(t)=i\}}\\
&+ h_2 \dot{\xi}^{\mathrm{T}} R_4 \dot{\xi} 1_{\{\theta(t)=i\}} - \int_{t-h_2-\tau_2}^{t-\tau_2} \dot{\xi}^{\mathrm{T}} R_4 \dot{\xi} \,\mathrm{d}s 1_{\{\theta(t)=i\}}.
\end{aligned} \tag{53}$$

By Lemma 3, V_E' can be rewritten as

$$\begin{aligned}
V_E' \le {}& \dot{\xi}^{\mathrm{T}} \left(\tau_1 R_1 + \tau_2 R_3 + h_1 R_2 + h_2 R_4\right) \dot{\xi} 1_{\{\theta(t)=i\}}\\
&- \xi^{\mathrm{T}} \left(\tau_1^{-1} R_1 + \tau_2^{-1} R_3\right) \xi 1_{\{\theta(t)=i\}}\\
&+ \xi^{\mathrm{T}} \left(\tau_1^{-1} R_1\right) \xi_{\tau_1} 1_{\{\theta(t)=i\}}\\
&- \xi_{h_1\cdot\tau_1}^{\mathrm{T}} \left(h_1^{-1} R_2\right) \xi_{h_1\cdot\tau_1} 1_{\{\theta(t)=i\}}\\
&+ \xi_{\tau_1}^{\mathrm{T}} \left(\tau_1^{-1} R_1\right) \xi 1_{\{\theta(t)=i\}}\\
&- \xi_{\tau_1}^{\mathrm{T}} \left(\tau_1^{-1} R_1 + h_1^{-1} R_2\right) \xi_{\tau_1} 1_{\{\theta(t)=i\}}\\
&+ \xi_{\tau_2}^{\mathrm{T}} \left(\tau_2^{-1} R_3\right) \xi 1_{\{\theta(t)=i\}}\\
&- \xi_{h_2\cdot\tau_2}^{\mathrm{T}} \left(h_2^{-1} R_4\right) \xi_{h_2\cdot\tau_2} 1_{\{\theta(t)=i\}}\\
&+ \xi^{\mathrm{T}}(t) \left(\tau_2^{-1} R_3\right) \xi_{\tau_2} 1_{\{\theta(t)=i\}}\\
&- \xi_{\tau_2}^{\mathrm{T}} \left(\tau_2^{-1} R_3 + h_2^{-1} R_4\right) \xi_{\tau_2} 1_{\{\theta(t)=i\}}\\
&+ \xi_{\tau_1}^{\mathrm{T}} \left(h_1^{-1} R_2\right) \xi_{h_1\cdot\tau_1} 1_{\{\theta(t)=i\}}\\
&+ \xi_{h_1\cdot\tau_1}^{\mathrm{T}} \left(h_1^{-1} R_2\right) \xi_{\tau_1} 1_{\{\theta(t)=i\}}\\
&+ \xi_{\tau_2}^{\mathrm{T}} \left(h_2^{-1} R_4\right) \xi_{h_2\cdot\tau_2} 1_{\{\theta(t)=i\}}\\
&+ \xi_{h_2\cdot\tau_2}^{\mathrm{T}} \left(h_2^{-1} R_4\right) \xi_{\tau_2} 1_{\{\theta(t)=i\}}.
\end{aligned} \tag{54}$$

Substituting $\dot{\xi} = \tilde{A}\xi + \tilde{B}_1 \xi_{\tau_1} + \tilde{B}_2 \xi_{\tau_2}$ and V_E' into Eq. (52) and noting that $\dot{\bar{V}} = \sum_{i=1}^{N} \dot{V}^i(\cdot)$ and $\sum_{i=1}^{N} \gamma_{ji} = 0$,

$$\begin{aligned}
\dot{\bar{V}} \le E\big\{&\xi^{\mathrm{T}} \left(\Upsilon_2 + Q_1 + Q_3 - \tau_1^{-1} R_1 - \tau_2^{-1} R_3 + \tilde{A}^{\mathrm{T}} \Omega_2 \tilde{A}\right) \xi\\
&+ \xi_{\tau_1}^{\mathrm{T}} \left(\tilde{B}_{1\mathrm{L}}^{\mathrm{T}} \Omega_2 \tilde{B}_{1\mathrm{L}} - (1-\kappa_1) Q_1 - h_1^{-1} R_2 - \tau_1^{-1} R_1\right) \xi_{\tau_1}\\
&+ \xi_{\tau_2}^{\mathrm{T}} \left(\tilde{B}_{2\mathrm{L}}^{\mathrm{T}} \Omega_2 \tilde{B}_{2\mathrm{L}} - (1-\kappa_2) Q_3 - \tau_2^{-1} R_3 - h_2^{-1} R_4\right) \xi_{\tau_2}\\
&- \xi_{h_1\cdot\tau_1}^{\mathrm{T}} \left(h_1^{-1} R_2\right) \xi_{h_1\cdot\tau_1} - \xi_{h_2\cdot\tau_2}^{\mathrm{T}} \left(h_2^{-1} R_4\right) \xi_{h_2\cdot\tau_2}\\
&+ \xi_{\tau_1}^{\mathrm{T}} \left(\tilde{B}_{1\mathrm{L}}^{\mathrm{T}} P + \tau_1^{-1} R_1 + \tilde{B}_{1\mathrm{L}}^{\mathrm{T}} \Omega_2 \tilde{A}\right) \xi\\
&+ \xi_{\tau_2}^{\mathrm{T}} \left(\tilde{B}_{2\mathrm{L}}^{\mathrm{T}} P + \tilde{B}_{2\mathrm{L}}^{\mathrm{T}} \Omega_2 \tilde{A} + \tau_2^{-1} R_3\right) \xi\\
&+ \xi^{\mathrm{T}} \left(P\tilde{B}_{1\mathrm{L}} + \tau_1^{-1} R_1 + \tilde{A}^{\mathrm{T}} \Omega_2 \tilde{B}_{1\mathrm{L}}\right) \xi_{\tau_1}\\
&+ \xi^{\mathrm{T}} \left(P\tilde{B}_{2\mathrm{L}} + \tilde{A}^{\mathrm{T}} \Omega_2 \tilde{B}_{2\mathrm{L}} + \tau_2^{-1} R_3\right) \xi_{\tau_2}\\
&+ \xi_{\tau_2}^{\mathrm{T}} \tilde{B}_{2\mathrm{L}}^{\mathrm{T}} \Omega_2 \tilde{B}_{1\mathrm{L}} \xi(t-\tau_1) + \xi_{\tau_1}^{\mathrm{T}} \left(h_1^{-1} R_2\right) \xi_{h_1\cdot\tau_1}\\
&+ \xi_{\tau_1}^{\mathrm{T}} \tilde{B}_{1\mathrm{L}}^{\mathrm{T}} \Omega_2 \tilde{B}_{2\mathrm{L}} \xi(t-\tau_2) + \xi_{h_1\cdot\tau_1}^{\mathrm{T}} \left(h_1^{-1} R_2\right) \xi_{\tau_1}\\
&+ \xi_{\tau_2}^{\mathrm{T}} \left(h_2^{-1} R_4\right) \xi_{h_2\cdot\tau_2} + \xi_{h_2\cdot\tau_2}^{\mathrm{T}} \left(h_2^{-1} R_4\right) \xi_{\tau_2}\big\},
\end{aligned} \tag{55}$$

where $\Upsilon_2 = \tilde{A}^{\mathrm{T}} P + P\tilde{A}$, $\Omega_2 = \tau_1 R_1 + \tau_2 R_3 + h_1 R_2 + h_2 R_4$, $\tilde{B}_{1\mathrm{L}} = -L_{\mathrm{p}}^{u} \otimes BK_{\mathrm{p}} Q^{\mathrm{p}}$, and $\tilde{B}_{2\mathrm{L}} = -L_{\mathrm{v}}^{u} \otimes BK_{\mathrm{v}} Q^{\mathrm{v}}$.

Substituting $\varsigma_i\phi(t), i \in \{1,2,\ldots,5\}$, which represent ξ, ξ_{τ_1}, ξ_{τ_2}, $\xi_{h_1\cdot\tau_1}$, and $\xi_{h_2\cdot\tau_2}$ respectively, into Eq. (55),

$$\dot{V}(\xi(t),k) \le E\left\{\phi^{\mathrm{T}}(t)\Theta\phi(t)\right\}, \tag{56}$$

where $\phi(t)$, Θ are determined from Eqs. (44) and (45).

With the new vector $\varsigma(t) = [\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5]^{\mathrm{T}}$, it is easy to obtain

$$\dot{V}(\xi(t),k) \le E\left\{\phi^{\mathrm{T}}(t)\varsigma^{\mathrm{T}}(t)\Psi\varsigma(t)\phi(t)\right\}, \tag{57}$$

and

$$\Psi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & 0 & 0 \\ \varphi_{21} & \varphi_{22} & 0 & \varphi_{24} & 0 \\ \varphi_{31} & 0 & \varphi_{33} & 0 & \varphi_{35} \\ 0 & \varphi_{42} & 0 & \varphi_{44} & 0 \\ 0 & 0 & \varphi_{53} & 0 & \varphi_{55} \end{bmatrix} + \Xi_{\tilde{A}\tilde{B}}^{\mathrm{T}}\Omega_2\Xi_{\tilde{A}\tilde{B}}, \tag{58}$$

where $\varphi_{11} = \Upsilon_2 + Q_1 + Q_3 - \tau_1^{-1}R_1 - \tau_2^{-1}R_3$, $\varphi_{13} = \varphi_{31}^{\mathrm{T}} = P\tilde{B}_{2\mathrm{L}} + \tau_2^{-1}R_3$, $\varphi_{22} = -(1-\kappa_1)Q_1 - h_1^{-1}R_2 - \tau_1^{-1}R_1$, $\varphi_{12} = \varphi_{21}^{\mathrm{T}} = P\tilde{B}_1 + \tau_1^{-1}R_1$, $\varphi_{33} = -(1-\kappa_2)Q_3 - \tau_2^{-1}R_3 - h_2^{-1}R_4$, $\varphi_{44} = -\varphi_{24} = -\varphi_{42} = -h_1^{-1}R_2$, $\varphi_{55} = -\varphi_{35} = -\varphi_{53} = -h_2^{-1}R_4$, and $\Xi_{\tilde{A}\tilde{B}} = [\tilde{A}\ \tilde{B}_{1\mathrm{L}}\ \tilde{B}_{2\mathrm{L}}\ 0\ 0]$.

According to Lemma 1 and condition (47) in Theorem 2, it is easy to obtain

$$\Psi < 0. \tag{59}$$

Thus, we can obtain $\dot{V}(\xi(t),k) < 0$ for $t > 0$. Moreover, there exists a constant $\rho > 0$ satisfying

$$\dot{V}(\xi(t),k) < -\rho\|\xi(t)\|^2. \tag{60}$$

This means that system (9) is asymptotically stable with two independent topologies in the case of two different time-varying delays.

Remark 3 $\Xi_{\tilde{A}\tilde{B}}$ is defined to construct a square matrix by $\Xi_{\tilde{A}\tilde{B}}^{\mathrm{T}}\Xi_{\tilde{A}\tilde{B}}$ and is conveniently used through Lemma 1. The new matrix, $\Xi_{\tilde{A}\tilde{B}}^{\mathrm{T}}\Omega_2\Xi_{\tilde{A}\tilde{B}}$, has the same dimensions as $[\varphi_{ij}]$. Moreover, regarding $[\varphi_{ij}]$, $\Xi_{\tilde{A}\tilde{B}}^{\mathrm{T}}$, $\Xi_{\tilde{A}\tilde{B}}$, and Ω_2 as four matrix blocks that are similar to the form of Eq. (10), the matrix in inequality (47) can be constructed

4. Simulation results

In this section, various examples that illustrate the feasibility of the obtained results are presented. The same conditions and control input will be considered with two models respectively, double-integrator dynamic model (9) which is used to illustrate the stability of the theorems and nonlinear AUV model (1) which is used to illustrate the effectiveness of theorems in the system of multi-AUVs. The parameters of the nonlinear AUV mathematical model are the same as those in Ref. [3]. The adjacency matrices of communication topologies G_{p}^u and G_{v}^u are taken as binary 0–1 matrices.

Consider the system with five members ($n = 5$). The topologies (N=4) randomly switch between $G_{\mathrm{p}i}$ and $G_{\mathrm{v}i}$ following the Markovian process $\theta(t)$. The details of the topologies are presented in Fig. 1. Note that $G_{\mathrm{p}i}$ does not contain a spanning tree, whereas G_{p}^u, and G_{v}^u with $G_{\mathrm{v}i}$ do.

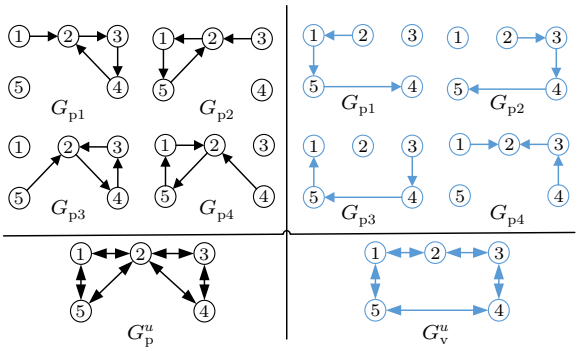


Fig. 1. (color online) All possible directed topologies graphs.

The transition probability matrix Π and state space U are designed respectively as

$$\Pi = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 2 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad U = \{1,2,\ldots,4\}.$$

The initial distribution of the continuous-time Markov process is given by its invariant distribution $\pi = [2/6\ 1/6\ 1/6\ 2/6]^{\mathrm{T}}$. With the LMI tools in MATLAB, $k_\alpha = 0.2$ and $k_\beta = 0.5$.

$$\bar{P} = \begin{bmatrix} 1.504 & -0.5013 \\ -0.5013 & 0.5013 \end{bmatrix} \otimes I_5.$$

Example 1 In this case, the time delays in double communication topologies are assumed to be the same. According to Theorem 1, to achieve consensus, the same time-varying delays in communication topologies are assumed to be $\tau(t) = 0.1(1+\sin(2t))$. The Markovian states of the switching topologies are shown in Fig. 2. In the simulation with double-integrator model as shown in Figs. 3 and 4, the position and velocity states are stable and converge to the common states.

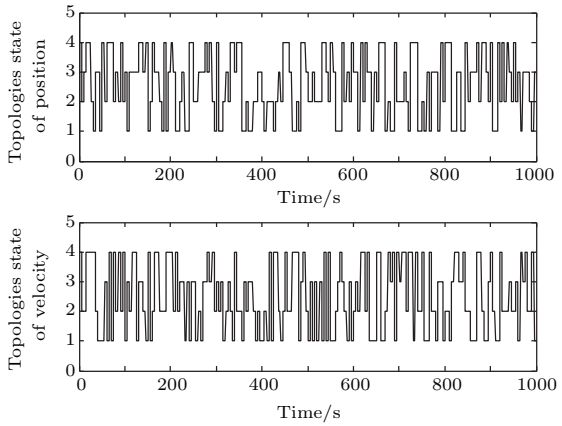


Fig. 2. Markov states of the switching topologies in the case of the same delays.

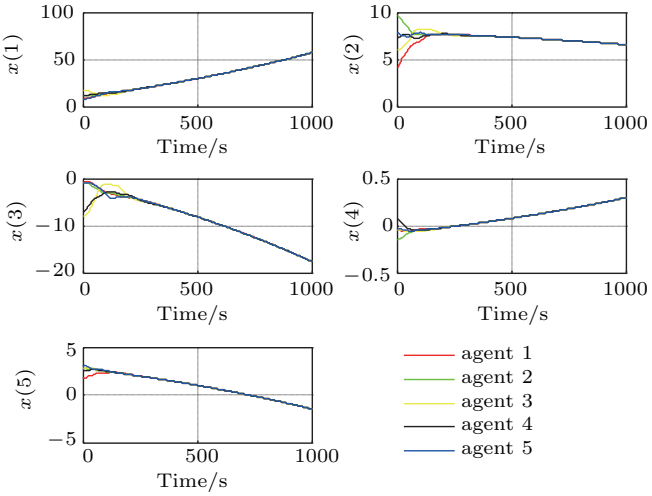


Fig. 3. (color online) Position states with double-integrator model and the same delays.

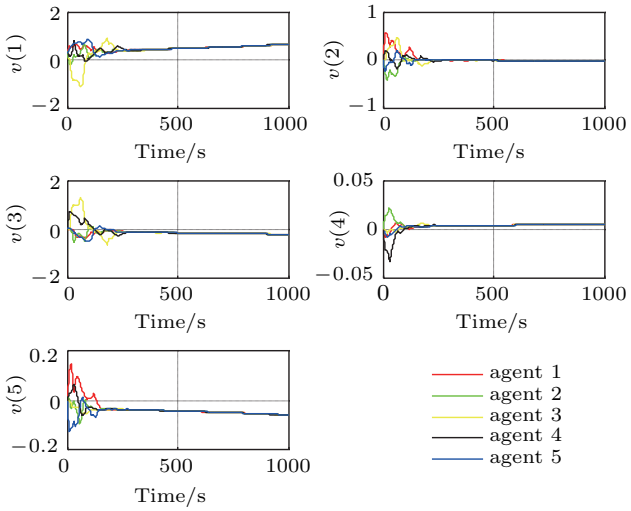


Fig. 4. (color online) Velocity states with double-integrator model and the same delays.

In the simulation with nonlinear AUV model under the same initialization, Figures 5 and 6 show the position and velocity states approach to the stable and common states, and figure 7 shows the trajectories in three-dimensional (3D) space.

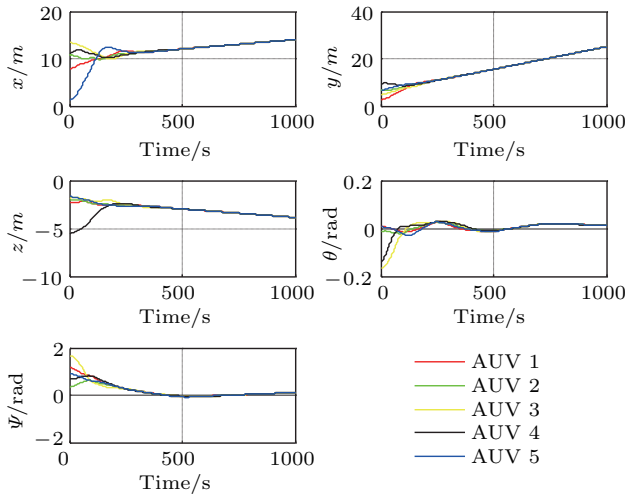


Fig. 5. (color online) Position and attitude states of the AUVs in the case of same delays.

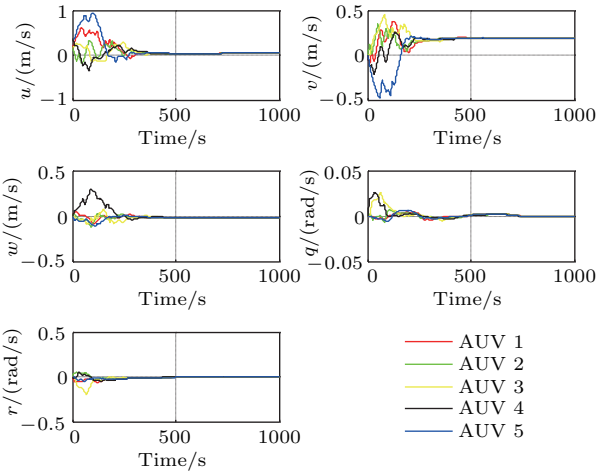


Fig. 6. (color online) Velocity states of the AUVs in the case of the same delays.

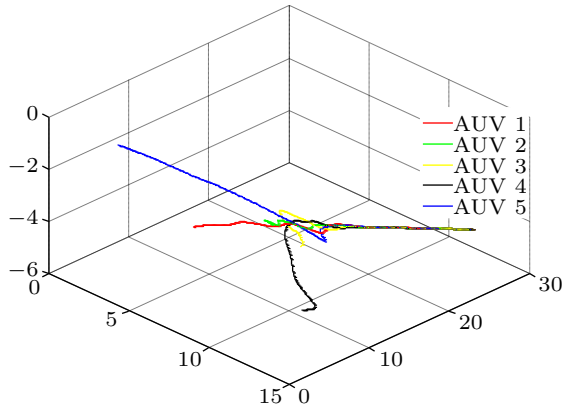


Fig. 7. (color online) 3D trajectories of all the AUVs in the case of the same delays.

Example 2 In this scenario, we assume that the time delays in position and velocity topologies are different. In accordance with Theorem 2, to achieve consensus, the different time-varying delays in double topologies are assumed to be $\tau_1(t) = 0.1\,(1+\sin(2t))$ and $\tau_2(t) = 0.1\,(1+\cos(2t))$. The Markovian states of the switching topologies in the case of different delays are shown in Fig. 8. In the simulation with double-integrator model as shown in Figs. 9 and 10, the position and velocity states are stable and converge to the common states.

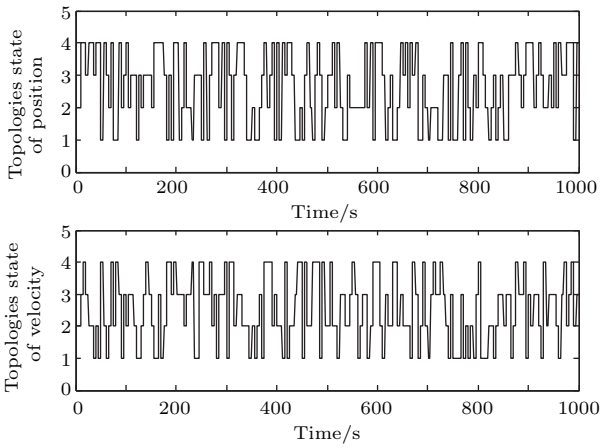


Fig. 8. Markov states of the switching topologies in the case of different delays.

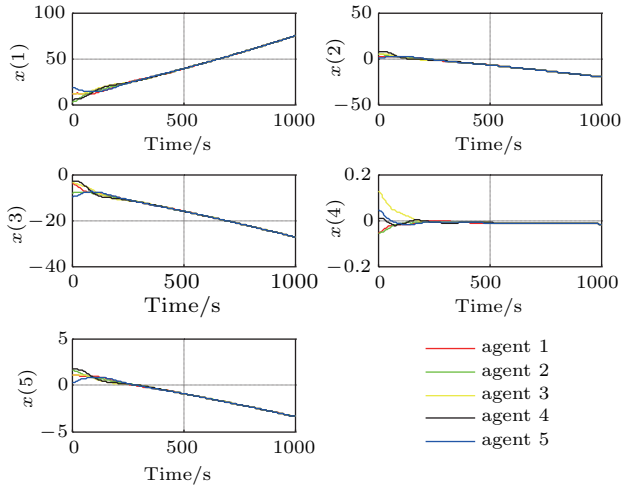


Fig. 9. (color online) Position states with double-integrator model and different delays.

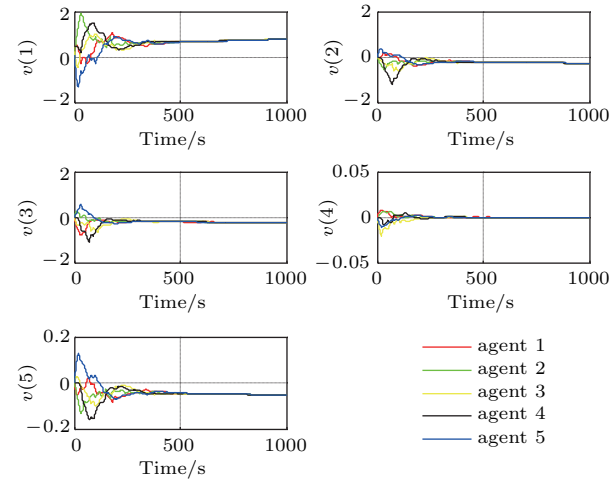


Fig. 10. (color online) Velocity states with double-integrator model and different delays.

In the simulation with nonlinear AUV model under the same initialization the processes of approaching to stable position and velocity states are shown in Figs. 11 and 12. This process requires longer time to adjust motion states than the scenario of the same time-varying delays. Figure 13 also shows the trajectories of all AUVs in 3D space.

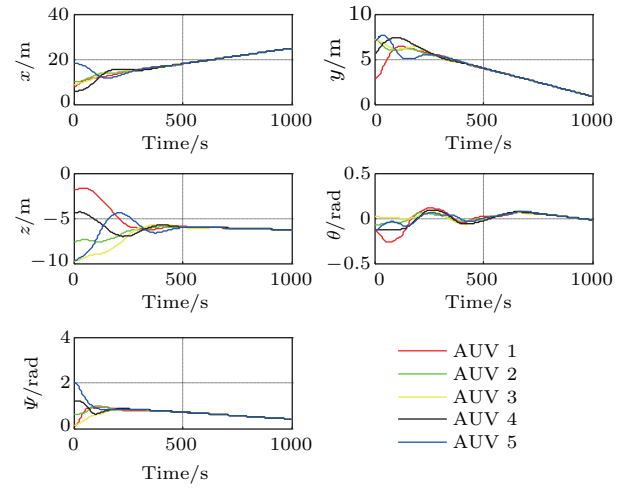


Fig. 11. (color online) Position and attitude states of the AUVs in the case of different delays.

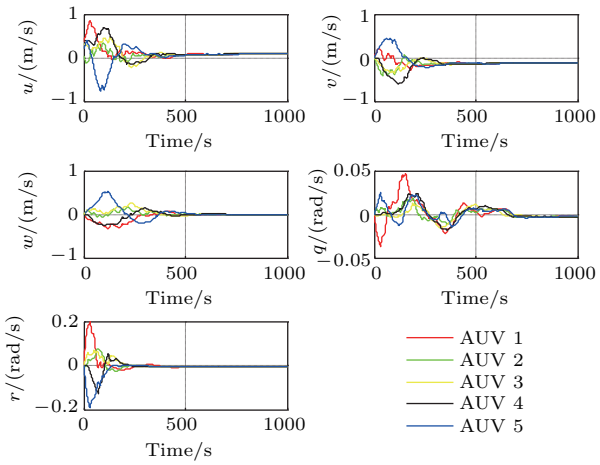


Fig. 12. (color online) Velocity states of the AUVs in the case of different delays.

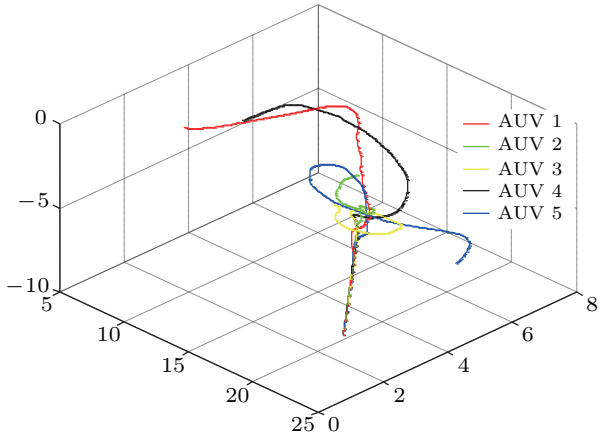


Fig. 13. (color online) 3D trajectories of all the AUVs in the case of different delays.

5. Conclusions

In this paper, the control problems of multi-AUVs are first discussed by using consensus theory under double independent Markovian switching topologies with the same and different time-varying delays . Subsequently, we propose and analyze the stability conditions for leaderless multi-AUVs based on the Lyapunov–Krasovskii theory and state feedback linearization. With the objective of transforming multi-AUVs control problem into consensus problem, we considere the fact that this special consensus can guarantee convergence of the overall formation. Simulation results obtained validate the theoretical results presented. In the future work, we plan to focus on the leader-following scenario with the influence of time-varying delays. In addition, we plan to consider multi-leaders AUVs system with switching formation structure.

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