Carlson iterating rational approximation and performance analysis of fractional operator with arbitrary order

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The performance analysis of the generalized Carlson iterating process, which can realize the rational approximation of fractional operator with arbitrary order, is presented in this paper. The reasons why the generalized Carlson iterating function possesses more excellent properties such as self-similarity and exponential symmetry are also explained. K-index, P-index, O-index, and complexity index are introduced to contribute to performance analysis. Considering nine different operational orders and choosing an appropriate rational initial impedance for a certain operational order, these rational approximation impedance functions calculated by the iterating function meet computational rationality, positive reality, and operational validity. Then they are capable of having the operational performance of fractional operators and being physical realization. The approximation performance of the impedance function to the ideal fractional operator and the circuit network complexity are also exhibited.

Keywords: fractional calculus, fractional operator, generalized Carlson iterating process, approximation error

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1. Introduction

Fractional calculus has been widely used in the areas of viscoelasticity,[1] diffusion,[2] random processes,[3] nonlinear dynamics and chaos,[4] etc. Particularly, the rational approximation problem of fractional operator s^{μ} has also become a hot spot of research in recent years, where s is the complex frequency variable or the operational variable in the Laplace transform domain and μ is called the operational order whose value is the fraction or the rational number, namely $\mu \in \mathbb{Q}$. According to network synthesis, the rational impedance function sequence $\{Z_k(s)\}$ (the iterating time $k \in \mathbb{N}$) which could be realized with circuit elements, has been used to implement the approximation of the irrational fractional operator function $I^{(\mu)}(s)$ under certain conditions and over a certain frequency range:

$$Z_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{n_k} \beta_{ki} s^i}{\sum_{i=0}^{d_k} \alpha_{ki} s^i} \xrightarrow{k\to\infty} I^{(\mu)}(s) = F^{(\mu)} s^{\mu}, \quad (1)$$

where these positive integer parameters n_k and d_k represent the highest degree of numerator polynomial $N_k(s)$ and denominator polynomial $D_k(s)$ respectively, β_{ki} and α_{ki} express correspondingly coefficients of $N_k(s)$ and $D_k(s)$. The two-terminal circuit element whose impedance function is $I^{(\mu)}(s)$, is referred as the μ-order ideal fractor. Its lumped characteristic parameter $F^{(\mu)}$ is called the fractance quantity.[5–7] Fractance is the abbreviation of fractional-order impedance. The unit of the physical quantity $F^{(\mu)}$ in the SI unit is $\left[F^{(\mu)}\right] = \mathrm{m}^2 \cdot \mathrm{kg} \cdot \mathrm{s}^{-3+\mu} \cdot \mathrm{A}^{-2} = \Omega \cdot \mathrm{s}^{\mu}$, which is named the Ohm μ-order second. The dimension of the physical quantity $F^{(\mu)}$ is $\dim F^{(\mu)} = \mathrm{L}^2 \cdot \mathrm{M} \cdot \mathrm{T}^{-3+\mu} \cdot \mathrm{I}^{-2}$. The two-terminal circuit network of impedance function $Z_k(s)$ in Eq. (1) is called the fractance approximation circuit,[7] which takes on the operational characteristics of ideal fractional operator in a certain frequency range.

In the early 1950s and 1960s, the regular Newton iterating process was proposed by Carlson and Halijak to realize the rational approximation of fractional operator $s^{\pm 1/n}$ (n denotes an integer equal to or greater than 2).[8,9] The binomial expansion method was put forward by Steiglitz *et al.* to implement the rational approximation of the negative half order operator.[10,11] In the modeling study of constant-argument immittance, Roy made use of the continued fraction expansion to achieve the rational approximation of fractional operator s^{μ}.[12] From the beginning of this century, $1/n$ order fractance approximation circuit through the Carlson regular Newton iterating process, Carlson iterating process for short, was designed in Ref. [13]. The generalization of the binomial expansion method to obtain the rational approximation impedance function sequence for an arbitrary order fractional operator was presented in Ref. [14]. The rational approximation of fractional operator s^{μ} by variable substitution was also achieved by Krishna *et al.* based on the study of Roy.[15,16] Using the continued fraction expansion method to design $-1/2^n$ order fractance approximation circuit was accomplished in Ref. [17]. There are also other methods to implement the rational approximation of

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fractional operator, such as the elliptic function approximation method,[18,19] the Padé rational approximation method,[20,21] and so on.

In many rational approximation methods, the Carlson iterating process about operator $s^{\pm 1/n}$ has made more and more people attach comprehensive importance to it. Nowadays, when mentioning the study history of fractance and fractance approximation circuit, people would date back to the Carlson iterating process. The impedance function sequence $\{Z_k(s)\}$ $(k \in \mathbb{N})$ acquired by the Carlson iterating process could approximate to the operator $s^{\pm 1/n}$ and extend to both high and low frequency uniformly, smoothly and steadily.[7,13] The K-index[7] is equal to $2n\lg((n+1)/(n-1))$. That is to say, for every one increase in the iterating time k, the increasing frequency band is almost up to 1.7–1.9 orders of magnitude. It is certainly useful that the Carlson iterating process could realize the rational approximation of fractional operator $s^{\pm 1/n}$.

The generalized Carlson iterating process which is to achieve the rational approximation of arbitrary order fractional operator $s^{\pm j/n}$ (j is also an integer valued from 1 to $(n-1)$) now has been verified in Refs. [7] and [22]. Performance analysis for the generalized Carlson iterating process, is discussed in this paper. We start with introducing the rational approximation principle about the generalized Carlson iterating process. Next the complexity index of fractance approximation circuit, K-index, P-index, and O-index are introduced to contribute to performance analysis. Then experiments are carried out to acquire these approximation performance indices for n varying from 2 to 5. Finally, we summarize the work of this paper, make an outlook and point out some relative topics in the further research.

2. The Carlson iterating rational approximation principle

Considering the following case in Eq. (1)

$$F^{(\mu)} \equiv 1,\ \mu = \pm\frac{j}{n},\ (n, j \in \mathbb{Z}^+, n \geq 2, 1 \leq j \leq n-1), \tag{2}$$

there is the generalized Carlson fratance approximation problem:

$$Z_k(s) \xrightarrow{k\to\infty} Z(s) = s^{\pm j/n}. \tag{3}$$

$Z(s)$ is called the Carlson limit impedance function which is a known function. $Z_k(s)$, however, is an unknown function which needs to be determined. If $Z_k(s)$ is substituted by an unknown number x_k, and the certain operator $s^{\pm 1}$ is replaced by a known positive real number a, namely $a = s^{\pm 1}$, equation (3) can be written in the following form

$$x_k \xrightarrow{k\to\infty} r = \sqrt[n]{a^j}, (x_k \in \mathbb{R}^+, a \in \mathbb{R}^+). \tag{4}$$

The Carlson fractance approximation problem therefore can be converted into the algebra iterating solution of the arithmetic root $r = \sqrt[n]{a^j}$ of the n-th two-term equation

$$f(x) = x^n - a^j = 0, (x \in \mathbb{R}^+, a \in \mathbb{R}^+). \tag{5}$$

The classical Newton iterating formula[8,23] of solving the arithmetic root $r = \sqrt[n]{a^j}$ is

$$\begin{aligned} x_{k+1} &= F_{\mathrm{N}}^{[j]}(x_k) \\ &= x_k - \frac{f(x_k)}{f'(x_k)} = x_k\frac{(n-1)x_k^n + a^j}{nx_k^n}, \\ &\quad (k \in \mathbb{N}). \end{aligned} \tag{6}$$

$F_{\mathrm{N}}^{[j]}(x)$ is the generalized Newton iterating function. To speed the convergence of the Newton iterating process, the predistortion function has been introduced

$$g_m(x) = x^m > 0,\ \ (x \in \mathbb{R}^+, 0 \leq m \leq n-1). \tag{7}$$

So we obtain an equivalent form of Eq. (5):

$$f_{\mathrm{C}}^{[j]}(x) = \frac{f(x)}{g_m(x)} = x^{n-m} - a^j x^{-m} = 0, \tag{8}$$

and a new iterating formula

$$x_{k+1} = F^{[j]}(x_k) = x_k\frac{(n-m-1)x_k^n + (m+1)a^j}{(n-m)x_k^n + ma^j},\ (k \in \mathbb{N}). \tag{9}$$

If $m = (n-1)/2$ is substituted into Eq. (9), the numerical iterating solution of the arithmetic root r, namely the generalized Carlson iterating formula, is acquired

$$x_{k+1} = F_{\mathrm{C}}^{[j]}(x_k) = x_k\frac{(n-1)x_k^n + (n+1)a^j}{(n+1)x_k^n + (n-1)a^j},\ \ (k \in \mathbb{N}). \tag{10}$$

$F_{\mathrm{C}}^{[j]}(x)$ is the generalized Carlson iterating function. Proceeding further, we replace a with $1/s$ in Eq. (10) and obtain the Carlson fractance rational approximation impedance function sequence of fractional operator $s^{-j/n}$

$$\begin{aligned} Z_{k+1}(s) &= F_{\mathrm{C}}^{[j]}(Z_k(s)) \\ &= Z_k(s)\frac{(n-1)s^j Z_k^n(s) + (n+1)}{(n+1)s^j Z_k^n(s) + (n-1)}, \\ &\quad (k \in \mathbb{N}). \end{aligned} \tag{11}$$

When $j = 1$, there exists $F_{\mathrm{N}}(x) = F_{\mathrm{N}}^{[1]}(x)$ and $F_{\mathrm{C}}(x) = F_{\mathrm{C}}^{[1]}(x)$. $F_{\mathrm{N}}(x)$ and $F_{\mathrm{C}}(x)$ are the classical Newton and Carlson iterating functions which have formerly only been used to accomplish the rational approximation of the fractional operator $s^{\pm 1/n}$. $F_{\mathrm{N}}(x)$ and $F_{\mathrm{C}}(x)$ are the corresponding particular cases of $F_{\mathrm{N}}^{[j]}(x)$ and $F_{\mathrm{C}}^{[j]}(x)$.

Let $n = 3$ and $a = 2^n$, the Newton iterating function and the Carlson iterating function are depicted separately in Figs. 1(a) and 1(b). The both possess basic mathematical properties of the fractance approximation iterating function, such

as positive root uniqueness, local convergence, global convergence, and so on.[7]

In comparison with the Newton iterating process of the second order, the Carlson iterating process is of the third order and has global regularity, namely a strict global monotonic increasing property. So the convergence rate at the root is the largest possible one in the set of predistortion functions and no overshoot or undershoot occurs at the beginning of the process.[8] Besides, the Carlson iterating function also takes on some other more excellent geometrical features.

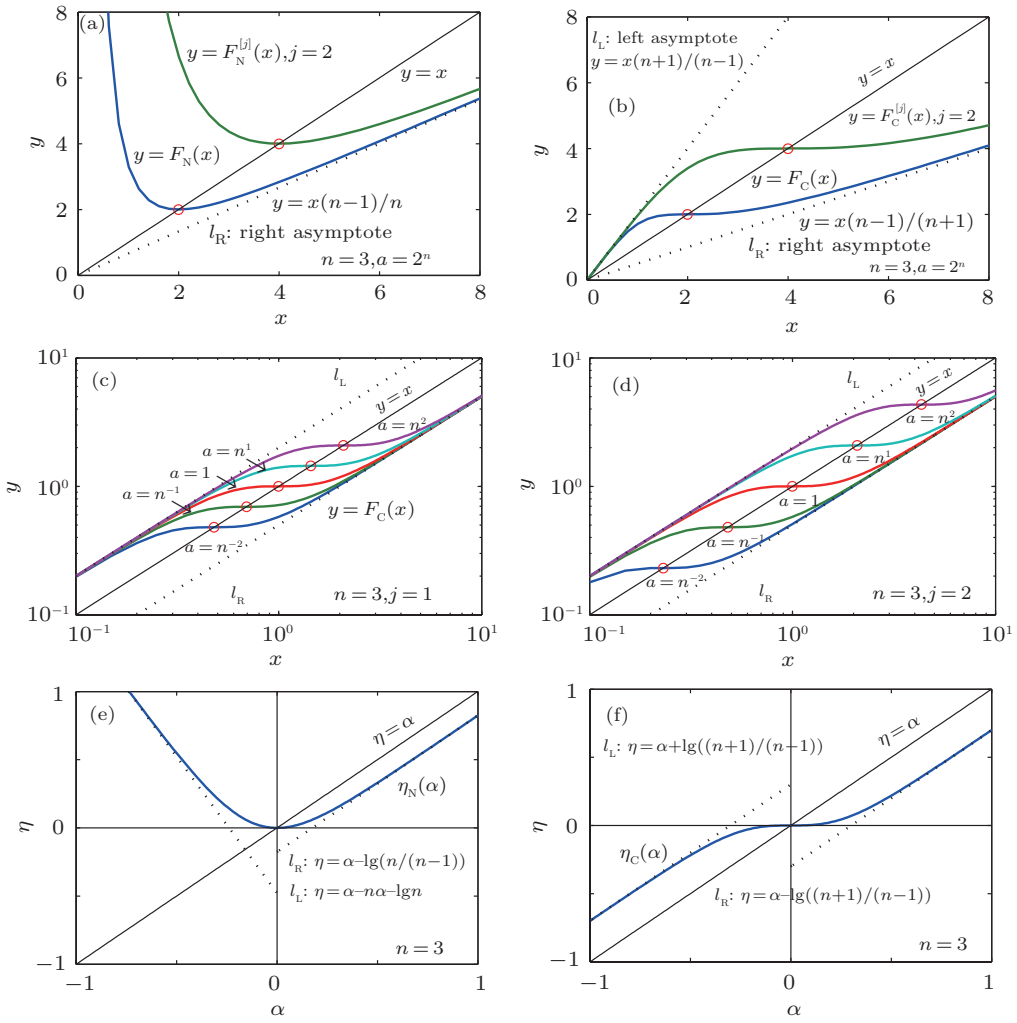


Fig. 1. (color online) Iterating function curves: (a) $F_N^{[j]}(x)$ (linear coordinate); (b) $F_C^{[j]}(x)$ (linear coordinate); (c) $F_C(x)$ (dual logarithmic coordinate); (d) $F_C^{[j]}(x)$ (dual logarithmic coordinate); (e) $\eta_N(\alpha)$; (f) $\eta_C(\alpha)$.

2.1. Self-similarity

For different positive real numbers a, these Carlson iterating function curves shown in Figs. 1(b), 1(c), and 1(d) are self-similar. This feature is specially obvious in the dual logarithmic coordinate, as can be seen in Figs. 1(c) and 1(d). This kind of self-similarity can make the Carlson iterating function normalized. Let $x=r\chi$ (χ called the normalized variable) and $r=\sqrt[n]{a^j}\Leftrightarrow a^j=r^n$, we obtain the normalized Carlson iterating function

$$F_{NC}(\chi)=\frac{F_C^{[j]}(r\chi)}{r}=\chi\frac{(n-1)\chi^n+(n+1)}{(n+1)\chi^n+(n-1)},\quad(\chi\in\mathbb{R}^+). \tag{12}$$

Therefore, the shapes of these Carlson iterating function curves are the same for all values of positive real number a in the dual logarithmic coordinate.

2.2. Exponential symmetry

Let $\chi=10^\alpha$ ($\alpha\in\mathbb{R}$) and we take the logarithmic operation on each side of Eq. (12), so the exponential form of $F_{NC}(\chi)$ is

$$\eta_C(\alpha)=\lg[F_{NC}(10^\alpha)]=\alpha+\lg\frac{(n-1)10^{n\alpha}+(n+1)}{(n+1)10^{n\alpha}+(n-1)}. \tag{13}$$

Obviously, there exists $\eta_C(-\alpha)=-\eta_C(\alpha)$, thus the function $\eta_C(\alpha)$ is odd-symmetric, as shown in Fig. 1(f). The exponential form $\eta_N(\alpha)$ of the normalized Newton iterating function is illustrated in Fig. 1(e). The exponential symmetry indicates that the frequency band of the impedance function calculated by the Carlson iterating function approximating to fractional

operator $s^{\pm j/n}$, is extended to both high and low frequencies uniformly, smoothly, and steadily.[7]

3. Theoretical foundations for performance analysis of Carlson iterating rational approximation

It has been verified in Ref. [22] that the impedance function $Z_k(s)$ calculated by the generalized Carlson iterating function satisfies these following properties:[7,24,25]

(i) Computational rationality

There exists no irrational computation and only rational computation of complex frequency variable s in the expression of $Z_k(s)$ when choosing the rational initial impedance function.

(ii) Positive reality principle

Both numerator polynomial $N_k(s)$ and denominator polynomial $D_k(s)$ should be strict Hurwitz polynomials. In other words, all zeros and poles of the impedance function $Z_k(s)$ are located in the negative real axis of the s complex plane or the left-half plane of the s complex plane in conjugate pairs.

(iii) Operational validity

The impedance function takes on the operational characteristics of fractional operator $s^{\pm j/n}$ over a certain complex frequency range.

Let

$$s = \mathrm{j}\Omega = \mathrm{j}2\pi 10^{\varpi} \Leftrightarrow \varpi = \lg\frac{\Omega}{2\pi},\ \ (\Omega \in \mathbb{R}^+, \varpi \in \mathbb{R}), \quad (14)$$

where Ω is the analog frequency variable and ϖ is the frequency exponent variable, so the operational validity requires

$$Z_k(\mathrm{j}\Omega) \xrightarrow{k\to\infty} I^{(\mu)}(\mathrm{j}\Omega) = (\mathrm{j}\Omega)^{\mu}, \quad (15)$$

and it can be divided into three parts as follows.

(I) Validity of the amplitude-frequency characteristics

$$\begin{aligned}\Lambda_k(\varpi) &= \lg\left|Z_k(\mathrm{j}2\pi 10^{\varpi})\right| \xrightarrow{k\to\infty} \Lambda^{(\mu)}(\varpi)\\ &= \lg\left|I^{(\mu)}(\mathrm{j}2\pi 10^{\varpi})\right| = \mu\left[\varpi + \lg(2\pi)\right]; \end{aligned} \quad (16)$$

(II) Validity of the phase-frequency characteristics

$$\Theta_k(\varpi) = \arg\left\{Z_k(\mathrm{j}2\pi 10^{\varpi})\right\} \xrightarrow{k\to\infty} \Theta^{(\mu)}(\varpi) = \frac{\pi}{2}\mu; \quad (17)$$

(III) Validity of the order-frequency characteristics

$$M_k(\varpi) = \frac{\mathrm{d}\Lambda_k(\varpi)}{\mathrm{d}\varpi} \xrightarrow{k\to\infty} M^{(\mu)}(\varpi) = \frac{\mathrm{d}\Lambda^{\mu}(\varpi)}{\mathrm{d}\varpi} = \mu. \quad (18)$$

Take $\mu = -2/3$ for example, the operational validity is shown in Fig. 2. The solid lines typify the frequency–domain characteristics of the impedance function $Z_k(s)$ with the iterating time k increasing. The dotted lines represent the frequency–domain characteristics of the ideal operator $s^{-2/3}$. The operational nature of fractional order elements and systems, is precisely characterized by the order–frequency characteristics and the phase–frequency characteristics.

The impedance function meets these properties mentioned above, so it has physical realization and the operational characteristics of fractional operator $s^{\pm j/n}$. In this section, a necessary preliminary to describe the approximating degree of the impedance function to the ideal fractional operator, the performance indices are introduced. We also bring in an index to represent the complexity of the fractance approximation circuit.

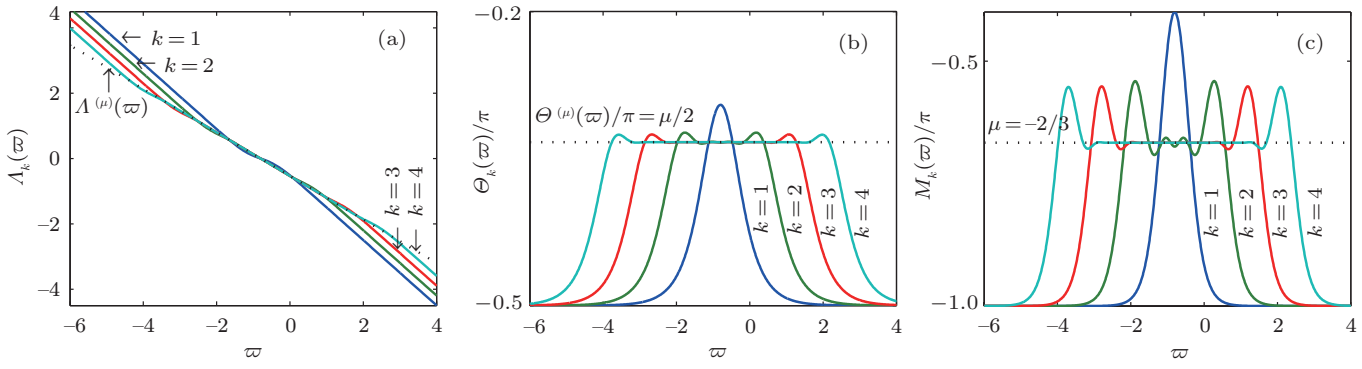


Fig. 2. (color online) Frequency–domain characteristics curves of the impedance function $Z_k(s)$ with $\mu = -2/3$ and $Z_0(s) = 1/s$: (a) amplitude–frequency characteristics; (b) phase–frequency characteristics; (c) order–frequency characteristics.

3.1. Approximation performance indices: *K*-index, *P*-index, and *O*-index

In order to quantitatively describe the approximation degree of the frequency–domain characteristics $Z_k(\mathrm{j}2\pi 10^{\varpi})$ to the frequency characteristics $I^{(\mu)}(\mathrm{j}2\pi 10^{\varpi})$, the approximation performance indices are introduced, including the slope index (K-index), the phase frequency index (P-index), and the order frequency index (O-index). Firstly, we explain the phase frequency approximation bandwidth exponent $B_{\mathrm{P}i}(k)$, the order frequency approximation bandwidth exponent $B_{\mathrm{O}i}(k)$ and

some other related concepts. The phase frequency approximation bandwidth exponent $B_{\mathrm{P}i}(k)$ is the difference between the upper limit frequency exponent $\varpi_{\mathrm{PU}i}(k)$ and the lower limit frequency exponent $\varpi_{\mathrm{PL}i}(k)$

$$B_{\mathrm{P}i}(k) = \varpi_{\mathrm{PU}i}(k) - \varpi_{\mathrm{PL}i}(k). \tag{19}$$

In this frequency interval, the phase frequency relative error function $r_{\theta k}(\varpi)$

$$r_{\theta k}(\varpi) = \left(1 - \frac{\Theta_k(\varpi)}{\pi\mu/2}\right) \times 100\% \tag{20}$$

satisfies the relationship

$$|r_{\theta k}(\varpi)| \leq r_i, \quad (r_i = 1\%, 5\%, 10\%, 20\%, i = 0, 1, 2, 3), \tag{21}$$

where i represents the approximation precision grade index.

The order frequency approximation bandwidth exponent $B_{\mathrm{O}i}(k)$ is also the difference between the upper limit frequency exponent $\varpi_{\mathrm{OU}i}(k)$ and the lower limit frequency exponent $\varpi_{\mathrm{OL}i}(k)$

$$B_{\mathrm{O}i}(k) = \varpi_{\mathrm{OU}i}(k) - \varpi_{\mathrm{OL}i}(k). \tag{22}$$

In this frequency interval, the order frequency relative error function $r_{\mu k}(\varpi)$

$$r_{\mu k}(\varpi) = \left(1 - \frac{M_k(\varpi)}{\mu}\right) \times 100\% \tag{23}$$

satisfies the relationship

$$\left|r_{\mu k}(\varpi)\right| \leq r_i, \quad (r_i = 1\%, 5\%, 10\%, 20\%, i = 0, 1, 2, 3). \tag{24}$$

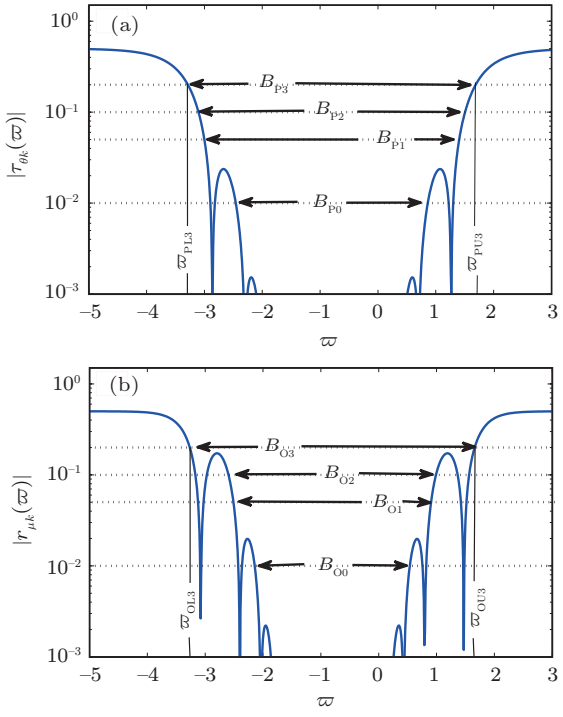


Fig. 3. (color online) Relative error function curves with $\mu = -2/3$, $Z_0(s) = 1/s$, and $k = 3$: (a) phase-frequency; (b) order-frequency.

The phase frequency and order frequency relative error function curves are depicted in Fig. 3 separately. The mathematical meaning of $B_{\mathrm{P}i}(k)$, $B_{\mathrm{O}i}(k)$, $\varpi_{\mathrm{PU}i}(k)$, $\varpi_{\mathrm{PL}i}(k)$, $\varpi_{\mathrm{OU}i}(k)$, and $\varpi_{\mathrm{OL}i}(k)$ are also illustrated in Fig. 3.

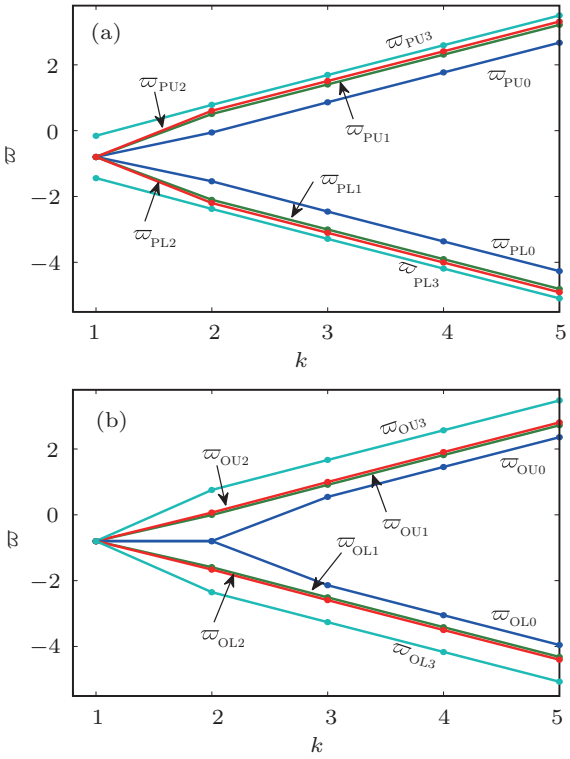


Fig. 4. (color online) Upper limit frequency exponent and lower limit frequency exponent with $\mu = -2/3$ and $Z_0(s) = 1/s$: (a) phase-frequency; (b) order-frequency.

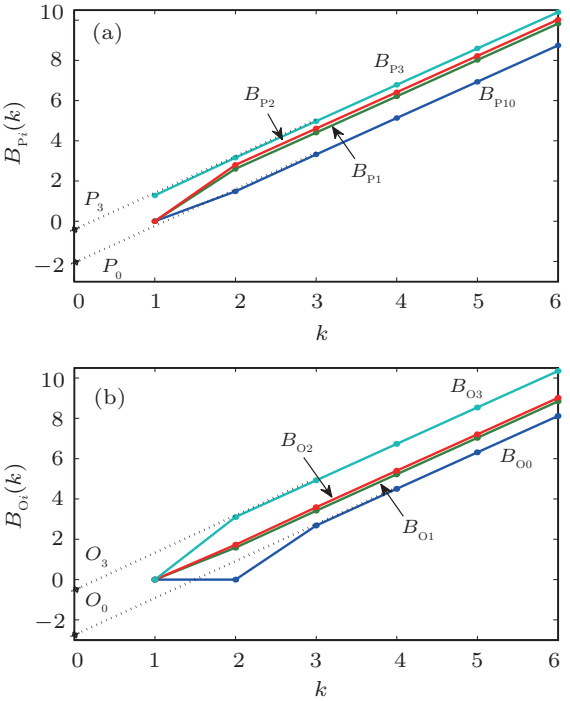


Fig. 5. (color online) Approximation bandwidth exponent curves with $\mu = -2/3$ and $Z_0(s) = 1/s$: (a) phase-frequency; (b) order-frequency.

The upper limit and lower limit frequency exponent curves with the iterating time k increasing are shown in Fig. 4.

$B_{\mathrm{P}i}(k)$ and $B_{\mathrm{O}i}(k)$ depicted in Fig. 5, are the measure values of magnitude orders of approximation bandwidth. When the iterating time k is greater than a certain positive integer, both curves of $B_{\mathrm{P}i}(k)$ and $B_{\mathrm{O}i}(k)$ are parallel respectively and have the following simple relationship

$$B_{\mathrm{P}i}(k) \approx kK_{\mathrm{P}} + P_i, \tag{25a}$$

$$B_{\mathrm{O}i}(k) \approx kK_{\mathrm{O}} + O_i, \tag{25b}$$

where K_{P} and K_{O} identify the slopes, as shown in Fig. 6. In general, the slope index, namely the K-index, identifies the increasing rate of frequency bandwidth exponent with the iterating time k varying and contents the relationship

$$K \approx K_{\mathrm{P}} \approx K_{\mathrm{O}}. \tag{26}$$

The geometrical interpretation of P_i and O_i are the intersection points of the vertical coordinate and the extended line segments of $B_{\mathrm{P}i}(k)$ and $B_{\mathrm{O}i}(k)$ curves respectively, as shown in Fig. 5. To describe the whole approximation performance, the P-index and O-index are denoted by the arithmetic mean of P_i and O_i separately,

$$P = \frac{1}{4}\sum_{i=0}^{3} P_i, \quad O = \frac{1}{4}\sum_{i=0}^{3} O_i. \tag{27}$$

The greater the values of K-index, P-index, and O-index, the better the approximation performance of the impedance function $Z_k(s)$ to the ideal fractional operator s^{μ}.

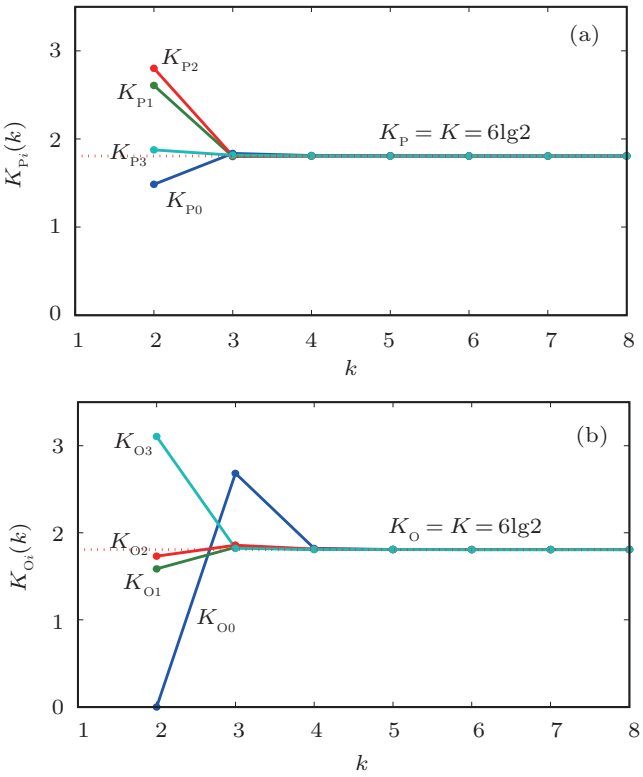


Fig. 6. (color online) Slopes of approximation bandwidth exponent curves with $\mu = -2/3$ and $Z_0(s) = 1/s$: (a) phase-frequency; (b) order-frequency.

3.2. Iterating algorithm of coefficient vectors and complexity of fractance approximation circuit

The coefficient vectors $\boldsymbol{\beta}_k$ and $\boldsymbol{\alpha}_k$ of numerator polynomial $N_k(s)$ and denominator polynomial $D_k(s)$ are given as follows:

$$\boldsymbol{\beta}_k = \left[\, \beta_{k,nk} \;\; \beta_{k,nk-1} \;\; \cdots \;\; \beta_{k,1} \;\; \beta_{k,0} \,\right], \tag{28a}$$

$$\boldsymbol{\alpha}_k = \left[\, \alpha_{k,nk} \;\; \alpha_{k,nk-1} \;\; \cdots \;\; \alpha_{k,1} \;\; \alpha_{k,0} \,\right], \tag{28b}$$

where the degrees are in descending arrangement.

Let $Z_k(s) = N_k(s)/D_k(s)$ used in Eq. (11) and we can rewrite the rational approximation impedance function sequence of fractional operator $s^{-j/n}$ in the other form:

$$\begin{aligned} Z_{k+1}(s) &= F_{\mathrm{C}}^{[j]}(Z_k(s)) \\ &= \frac{N_k(s)}{D_k(s)} \frac{(n-1)s^j N_k^n(s) + (n+1)D_k^n(s)}{(n+1)s^j N_k^n(s) + (n-1)D_k^n(s)}. \end{aligned} \tag{29}$$

Corresponding to this, the iterating formulae of coefficient vectors $\boldsymbol{\beta}_k$ and $\boldsymbol{\alpha}_k$ are obtained,

$$\boldsymbol{\beta}_{k+1} = \boldsymbol{\beta}_k * ((n-1)^j \overrightarrow{\boldsymbol{\beta}_k^{*n}} + (n+1)^j \overrightarrow{\overline{\boldsymbol{\alpha}_k^{*n}}}), (k \in \mathbb{N}), \tag{30a}$$

$$\boldsymbol{\alpha}_{k+1} = \boldsymbol{\alpha}_k * ((n+1)^j \overrightarrow{\boldsymbol{\beta}_k^{*n}} + (n-1)^j \overrightarrow{\overline{\boldsymbol{\alpha}_k^{*n}}}), (k \in \mathbb{N}), \tag{30b}$$

where the operation symbol $*$ denotes convolution and

$$\boldsymbol{\beta}_k^{*n} = \underbrace{\boldsymbol{\beta}_k * \boldsymbol{\beta}_k * \cdots * \boldsymbol{\beta}_k}_{(n-1)\text{-times convolution}}, {}^j\overrightarrow{\boldsymbol{\beta}_k^{*n}} = [\boldsymbol{\beta}_k^{*n}, \underbrace{0, \ldots, 0}_{j\times 0}], \tag{31a}$$

$$\boldsymbol{\alpha}_k^{*n} = \underbrace{\boldsymbol{\alpha}_k * \boldsymbol{\alpha}_k * \cdots * \boldsymbol{\alpha}_k}_{(n-1)\text{-times convolution}}, {}^j\overrightarrow{\overline{\boldsymbol{\alpha}_k^{*n}}} = [\underbrace{0, \ldots, 0}_{j\times 0}, \boldsymbol{\alpha}_k^{*n}]. \tag{31b}$$

The aim of using the zero padding operation is to utilize the matrix operations, which can reduce the amount of calculation. The degrees n_k and d_k of the impedance function $Z_k(s)$ are decided by the degrees n_0 and d_0 of the initial impedance, the iterating time k and the operational order μ, then there exists

$$n_0 = d_0 \to n_k = d_k = (n+1)^k(n_0 + \mu) - \mu, \tag{32a}$$

$$n_0 = d_0 + 1 \to n_k = d_k + 1 = (n+1)^k(n_0 + \mu) - \mu, \tag{32b}$$

$$n_0 = d_0 - 1 \to n_k = d_k - 1 = (n+1)^k(n_0 + 1) - 1. \tag{32c}$$

According to the network synthesis theory, we could use available electrical and electronic elements, such as resistors or capacitors, to design the fractance approximation circuit of impedance function $Z_k(s)$. The degree of the impedance function $Z_k(s)$

$$\varsigma_k = \max\{n_k, d_k\} \tag{33}$$

indicates how many independent dynamic elements such as capacitors and inductors are used in the network. So we make ς_k as a measure index of the complexity of the fractance approximation circuit.

4. Performance analysis

4.1. Performance analysis of Carlson iterating rational approximation

Given the appropriate initial impedance $Z_0(s)$, the impedance function $Z_k(s)$ and coefficient vectors β_k and α_k are calculated by the iterating formulae.

Firstly, take $\mu = -j/n = -2/3$ for example. Let

$$Z_0(s) = \frac{N_0(s)}{D_0(s)} = \frac{1}{s}, \quad (n_0 = 0, d_0 = 1), \tag{34}$$

the coefficient vectors and the zero-pole distributions of impedance functions with the iterating time k equaling several initial values, are listed in the following Table 1. Except there is a pole in the origin, other zeros and poles are distributed in the left-half plane of the s complex plane. Thus these impedance functions meet the positive reality principle. As can be seen from Fig. 2, the frequency bandwidths of the impedance functions approximating to the fractional operator $s^{-2/3}$ are gradually increasing with the iterating time k increasing. Therefore, these impedance functions possess the operational validity.

Table 1. Coefficient vectors and zero-pole distributions of impedance functions with $\mu = -2/3$ and $Z_0(s) = 1/s$.

iterating time k	β_k and α_k (before canceling the commen terms)	β_k and α_k (after canceling the common terms)	zeros	poles
$k=0$	$\beta_0 = [1]$, $\alpha_0 = [1,0]$	$\beta_0 = [1]$, $\alpha_0 = [1,0]$	–	0
$k=1$	$\beta_1 = [2,1,0,0]$, $\alpha_1 = [1,2,0,0,0]$	$\beta_1 = [2,1]$, $\alpha_1 = [1,2,0]$	-0.50	$0,-2.0$
$k=2$	$\beta_2 = [4,42,92,80,24,1,0,0,0,0,0,0,0,0,0,0]$, $\alpha_2 = [1,24,80,92,42,4,0,0,0,0,0,0,0,0,0,0,0]$	$\beta_2 = [4,42,92,80,24,1]$, $\alpha_2 = [1,24,80,92,42,4,0]$	$-7.9, -1.0+\mathrm{j}0.48, -1.0-\mathrm{j}0.48, -0.50, -0.049$	$-20, -2.0, -0.80+\mathrm{j}0.37, -0.80-\mathrm{j}0.37, -0.13, 0$

From the above discussion, the impedance function $Z_k^{(-2/3)}(s)$ calculated by the iterating formula with the initial impedance $Z_0(s) = 1/s$, meets all these properties which the fractance impedance function should own, and it is capable of having the operational performance of operator $s^{-2/3}$ and being physically realized.

The K-index as shown in Fig. 6 is

$$\begin{aligned} K &\approx K_{\mathrm{P}} \approx K_{\mathrm{O}} \\ &\approx 2n\lg((n+1)/(n-1)) \\ &\approx 6\lg 2 \approx 1.8062. \end{aligned} \tag{35}$$

This means that if the iterating time k increases by 1, then the approximation frequency bandwidth exponent will increase by 1.8062. The P-index and O-index are respectively equal to the following values

$$\begin{aligned} P &= \frac{1}{4}\sum_{i=0}^{3} P_i \approx -1.0885, \\ O &= \frac{1}{4}\sum_{i=0}^{3} O_i \approx -1.7573. \end{aligned} \tag{36}$$

From Table 1, the degree of denominator polynomial is higher than that of the nominator polynomial of the impedance function, that is

$$n_k = d_k - 1 = (n+1)^k(n_0+1) - 1 = 4^k - 1. \tag{37}$$

After the common terms between numerator and denominator polynomials have been cancelled, there exists

$$\begin{aligned} n_k &= d_k - 1 \\ &= (n-j)((n+1)^k - 1)/n \\ &= (4^k - 1)/3. \end{aligned} \tag{38}$$

Thus the complexity of fractance approximation circuit with impedance function $Z_k^{(-2/3)}(s)$

$$\varsigma_k = \max\{n_k, d_k\} = d_k = (4^k - 1)/3 + 1. \tag{39}$$

Taking nine different kinds of operational orders $\mu = -j/n$ with $2 \le n \le 5$ $(1 \le j \le n-1)$ into consideration and choosing an appropriate rational initial impedance function for a certain operational order, we analyze the approximation performance of impedance function $Z_k(s)$ to ideal fractional operator s^{μ}, acquire circuit complexity and make a summary as shown in Table 2. For $1/n$ order with a resistor as the initial impedance and $(n-1)/n$ order with a capacitor as the initial impedance, both have the same K-index which is approximately equal to $2n\lg((n+1)/(n-1))$. The latter uses one more capacitor than the former. With the increasing of the absolute values of operational orders, the K-index is decreasing. With the increasing of n, the complexity of the fractance approximation circuit also increases.

Table 2. Performance analyses of different fractional operators.

operational order μ	circuit complexity ς_k	K-index	P-index	O-index	$Z_0(s)$
$-1/2$	$(3^k-1)/2$	1.9085	-1.3431	-2.2409	$\beta_0=[1],\alpha_0=[1]$
$-1/3$	$(4^k-1)/3$	1.8062	-1.2950	-2.1843	$\beta_0=[1],\alpha_0=[1]$
$-2/3$	$(4^k-1)/3+1$	1.8062	-1.0885	-1.7573	$\beta_0=[1],\alpha_0=[1,0]$
$-1/4$	$(5^k-1)/4$	1.7748	-1.2796	-2.1634	$\beta_0=[1],\alpha_0=[1]$
$-3/4$	$(5^k-1)/4+1$	1.7748	-0.8801	-1.6081	$\beta_0=[1],\alpha_0=[1,0]$
$-1/5$	$(6^k-1)/5$	1.7069	-1.4226	-2.1539	$\beta_0=[1],\alpha_0=[1]$
$-2/5$	6^k	–	–	–	$\beta_0=[1,1],\alpha_0=[1,0]$
$-3/5$	6^k	–	–	–	$\beta_0=[1,1],\alpha_0=[1,0]$
$-4/5$	$(6^k-1)/5+1$	1.7069	-0.6261	-1.1919	$\beta_0=[1],\alpha_0=[1,0]$

4.2. Performance comparison

As we all know, the Liu-Kaplan chain fractance also can realize the rational approximation of fractional operator with arbitrary order. So performance comparison between Liu–Kaplan chain fractance and the generalized Carlson iterating rational approximation has been made in this paper, as shown in Figs. 7 and 8.

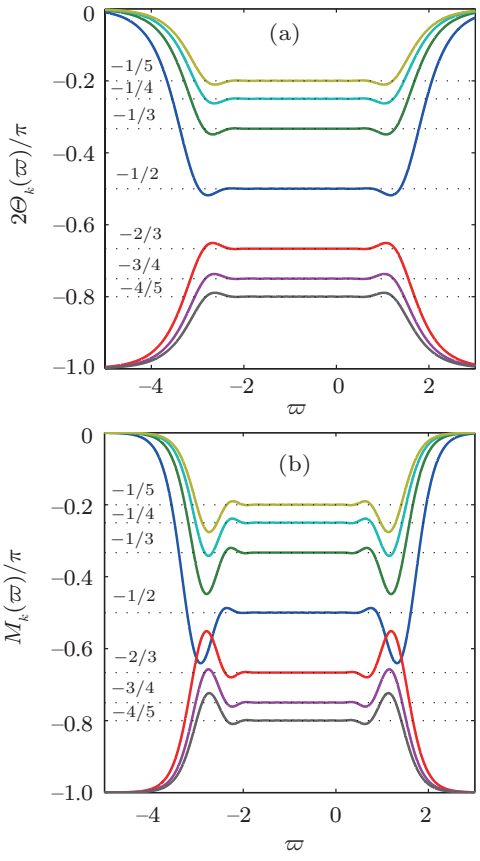


Fig. 7. (color online) The generalized Carlson iterating rational approximation with $k=3$: (a) phase–frequency characteristics; (b) order–frequency characteristics.

The approximation frequency–domain characteristics curves of the impedance function calculated by the generalized Carlson iterating function are relatively smooth over a certain frequency range. The Liu–Kaplan chain franctance exists as an operational oscillating phenomena.[26–29] The wider the length of operational frequency window of the Liu–Kaplan chain fractance, the larger the oscillating magnitude.

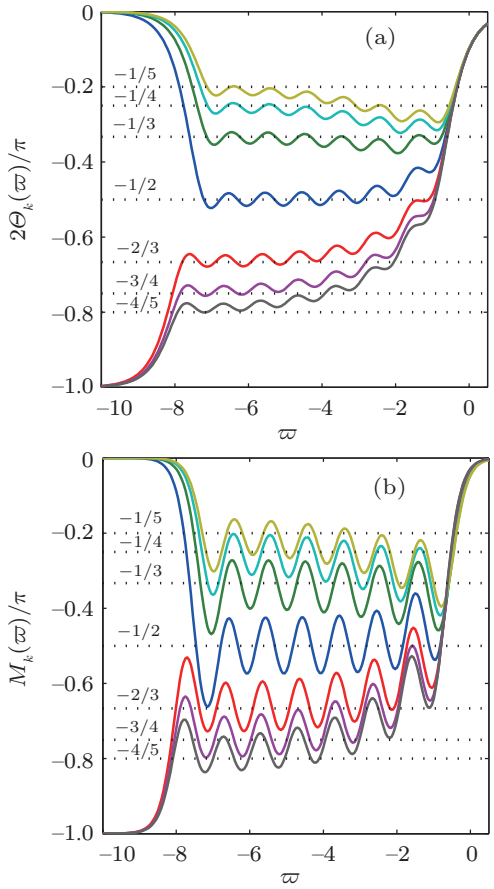


Fig. 8. (color online) Liu–Kaplan chain fractance with the length of operational frequency window equal to 1 and $\varsigma_k=8$: (a) phase-frequency characteristics; (b) order-frequency characteristics.

5. Conclusion

The performance analysis of the generalized Carlson iterating process of fractional operator with arbitrary order has been explored in this paper. In addition, the unit and dimension of the physical quantity $F^{(\mu)}$ are given in this paper, which is useful in the future study. For some special orders, $F^{(\mu)}$ are called in different particular forms, for example, $[F^{(1/2)}]=\Omega\cdot \mathrm{s}^{1/2}$ is called Ohm square root second (Ohm root second for short), $[F^{(1/3)}]=\Omega\cdot \mathrm{s}^{1/3}$ is Ohm cube root second, $[F^{(1/n)}]=\Omega\cdot \mathrm{s}^{1/n}$ is Ohm n-th root second, $[F^{(-1/2)}]=$

$\Omega \cdot s^{-1/2}$ is Ohm per square root second, $[F^{(-1/3)}] = \Omega \cdot s^{-1/3}$ is Ohm per cube root second, $[F^{(-1/n)}] = \Omega \cdot s^{-1/n}$ is Ohm per n-th root second. In further researches, some subjects are to be solved as follows.

i) We attempt to obtain the analytical expressions of zeros and poles of the impedance function, so it is easy to judge the stability and analyze the approximation performance of the circuit network to the fractional operator from zero-pole analytical expressions.

ii) The K-index of the Carlson iterating process approximating to the fractional operator $s^{\pm 1/n}$ has already been known, how about the K-index with j equal to or greater than 2? Investigating the theoretical expression of the K-index should receive significant attention.

iii) We utilize passive or active elements to design an arbitrary order fractance approximation circuit which is used as a fractional order element to realize arbitrary order fractional calculus. The best situation is to make fractional order capacitors and inductors available in the market, just like normal capacitors and inductors are. Furthermore, a great deal of interest exists in modelling and studying fractional order phenomena and processes in the nature.

References

[1] Hu K X and Zhu K Q 2009 *Chin. Phys. Lett.* **26** 108301
[2] Kumar R and Gupta V 2013 *Chin. Phys. B* **22** 074601
[3] Ghany H A and Hyder A A 2014 *Chin. Phys. B* **23** 060503
[4] Wang F Q and Ma X K 2013 *Chin. Phys. B* **22** 120504
[5] Yu B, Yuan X and Tao L 2015 *J. Electr. Inf. Technol.* **37** 21
[6] Pu Y F and Yuan X 2016 *IEEE Access* **4** 1872
[7] Yuan X 2015 *Mathematical Principles of Fractance Approximation Circuits* (Beijing: Science Press) pp. 218–256
[8] Carlson G and Halijak C 1962 *IRE Trans. Circuit Theory* **9** 302
[9] Carlson G and Halijak C 1964 *IEEE Trans. Circuit Theory* **11** 210
[10] Steiglits K 1964 *IEEE Trans. Circuit Theory* **11** 160
[11] Halijak C 1964 *IEEE Trans. Circuit Theory* **11** 494
[12] Roy S D 1967 *IEEE Trans. Circuit Theory* **14** 264
[13] Liao K, Yuan X, Pu Y F and Zhou J L 2006 *J. Sichuan Univ. (Nat. Sci. Edn.)* **43** 104
[14] Ren Y and Yuan X 2008 *J. Sichuan Univ. (Nat. Sci. Edn.)* **45** 1100
[15] Krishna B T and Reddy K V V S 2008 *Acta Passive Electron. Compon.* **2008** 369421
[16] Krishna B T 2011 *Signal Process.* **91** 386
[17] Liu Y, Pu Y F, Shen X D, Zhou J L 2012 *J. Sichuan Univ. (Eng. Sci. Edn.)* **44** 153
[18] Chen Y Q and Vinagre B M and Podlubny I 2004 *Nonlinear Dynam.* **38** 155
[19] Onat C, Sahin M and Yaman Y 2012 *Aircr. Eng. Aerosp. Tec.* **84** 203
[20] Sun H H, Abdelwahab A A and Onaral B 1984 *IEEE T. Automat. Control* **29** 441
[21] Zou D, Yuan X, Tao C Q and Yang Q 2013 *J. Sichuan Univ. (Nat. Sci. Edn.)* **50** 293
[22] He Q Y and Yuan X 2016 *Acta Phys. Sin.* **65** 160202 (in Chinese)
[23] Editorial Board 2002 *Mathematics Dictionary* (Taiyuan: Education Press)
[24] Valkenburg V M E (translated by Yang X J, Zheng J L, Yang W L) 1982 *Network Synthesis* (Beijing: Science Press) pp. 222–225 (in Chinese)
[25] Zu Y X and Lu Y Q 2007 *Network Analysis and Synthesis* (Beijing: China Machine Press) pp. 111–120 (in Chinese)
[26] Liu S H 1985 *Phys. Rev. Lett.* **55** 529
[27] Kaplan T and Gray L J 1985 *Phys. Rev. B* **32** 7360
[28] Kaplan T, Liu S H and Gray L J 1986 *Phys. Rev. B* **34** 4870
[29] Kaplan T, Gray L J and Liu S H 1987 *Phys. Rev. B* **35** 5379