Validation of the Wiedemann–Franz law in a granular s-wave superconductor in the nanometer scale*

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The present study tries to evaluate the validity of the Wiedemann–Franz law in a granular s-wave superconductor in the presence of concentrated impurities. By using Green's function method and the Kubo formula technique, three distinct contributions of the Aslamazov–Larkin, the Maki–Thompson and, the density of states are calculated for both the electrical conductivity and the thermal conductivity in a granular s-wave superconductor. It is demonstrated that these different contributions to the fluctuation conductivity depend differently on the tunneling because of their different natures. This study examines the transport in a granular superconductor system in three dimensions in the limit of large tunneling conductance, which makes it possible to ignore all localization effects and the Coulomb interaction. We find that the tunneling is efficient near the critical temperature and that there is a crossover to the characteristic behavior of a homogeneous system. When it is far from the critical temperature, the tunneling is not effective and the system behaves as an ensemble of real zero-dimensional grains. The results show that the Wiedemann–Franz law is violated in both temperature regions.

Keywords: granular superconductor, s-wave, impurity vertex, fluctuation propagator

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1. Introduction

Electrical conductivity and thermal conductivity of an ordinary metal are interrelated to each other via a universal relation [1] known as the Wiedemann–Franz (WF) law

$$\frac{\kappa}{\sigma} = L_0 T, \tag{1}$$

where $L_0 = \pi^2/3e^2$ is the Lorentz number, T is the temperature, and e is the electron charge. The units used in the present study are $k_B = \hbar = 1$. The WF law holds as long as scattering processes are quasi-elastic, i.e., only the weakly energy-dependent. Equation (1) is a direct consequence of Fermi liquid theory. This theory describes the properties of a weakly disordered, interacting electron gas, provided that screening renders the Coulomb interaction sufficiently weak and short-ranged. Under these conditions, the low-laying excitations of the metal are non-interacting fermionic quasi-particles that carry both charge and energy, and as a consequence the WF law is valid.[2] In other words, no additional information is obtained from a measurement of the thermal conductivity that is not already present in the electrical conductivity as long as interaction effects result only in the renormalization of electron spectral parameters.[3–6]

When the size of a system is reduced, its mesoscopic behavior will be extensively specified by the finite electron number. In this case, the quantum fluctuation and the quantum tunneling processes can happen. A granular s-wave superconductor at the nanometer-scale can be formed as a random network of superconducting grains that are coupled by Josephson weak links and coated with an insulator layer.[7,8] Depending on the material, the size of grains can range from several to hundreds of nanometers. The small thickness of the insulator enables the electrons to tunnel from one grain to another. This tunneling determines the property of the entire system. It is assumed that there are impurities in each granule or the shape of each granule is not simple and electrons are randomly scattered by the boundaries. In other words, it is assumed that the electron motion is chaotic in an isolated granule.[9–11] Each isolated grain can be characterized by the mean level spacing δ. Provided the hopping amplitudes are not very large and macroscopic transport in the system of the granules is determined by the ratio of hopping amplitudes t to δ, it is assumed that[12]

$$1 \ll g_T \ll E_T/\delta, \tag{2}$$

where $g_T = \left(\frac{\pi}{2}\frac{t}{\delta}\right)^2$ is the tunneling dimensionless conductance, $E_T = D/R^2$ is the Thouless energy of a single grain, D is the diffusion coefficient, and R is the radius of the grain. It is instinctively clear that the discreteness of the spectrum in a single grain in the limit $t \gg \delta$ is not perceived, and that the electron motion is diffusive through many grains. This limit corresponds to the macroscopically weak disorder. In the opposite limit $t \ll \delta$, the electrons are almost completely localized in granules, and this is the strong disorder limit. The metal–insulator transition occurs at values of the macroscopic conductance g_T of the order of unity. At such values, calculations are very difficult. The problem becomes even more complicated due to the Coulomb interaction. At small values

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of g_T the system must be an insulator. Therefore, the present study considers the region of large conductance $g_T \gg 1$, where the system without interactions would be a good metal.[13,14]

Theory of superconducting fluctuations was developed long ago near the transition into the superconducting state.[15–18] The correction to electrical and thermal conductivity due to superconducting fluctuations leads to three distinct contributions of Aslamazov–Larkin (AL), Maki–Thompson (MT), and density of state (DOS) for each conductivity.[6,19,20] In the first one, above the transition temperature T_c, non-equilibrium Cooper pairs are formed and open a new channel of charge transfer. The second contribution of fluctuations comes from a coherent scattering of the electrons forming a Cooper pair on impurities. Finally, the rearrangement of the states close to the Fermi energy leads to DOS contribution since electrons involved in the pair transport are no longer available for the single particle transport. Both the AL and MT terms lead to an enhancement of conductivity above T_c. However, the DOS correction is of the opposite sign.[18,20]

In region $g_T \gg 1$, all the effects of the weak localization and the charging effects have to be small, and for this reason they were ignored.[13,14] The present study considers all fluctuation corrections of the first order in the case of arbitrary impurity concentration in limit $T\tau \ll 1$ in granular superconductor, where τ is the scattering mean free time. Using the Green's function technique and the Kubo formula, the present study obtains MT, AL and DOS corrections to the electrical and thermal conductivity in granular s-wave superconductors. Then it will deal with the validation of WF law in granular s-wave superconductors both near to and far from the critical temperature in a zero field. The aim of this study is to investigate the WF law in a granular s-wave superconductor with zero dimension. The reason for studying the WF law in the granular superconductors is the existence of a singular correction due to the fluctuation in these materials. Also, the tunneling of the Cooper pairs leads to variations in the thermal and electrical conductivity in the granular superconductors. The results obtained in this paper agree with the experimental work.[15]

2. Formulation of the problem

A simplified model will be considered, in which superconductor grains form a regular lattice. The grains are disordered due to impurities or irregular boundaries. The grains are coupled to each other and are not perfect. The impurities can be both inside the grain and on the surface. It is assumed that electrons can hop from one grain to another grain and can interact with phonons. The Hamiltonian of the system can be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_p + \mathcal{H}_T. \tag{3}$$

Here $\mathcal{H}_0$ and $\mathcal{H}_p$ describe the free electron gas and the BCS pairing Hamiltonian inside each grain respectively, reading

$$\mathcal{H}_0 = \sum_{i,k} \varepsilon_{i,k} a^\dagger_{i,k} + \mathcal{H}_{imp}, \tag{4}$$

$$\mathcal{H}_p = -\lambda \sum_{i,kk'} a^\dagger_{i,k} a^\dagger_{i,-k} a_{i,-k'} a_{i,k'}, \tag{5}$$

where i stands for the numbers of the grains, $a^\dagger_{i,k}(a_{i,k})$ is the creation (annihilation) operator of an electron in the state $k = (\boldsymbol{k}, \uparrow)$ or $-k = (-\boldsymbol{k}, \downarrow)$, $\mathcal{H}_{imp}$ describes the electron elastic scattering with impurities, and λ is an interaction constant. The interaction term $\mathcal{H}_p$ in Eq. (3) contains only diagonal terms.[21–23] This simple description is correct under the condition that the off-diagonal terms are really small. The term $\mathcal{H}_T$ in Eq. (3) describes tunneling from one grain to another, which can be written as[19]

$$\mathcal{H}_T = \sum_{i,j,p,q} t^{pq}_{ij} a^\dagger_{ip} a_{jq} + \text{h.c}, \tag{6}$$

where summation is performed over the states p, q of each grain (the spin is conserved) and over the neighboring grain i and j, and t^{pq}_{ij} is the matrix elements. In Eq. (6) it is assumed that the tunneling amplitude is uniform and the momentum is independent, $t^{pq}_{ij} \equiv t$.

The Kubo formula is used to drive the thermal conductivity in granular superconductors. In the Matsubara imaginary time formalism we have[24,25]

$$\kappa(\omega) = \frac{\mathrm{i}}{\omega_\nu T} \left[\Pi^{(\text{heat})}(\omega_\nu)\right], \tag{7}$$

where $\Pi^{(\text{heat})}(\omega_\nu)$ is the linear response operator to an applied temperature gradient and $\omega_\nu = 2\pi T\nu$ is the bosonic frequency of the external field in the Matsubara representation.[26] Analogously, the electrical conductivity can be written as[24,25]

$$\sigma(\omega) = \frac{\mathrm{i}}{\omega_\nu} \left[Q^{(\text{elec})}(\omega_\nu)\right]. \tag{8}$$

The electromagnetic response operator $Q^{(\text{elec})}(\omega_\nu)$, defined on Matsubara frequencies, can be presented as the correlator of two one-electron Green's functions averaged over impurity positions, accounting for interactions, in our case the particle–particle in the Cooper channel. In the presence of magnetic field $\boldsymbol{A}(\boldsymbol{r})$, the Green's functions change and contain the effects of orbital quantization. However, in the presence of strong disorder $\omega_c\tau \ll 1$ (ω_c is the cyclotron frequency) or at relatively high temperature, the discrete levels are smeared out and the effects of the magnetic field can be treated semiclassically. This means that the Green's function in the coordinate representation is

$$\Im\left(\mathrm{i}\omega_n, \boldsymbol{r}, \boldsymbol{r}'\right) = \Im^{(0)}\left(\mathrm{i}\omega_n, \boldsymbol{r} - \boldsymbol{r}'\right) \exp\left(\frac{\mathrm{i}e}{c} \int_{\boldsymbol{r}}^{\boldsymbol{r}''} \boldsymbol{A}\,\mathrm{d}\boldsymbol{r}\right),$$

where $\Im^{(0)}(\mathrm{i}\omega_n, \boldsymbol{r}-\boldsymbol{r}')$ is the Green's function without magnetic field. In the zero-order approximation in the superconducting fluctuations, the disorder average Green's function $\Im^0$ in the momentum representation can be written as[26]

$$\Im^0(\mathrm{i}\omega_n, \boldsymbol{p}) = \left(\mathrm{i}\omega_n - \xi(\boldsymbol{p}) + \frac{\mathrm{i}}{2\tau \mathrm{sgn}(\omega_n)}\right)^{-1}, \tag{9}$$

where $\omega_n = (2n+1)\pi T$ is the fermion frequency, $\xi(\boldsymbol{p}) = \varepsilon(\boldsymbol{p}) - \varepsilon_{\mathrm{F}}$ is the electron energy counted from the Fermi level, $\frac{1}{2\tau}\mathrm{sgn}(\omega_n)$ is the self-energy taking into account the coherent scattering of electron on impurities in the bulk metal, sgn is the sign function, and τ is the scattering mean free time.

For the calculation of superconducting fluctuation corrections to the electrical conductivity in granular superconductors, we introduce the propagator of the superconducting fluctuations presented as[12,17]

$$\wp(\Omega_k, \boldsymbol{q}) = -\frac{1}{\nu_0}\left[\ln\left(\frac{\left[(|\Omega_k| + \gamma_0(H))^2 + \frac{v_{\mathrm{F}}^2 q^2}{3}\right]^{1/2}}{\Delta_0}\right) + \alpha(\boldsymbol{q})\right]^{-1}, \tag{10}$$

where $\Omega_k = 2k\pi T$ is the bosonic frequency, ν_0 is the density of states on the Fermi surface,

$$\alpha(\boldsymbol{q}) = \left(\frac{8\pi}{3}\right)\left[\sum_{i=1}^{3} g_{\mathrm{T}}\left(\frac{\delta}{\gamma_0(H)}\right)(1-\cos(q_i d))\right]$$

describes the tunneling of electrons from one grain to another grain, d is the distance of the center of neighboring grain, $\varepsilon_0(H) = (2/5)(\phi/\phi_0)^2 E_{\mathrm{T}}$ with ϕ the magnetic flux through the grain, $E_{\mathrm{T}} = D/R^2$ is the Thouless energy of a single grain, D is the diffusion coefficient, and R is the radius of the grain.

The existence of BCS pairing potential, and therefore, the possibility for two electrons to form a Cooper pair, introduces a new correction. The latter takes into account the coherent scattering of the electrons forming Cooper pair on the same impurity. The impurity vertex for electrical conductivity can be written as[12,27]

$$C(\boldsymbol{q}, \omega_n, \Omega_\mu - \omega_n) = 2\pi\nu_0\tau \frac{\left|2\omega_n - \Omega_k + \frac{1}{\tau}\right|}{\left|2\omega_n - \Omega_k + \frac{\left[\frac{v_{\mathrm{F}}^2}{3} + (\varepsilon_0(H) + \alpha(\boldsymbol{q}))^2\right]^{1/2}}{2\tau}\right|^2}. \tag{11}$$

As in conventional bulk superconductors, we can write the corrections to the classical electrical conductivity as a sum of corrections to σ_{DOS}, σ_{AL}, and σ_{ML}. Figure 1 represents the Feynman diagrams describing these contributions.[10] First we consider the correction to the conductivity due to suppression of the density of state (diagrams 5–10 of Fig. 1). These (DOS) diagrams arise from corrections to the normal quasiparticles density of states due to fluctuations of the normal quasiparticles into the superconducting state. In the dirty limit, the calculation of the contributions to the longitudinal fluctuation conductivity from such diagrams was discussed previously.[27] Diagrams 9 and 10 arise from averaging diagrams 5 and 6 over impurity positions. It was shown that, for longitudinal fluctuation conductivity, diagrams 9 and 10 are less temperature dependent and can therefore be ignored. In the dirty limit, diagrams 7 and 8 were shown to be equal to $1/3$ time as diagrams 5 and 6, which are evidently equal to each other. In the clean limit, diagrams 7 and 8 can be ignored relative to diagrams 5 and 6. For general impurity scattering, the ratio of these diagrams depends on τ. As we are interested in the results for arbitrary impurity concentration, we shall evaluate all these diagrams separately.

The total electrical conductivity σ can be written as[12,27]

$$\sigma = \sigma_{\mathrm{DOS}} + \sigma_{\mathrm{AL}} + \sigma_{\mathrm{MT}}. \tag{12}$$

In the following sections, we consider the contributions to the fluctuation electrical conductivity that arise from diagrams of Fig. 1.

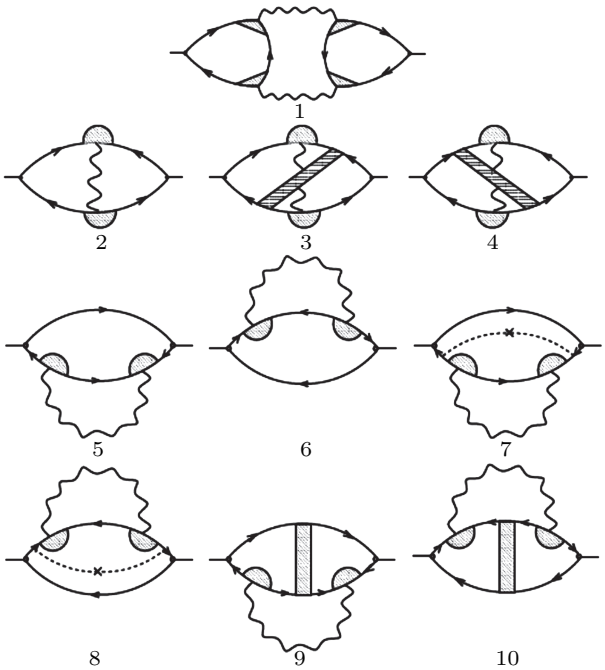


Fig. 1. Feynman diagrams for the leading order contribution to the fluctuation conductivity of granular metals. Wavy lines are fluctuation propagators corresponding to Cooper pairs, solid lines with arrows are normal-state Green's functions corresponding to normal state quasiparticles, and shaded semicircles are vertex corrections arising from impurities. Dashed lines with central crosses are additional impurity renormalizations and shaded blocks are impurity ladders. Diagram 1 is the Aslamazov–Larkin (AL) contribution, diagram 2 is the Maki–Thompson (MT), and diagrams 5–8 are the density of states (DOS). Diagrams 3, 4, and 9, 10 arise when one averages the DOS and MT diagrams over impurities.

3. Correction to the electrical conductivity due to suppression of DOS

The general expression for the DOS contribution to $\sigma_{\rm DOS}$ from diagram 5 is given by[26,27]

$$\sigma_{\rm DOS} = \sigma_0 (4T)^{-1} \int_{-\infty}^{+\infty} \left[\frac{\nu(\omega_n)}{\nu_0}\right]^2 \cosh^{-2}\left(\frac{\omega}{2T}\right) {\rm d}\omega, \quad (13)$$

where

$$\begin{aligned}\nu(\omega_n) = & \left(\frac{1}{\pi}\right)\left(\frac{1}{(2\pi\nu_0\tau)}\right)^2 \int \frac{{\rm d}^3 q}{(2\pi)^3} T\sum_{\Omega_k} \wp(\Omega_k, q) \\ & \times \int \frac{{\rm d}^3 q}{2\pi^3} C^2(\varepsilon_n, \Omega_k - \varepsilon_n) \Im^2(\varepsilon_n, p) \\ & \times \Im(p-q, \Omega_k - \varepsilon_n) \Im(\Omega_k - \varepsilon_n, p). \end{aligned} \quad (14)$$

Diagram 6 gives an identical contribution. The contribution of the other diagrams (7–10 of Fig. 1) can be treated in the same manner. In Eq. (14), $\sigma_0 = \frac{8 g_{\rm T} e^2}{\pi R}$ is the classical electrical conductivity of granular metal. Equation (14) contains integration over the momentum in the single grain p and the quasi-momentum q. With inserting Eq. (14) into Eq. (13), one can write the DOS contribution to electrical conductivity in the absent magnetic field in the form of

$$\begin{aligned} & {\rm Re}\,\sigma^{\rm DOS}(\omega) \\ = & -\frac{T e^2 g_{\rm T}^2 \delta^2 d}{2\pi^2 \eta} \ln\left[\frac{2}{\sqrt{\varepsilon + r + \sqrt{\varepsilon}}}\right] \\ & \times \frac{1}{(\tau^{-2} + \omega^2)^2} \frac{4}{\tau}\left[\psi\left(\frac{1}{2}\right) - {\rm Re}\,\psi\left(\frac{1}{2} - \frac{{\rm i}\omega}{2\pi T}\right)\right] \\ & + \frac{1}{4\pi T\tau\omega(\tau^{-2} + \omega^2)} {\rm Im}\,\psi'\left(\frac{1}{2} - \frac{{\rm i}\omega}{2\pi T}\right) + \frac{1}{\omega(\tau^{-2} + \omega^2)} \\ & \times \left[{\rm Im}\,\psi\left(\frac{1}{2} - \frac{{\rm i}\omega}{2\pi T}\right) - 2\,{\rm Im}\,\psi\left(\frac{1}{2} - \frac{{\rm i}\omega}{2\pi T} + \frac{1}{4\pi T\tau}\right)\right], \end{aligned} \quad (15)$$

where

$$r = 4\eta \frac{\delta^2 g_{\rm T}^2}{v_{\rm F}^2}, \quad (16)$$

$$\eta = -\frac{\tau^2 v_{\rm F}^2}{3}\left[\psi\left(\frac{1}{2} + \frac{1}{4\pi T\tau}\right) - \psi\left(\frac{1}{2}\right) - \frac{1}{4\pi T\tau}\psi'\left(\frac{1}{2}\right)\right]. \quad (17)$$

From Eq. (15), for $T\tau \ll 1$ and $T\tau \gg 1$ limits, the fluctuation contribution DOS to the electrical conductivity can be written as

$$\sigma^{\rm DOS} = -\frac{3 e^2 g_{\rm T}^2 \delta^2 d T^2 \tau^2}{7\xi(3) v_{\rm F}^2} \ln\left[\sqrt{\frac{48\pi^3 T^2}{7\xi(3) g_{\rm T}^2 \delta^2}}\right], \quad T\tau \ll 1, \quad (18)$$

$$\sigma^{\rm DOS} = -\frac{e^2 g_{\rm T}^2 \delta^2 d}{4\pi^5 v_{\rm F}^2} \ln\left[\sqrt{\frac{24T}{g_{\rm T}^2 \delta^2 \tau}}\right] \psi''\left(\frac{1}{2}\right), \quad T\tau \gg 1. \quad (19)$$

4. Aslamazov–Larkin contribution to electrical conductivity

This correction originates from the ability of virtual Cooper pairs to carry an electrical current. In contrast to the one electron tunneling determining the DOS correction to the electrical conductivity $\sigma_{\rm Dos}$, the probability of tunneling of the Cooper pairs from one grain to another is proportional to t^4. According to the diagram 1 of Fig. 1, the quantity $\sigma_{\rm AL}$ for the response function $Q^{\rm AL}$ can be written as[24,28]

$$\sigma_{\rm AL}(\omega) = \frac{Q^{\rm AL}(\omega)}{-{\rm i}\omega}. \quad (20)$$

The analytical expression for AL contribution has the form[12,17]

$$\begin{aligned} Q^{\rm AL}(\omega_n) = & \frac{4}{3}\sum_{i=1}^{3} \int \frac{{\rm d}^3 q}{(2\pi)^3} T\sum_{\Omega_k} \wp(\Omega_k, q) \\ & \times \wp(\Omega_k - \omega_n, q) B_1^2(0, q), \end{aligned} \quad (21)$$

where $B_1(0, q)$ corresponds to one electron loop and can be written as

$$\begin{aligned} B_1(0, q) = & -4 \int \sin(q_i' d) \cos(q_i - q_i') d \frac{{\rm d}^3 q'}{(2\pi)^3} \frac{e d t^2 V^2}{(2\pi\nu_0\tau)^2} \\ & \times \int \frac{{\rm d}^3 p}{(2\pi)^6} T\sum_{\varepsilon_n} \Im(\omega_n p)\Im(q-p, -\omega_n)\Im(-\omega_n p) \\ & \times C^2(\omega_n). \end{aligned} \quad (22)$$

Integrating over the momentum p in Eq. (22) one can reduce the function $B_1(0, q)$ to the form

$$\begin{aligned} B_1(0, q) = & -\frac{8}{\pi^2} g_{\rm T} \frac{e d}{T_{\rm c}} 4 \int \sin(q_i d) \\ & \times \cos(q_i - q_i') d \frac{{\rm d}^3 q'}{(2\pi)^3}. \end{aligned} \quad (23)$$

Using Eqs. (19), (21), and (23) the fluctuation contribution $\sigma_{\rm AL}$ to the electrical conductivity can be written as

$$\begin{aligned} \sigma^{\rm (AL)} = & \frac{\pi^2 e^2 \eta^2}{d} \\ & \times \int \frac{{\rm d}^3 q}{(2\pi)^3} \frac{q^2}{[(\eta q^2 + \varepsilon)(\eta q^2 + \varepsilon + r)]^{3/2}} \\ = & \frac{e^2}{16 d} \frac{1}{[\varepsilon(\varepsilon + r)]^{1/2}}, \end{aligned} \quad (24)$$

where $\varepsilon = \ln\frac{T}{T_{\rm c}}$. From Eqs. (20)–(24), the fluctuation contribution $\sigma_{\rm AL}$ to the electrical conductivity can be written as

$$\begin{aligned} \sigma^{\rm (AL)} = & \frac{e^2 \delta^2 g_{\rm T}^2}{16\pi d(\varepsilon)^{1/2}} \\ & \times \left[-\frac{v_{\rm F}^2 \tau^2}{3}\left[\psi\left(\frac{1}{2} + \frac{1}{4\pi T\tau}\right)\right.\right. \\ & \left.\left. - \psi\left(\frac{1}{2}\right) - \frac{1}{4\pi T\tau}\psi'\left(\frac{1}{2}\right)\right]\right]. \end{aligned} \quad (25)$$

From Eq. (25), for $T\tau \gg 1$ and $T\tau \ll 1$ limits, one has

$$\sigma^{(\mathrm{AL})} = \frac{3e^2\pi\delta^2 g_{\mathrm{T}}^2 T^2}{7\xi(3)\nu_{\mathrm{F}}^2(\varepsilon)^{1/2}}, \quad T\tau \gg 1, \tag{26}$$

$$\sigma^{(\mathrm{AL})} = \frac{3}{2}\frac{e^2\delta^2 g_{\mathrm{T}}^2 T}{\pi^2\nu_{\mathrm{F}}^2(\varepsilon)^{1/2}\tau}, \quad T\tau \ll 1. \tag{27}$$

5. Maki–Thompson correction to electrical conductivity

This contribution usually increases the electrical conductivity. According to diagram 2 of Fig. 1, the analytical expression for MT contribution can be written as[8]

$$Q^{\mathrm{MT}}(\omega_n) = \frac{2}{3}\sum_{i=1}^{3}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}T\sum\wp(\Omega_k,\boldsymbol{q})B_2(\omega,\boldsymbol{q}), \tag{28}$$

where

$$\begin{aligned} &B_2(\omega_n,\boldsymbol{q}) \\ &= 4\int\sin(q_i'd)\sin\left(q_i-q_i'\right)d\frac{\mathrm{d}^3\boldsymbol{q}'}{(2\pi)^3}\frac{(edtV)^2}{(2\pi\nu_0\tau)^2} \\ &\times\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^6}T\sum_{\varepsilon_n}\Im(\varepsilon_n,\boldsymbol{p})\Im(\varepsilon_n,\boldsymbol{p})\Im(-\varepsilon_n+\omega_n,\boldsymbol{p}) \\ &\times C(\varepsilon_n)C(-\varepsilon_n+\omega_n,\varepsilon_n-\omega_n), \end{aligned} \tag{29}$$

$\boldsymbol{p}$ stands for the momentum in the granules, and $\boldsymbol{q}$, $\boldsymbol{q}'$ are quasimomenta. In evaluating the sum over the Matsubara frequency ε_n, it is useful to break up the sum into two parts. In the first part, ε_n is in the domains $[-\infty,-\omega_{\mathrm{v}}]$ and $[0,\infty]$. This gives rise to the regular part of the MT diagram. The second (anomalous) part of the MT diagram arises from the summation over ε_n. After integration over the momenta, moreover, one can write the function B both as a sum of an anomalous B^{an} and a regular B^{reg} contribution to the MT diagram

$$B_2(\omega_n,\boldsymbol{q}) = -39\frac{g_{\mathrm{T}}e^2}{d}\cos(q_id)(B^{\mathrm{an}}+B^{\mathrm{reg}}, \tag{30}$$

$$\begin{aligned} B^{\mathrm{an}} &= \frac{1}{4\pi(\omega_n+4\pi T\alpha(q)} \\ &\times\left[\psi\left(\frac{1}{2}+\frac{\omega_n}{2\pi T}+\alpha(q)\right)-\psi\left(\frac{1}{2}+\alpha(q)\right)\right], \end{aligned} \tag{31}$$

$$B^{\mathrm{reg}} = \frac{1}{8\pi\omega_n}\left[\psi\left(\frac{1}{2}+\frac{\omega_n}{2\pi T}+\alpha(q)\right)-\psi\left(\frac{1}{2}+\alpha(q)\right)\right]. \tag{32}$$

From Eqs. (28)–(32), explicit formula for the regular and anomalous part of electrical conductivity can be written as

$$\begin{aligned} \sigma_{(\mathrm{MT})}^{(\mathrm{an})} &= \frac{e^2d^2g_{\mathrm{T}}^2\delta^2r\tau}{4\pi}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\cos qd}{\left[\frac{1}{\tau_\varphi}+\frac{\nu_{\mathrm{F}}^2q^2}{3T}\left(\frac{1+2T\tau}{T\tau}\right)^{-1}\right]\left[\varepsilon+\eta q^2+\frac{r}{2}(1-\cos qd)\right]} \\ &= \frac{e^2}{4d(\varepsilon-\gamma_\varphi)}\ln\left(\frac{\varepsilon^{1/2}+(\varepsilon+r)^{1/2}}{\gamma_\varphi^{1/2}+(\gamma_\varphi+r)^{1/2}}\right), \end{aligned} \tag{33}$$

where

$$\gamma_\varphi = \frac{2\eta}{\nu_{\mathrm{F}}^2\tau\tau_\varphi} \to \frac{\pi}{8T\tau_\varphi}\begin{cases}1, & T\tau\ll 1,\\ 7\xi(3)/\left(2\pi^3\tau T\right), & 1\ll T\tau,\end{cases} \tag{34}$$

where τ_φ is a characteristic pair-breaking lifetime. The regular part of the MT contribution to the conductivity is

$$\begin{aligned} \sigma_{\mathrm{MT}}^{(\mathrm{reg})} &= -\frac{e^2d^2\pi rF(T\tau)}{4}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\cos qd}{\varepsilon+\eta q^2+\frac{r}{2}(1-\cos qd)} \\ &= -\frac{e^2}{2d}F(T\tau)\ln\left(\frac{2}{\varepsilon^{1/2}+(\varepsilon+r)^{1/2}}\right), \end{aligned} \tag{35}$$

where

$$\begin{aligned} F(T\tau) &= \frac{-\psi'\left(\frac{1}{2}+\frac{1}{4\pi T\tau}\right)+\psi'\left(\frac{1}{2}\right)+\frac{1}{4\pi T\tau}\psi''\left(\frac{1}{2}\right)}{\pi^2\left[\psi\left(\frac{1}{2}+\frac{1}{4\pi T\tau}\right)-\psi\left(\frac{1}{2}\right)-\frac{1}{4\pi T\tau}\psi'\left(\frac{1}{2}\right)\right]} \\ &\to \begin{cases}56\xi(3)/\pi^4\approx 0.346, & T\tau\ll 1,\\ \frac{8}{\pi}T\tau, & T\tau\ll 1.\end{cases} \end{aligned} \tag{36}$$

As can be seen, the terms σ_{AL} and $\sigma_{\mathrm{MT}}^{(\mathrm{an})}$ have positive contributions to the electrical conductivity, whereas the terms σ_{DOS} and $\sigma_{\mathrm{MT}}^{(\mathrm{reg})}$ give negative contributions.

From Eq. (12), one can obtain the total electrical conductivity in granular superconductors in two limits of $T\tau \gg 1$ and $T\tau \ll 1$. The function of superconducting fluctuation propagator to the thermal conductivity in the granular s-wave superconductor at every order in tunneling has the form[8,25]

$$\mathcal{L}_{\boldsymbol{K}}(\Omega_k) = -\frac{1}{\nu_0}\frac{1}{\chi_{\boldsymbol{K}}+\frac{\pi|\Omega_k|}{8T_{\mathrm{c}}}}, \tag{37}$$

where $\chi_{\boldsymbol{K}} \equiv \varepsilon + N\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}(1-\mathcal{F}_{\boldsymbol{K}})$, N is the number of the nearest neighbors, and Ω_k is the bosonic Matsubara's frequency reflecting the bosonic nature of Cooper pairs. Here $\mathcal{F}_{\boldsymbol{K}} = \frac{1}{N}\sum_{\boldsymbol{a}}\exp\{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{a}\}$ is the so-called lattice structure factor, where $\boldsymbol{a}$ is a vector connecting the nearest neighbor grains and $\boldsymbol{K}$ is the wave vector associated with the lattice grains. It can be shown that the evaluation of the structure function

$\mathcal{F}_{\boldsymbol{K}}$ leads to the same result as that given in the factor $\alpha(\boldsymbol{q})$ in Eq. (10).

One can introduce the vertex Cooper on correction to the thermal conductivity in granular superconductors in the form[16]

$$\Lambda\left(\boldsymbol{q},\omega_1,\omega_2\right)=\frac{1}{\tau}\frac{1}{|\omega_1-\omega_2|+D\boldsymbol{q}^2+\varepsilon_q\delta}, \tag{38}$$

where ω_1 and ω_2 are the energies of the electron involved in the Cooper pair, $\boldsymbol{q}$ is the momentum of the pair, and D is the diffusion coefficient. In Eq. (38), $\varepsilon_q=2g_{\mathrm{T}}\sum_a\left(1-\cos\boldsymbol{q}a\right)$ with a being the lattice vectors is the function of quasi-momentum q that appears due to the tunneling between the grains. In the approximation of an ensemble of zero dimensional grains $a\ll\xi_0$, the cooperon $\Lambda\left(\boldsymbol{q},\varepsilon_1,\varepsilon_2\right)$ provides the main contribution as $\boldsymbol{q}\to 0$, and the vertex correction reads $\Lambda\left(\boldsymbol{q},\omega_1,\omega_2\right)=\frac{1}{\tau}\frac{1}{|\omega_1-\omega_2|+\varepsilon_q\delta}$.

Now we calculate the corrections to thermal conductivity due to superconducting fluctuations for three different contributions, the AL, MT, and DOS. The diagram 1 of Fig. 1 describes the DOS correction. Solid lines are impurity-averaged single-electron Green's functions, the wavy line represents the fluctuation propagator, and the shaded areas are cooperon vertex corrections. At the lowest order of the tunneling in the fluctuation propagator, the DOS contribution involves two electrons forming a fluctuating Cooper pair inside one given grain, contrary to what happens for MT or AL diagrams in diagram 1 of Fig. 1 and the diagrams of Fig. 2.[12,28] It means that the DOS correction is the only contribution which is present even in the absence of tunneling. Therefore, in temperature regions far from the critical temperature, this term gives a significant contribution to thermal conductivity stressing the granular nature of the sample. The contribution to thermal conductivity can be evaluated by means of Eq. (7). The total thermal conductivity κ can be written as[12,27]

$$\kappa=\kappa_{\mathrm{DOS}}+\kappa_{\mathrm{AL}}+\kappa_{\mathrm{MT}}. \tag{39}$$

In the following sections we consider the contributions to the fluctuation thermal conductivity that arise from diagrams of Fig. 1.

6. Density of state correction to thermal conductivity

From diagram 2 of Fig. 1, the corresponding response function operator for DOS correction can be written as

$$Q^{(\mathrm{Dos})}\left(\omega_\nu\right)=T^2t^2a^2\sum_j\sum_{\Omega_\mu}\mathcal{L}_{ij}(\Omega_\mu)\Sigma(\Omega_\mu,\omega_\nu), \tag{40}$$

where

$$\Sigma(\Omega_\mu\omega_\nu)=\sum_{\varepsilon_n}\left(\varepsilon_n+\varepsilon_{n+\nu}\right)^2$$

$$\times\Lambda^2\left(\varepsilon_{n+\nu},\varepsilon_{-n-\nu-\mu}\right)I(\varepsilon_n,\Omega_\mu,\omega_\nu), \tag{41}$$

with

$$I\left(\varepsilon_n,\Omega_\mu,\omega_\nu\right)=\int\left(\mathrm{d}\boldsymbol{p}\right)\Im^2\left(\boldsymbol{p},\varepsilon_{n+\nu}\right)\Im\left(\boldsymbol{p},\varepsilon_{-n-\nu+\mu}\right)\times\int\left(\mathrm{d}\boldsymbol{p}'\right)\Im\left(\boldsymbol{p}',\varepsilon_n\right). \tag{42}$$

In Eq. (40), $\mathcal{L}_{ij}(\Omega_\mu)$ is the Fourier transform of the grain lattice of the propagator in Eq. (37), which can be written as

$$\mathcal{L}_{ij}(\Omega_\mu)=\sum_{\boldsymbol{K}}\mathrm{e}^{\mathrm{i}\boldsymbol{R}_{ij}\cdot\boldsymbol{K}}\mathcal{L}_{\boldsymbol{K}}(\Omega_\mu), \tag{43}$$

where $\boldsymbol{R}_{ij}$ is the vector between the two sites. For DOS diagram, $\boldsymbol{R}_{ij}=\boldsymbol{R}_{ii}$. The main contribution to the singular behavior comes from classical frequencies, $\left|\Omega_\mu\right|\ll T_{\mathrm{c}}$; therefore the so-called static limit $\Omega_\mu=0$ will be used in the calculation of correction. This will be true also for the Maki–Thompson correction. In this limit, the product of integrals in Eq. (42) can be evaluated by means of the contour integration

$$I\left(\varepsilon_n,o,\omega_\nu\right)=-2\left(\pi\nu_{\mathrm{F}}\tau\right)^2\left[\Theta\left(\varepsilon_n\varepsilon_{n+\nu}\right)-\Theta\left(-\varepsilon_n\varepsilon_{n+\nu}\right)\right], \tag{44}$$

where $\Theta(x)$ is the step function. Substituting Eq. (44) into Eq. (41) and with the sum being over the electronic Matsubara frequencies, the only linear contribution in ω_ν is given by

$$\Sigma(0,\omega_\nu)\simeq-\omega_\nu\pi\upsilon_{\mathrm{F}}^2. \tag{45}$$

From Eqs. (38) and (44), the DOS response function can be evaluated in the form

$$Q^{(\mathrm{Dos})}\left(\omega_\nu\right)=\left(\omega_\nu\right)\frac{8}{\pi}g_{\mathrm{T}}Ta^2\sum_{ij}\mathcal{L}_{ij}(0). \tag{46}$$

From Eq. (7), the DOS correction to thermal conductivity can be written as

$$\frac{\delta\kappa^{(\mathrm{Dos})}}{k_0}=-\frac{3}{\pi^2}\frac{1}{g_{\mathrm{T}}}\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}\left\langle\frac{1}{\chi_{\boldsymbol{K}}}\right\rangle K, \tag{47}$$

where $\langle\cdots\rangle\equiv V\int_0^{2\pi/d}(\ldots)(a^n/(2\pi)^n)\mathrm{d}^n\boldsymbol{K}$ and n is the system dimension. The lattice Fourier transform was used for defining the reduced temperature $\varepsilon=\ln(\frac{T}{T_{\mathrm{c}}})\simeq\frac{T-T_{\mathrm{c}}}{T_{\mathrm{c}}}$. Close to T_{c}, the integral in Eq. (47) obtains its main contribution from the small momentum region, and we recover the bulk behavior in the form

$$\delta\kappa^{(\mathrm{Dos})}\propto\frac{1}{g_{\mathrm{T}}}\begin{cases}\sqrt{\varepsilon}, & n=3;\\ \ln\dfrac{1}{\varepsilon}, & n=2;\\ \dfrac{1}{\sqrt{\varepsilon}}, & n=1.\end{cases} \tag{48}$$

7. Maki–Thompson correction to thermal conductivity

The MT diagram is shown in diagram 2 of Fig. 1. With respect to the DOS diagram, there is an important difference. In the case of the DOS diagram, the bubble represents the propagation of a particle and its corresponding hole, and the tunneling coefficients for vertices are t_{ij} and $t_{ij}^{*} = t_{ji} = -t_{ij}$. In MT diagrams, one has two incoming particles and the tunneling coefficients are the same. Besides, the electrons entering the diagram from the opposite side contribute with opposite sign energies. The linear response operator for the MT correction can be written as

$$Q^{(\mathrm{MT})}(\omega_\nu) = T^2 t^2 a^2 \sum_{i,j} \sum_{\Omega_\mu} \mathcal{L}_{ij}(\Omega_\mu) \Sigma(\Omega_\mu, \omega_\nu), \tag{49}$$

where

$$\Sigma\left(\Omega_\mu, \omega_\nu\right) = \sum_{\varepsilon_n} (\varepsilon_n + \varepsilon_{n+\nu})^2 \Lambda\left(\varepsilon_{n+\nu}, \varepsilon_{-n-\nu+\mu}\right) \times \Lambda\left(\varepsilon_n, \varepsilon_{-n+\mu}\right) I\left(\varepsilon_n, \Omega_\mu, \omega_\nu\right), \tag{50}$$

with

$$\left(\varepsilon_n, \Omega_\mu, \omega_\nu\right) = \int (\mathrm{d}\boldsymbol{p}) \Im\left(\boldsymbol{p}, \varepsilon_{n+\nu}\right) \Im\left(\boldsymbol{p}, \varepsilon_{-n-\nu+\mu}\right) \times \int \left(\mathrm{d}\boldsymbol{p}'\right) \Im\left(\boldsymbol{p}', \varepsilon_n\right) \Im\left(\boldsymbol{p}', \varepsilon_{-n+\mu}\right). \tag{51}$$

In the static limit, the response function in Eq. (49) can be written as

$$Q^{(\mathrm{MT})}(\omega_\nu) = \mathrm{i} 2\pi \left(a t \nu_{\mathrm{F}}\right)^2 \omega T \sum_{i,j} \mathcal{L}_{ij}(\Omega_\mu = 0). \tag{52}$$

For MT $\boldsymbol{R}_{ij} = \boldsymbol{a}$, with $\boldsymbol{a}$ being the distance between the two sites. From the linear response operator in Eq. (49), the MT correction can be written as

$$\frac{\delta \kappa^{(\mathrm{MT})}}{\kappa_0} = -\frac{3}{\pi^2} \frac{1}{g_{\mathrm{T}}} \frac{g_{\mathrm{T}} \delta}{T_{\mathrm{c}}} \left\langle \frac{\mathcal{F}_{\boldsymbol{K}}}{\chi_{\boldsymbol{K}}} \right\rangle_K. \tag{53}$$

As expected from the bulk behavior, the MT correction has the same singular behavior as the DOS but with an opposite sign. Moreover, because such a correction involves the coherent tunneling of the fluctuating Cooper pair from one site to the nearest neighbors, it is proportional to the lattice structure factor $\mathcal{F}_{\boldsymbol{K}}$. Due to this proportionality in the far from critical temperature T_{c}, the correction vanishes because $\langle \mathcal{F}_{\boldsymbol{K}} \rangle_K = 0$.

8. Aslamazov–Larkin correction to thermal conductivity

By considering all their possible combinations in pairs, the AL diagrams for a granular system can be built up by means of regions in Fig. 2. For the sake of simplicity, one can call the term first region in Fig. 2(a) as β_1, and the second region as β_2.

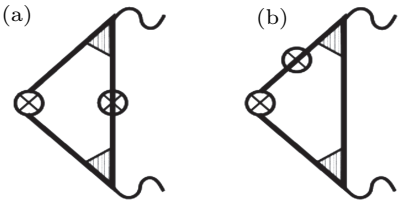


Fig. 2. Diagrams of the regions in the Aslamazov–Larkin contribution to the thermal conductivity. By taking into account all their possible combinations in pairs, the AL diagrams for a granular system can be constructed by means of the shown regions. Diagram (b) has a double multiplicity since the bare tunnel vertex can stay on both sides of the region.

The general expression of the linear response operator is

$$Q^{(\mathrm{AL})}(\omega_\nu) = T t^4 a^2 \sum_{\substack{l,i \\ j,m}} \sum_{\Omega_\mu} \mathcal{L}_{ij}(\Omega_{\mu+\nu}) \mathcal{L}_{ml}(\Omega_\mu) \times \beta_{\mathrm{left}}(\omega_\nu, \Omega_\mu) \beta_{\mathrm{right}}(\omega_\nu, \Omega_\mu), \tag{54}$$

where β_{left} and β_{right} can be either β_1 or β_2-type. The β_1-type region is

$$\beta_1\left(\omega_\nu, \Omega_\mu\right) = T \sum_{\varepsilon_n} (\varepsilon_n + \varepsilon_{n+\nu}) \Lambda\left(\varepsilon_{n+\nu}, \varepsilon_{\mu-n}\right) \times \Lambda\left(\varepsilon_n, \varepsilon_{\mu-n}\right) I^1\left(\varepsilon_n, \Omega_\mu, \omega_\nu\right), \tag{55}$$

where

$$I^1\left(\varepsilon_n, \Omega_\mu, \omega_\nu\right) = \int (\mathrm{d}\boldsymbol{p}) \Im\left(\boldsymbol{p}, \varepsilon_{n+\nu}\right) \Im\left(-\boldsymbol{p}, \varepsilon_{\mu-n}\right) \times \int \left(\mathrm{d}\boldsymbol{p}'\right) \Im\left(-\boldsymbol{p}', \varepsilon_{\mu-n}\right) \Im\left(\boldsymbol{p}', \varepsilon_n\right) = I_1^1 I_2^1. \tag{56}$$

Because of the pole structure of the fluctuation propagator in Eq. (37), one can ignore the ω_ν dependence and keep just the one in Ω_μ. The product in Eq. (49) has the form

$$I_1^1 I_2^1 = (2\pi \nu_{\mathrm{F}} \tau)^2 \left\{ \Theta\left(\Omega_\mu\right) \left[\Theta(\varepsilon_n - \Omega_\mu) + \Theta(-\varepsilon_n)\right] + \Theta\left(-\Omega_\mu\right) \left[\Theta(\varepsilon_n) + \Theta(-\varepsilon_n + \Omega_\mu)\right] \right\}, \tag{57}$$

where $\Theta(x)$ is the step function. The β_1-type region can be written as

$$\beta_1\left(0, \Omega_\mu\right) = 8 \left(\pi \nu_{\mathrm{F}}\right)^2 T \Omega_\mu \frac{1}{(4\pi T)^2} \psi'\left(\frac{1}{2} + \frac{\left|\Omega_\mu\right|}{4\pi T}\right) = \frac{\pi^2}{4} \frac{\nu_{\mathrm{F}}^2}{T} \Omega_\mu, \tag{58}$$

where ψ is the digamma function. The β_2-type region has the form

$$\beta_2(\omega_\nu, \Omega_\mu) = T \sum_{\varepsilon_n} (\varepsilon_n + \varepsilon_{n+\nu}) \Lambda\left(\varepsilon_{n+\nu}, \varepsilon_{\mu-n}\right) \times \Lambda\left(\varepsilon_n, \varepsilon_{\mu-n}\right) I^{\mathrm{r}}(\varepsilon_n, \Omega_\mu, \omega_\nu), \tag{59}$$

where

$$I\left(\varepsilon_n, \Omega_\mu, \omega_\nu\right) = \int (\mathrm{d}\boldsymbol{p}) \Im\left(\boldsymbol{p}, \varepsilon_{n+\nu}\right)$$

$$\times \int \left(\mathrm{d}\boldsymbol{p}'\right) \Im\left(\boldsymbol{p}', \varepsilon_{\mu-n}\right) \Im\left(\boldsymbol{p}', \varepsilon_n\right) \Im\left(\boldsymbol{p}', \varepsilon_{n+\nu}\right) = I_1^{\mathrm{r}} I_2^{\mathrm{r}}, \tag{60}$$

where the first integral can be written as

$$I_1^{\mathrm{r}} = -\mathrm{i}\pi\nu_{\mathrm{F}}\mathrm{sgn}(\varepsilon_n). \tag{61}$$

For I_2^{r}, one has to consider all the possible signs of Ω_μ and ε_n. The only non-vanishing contributions are

$$I_2^{\mathrm{r}} = \mathrm{i}2\pi\nu_{\mathrm{F}}\tau^2\left\{\Theta\left(-\Omega_\mu\right)\left[\Theta(\Omega_\mu-\varepsilon_n)-\Theta(\varepsilon_n)\right] + \Theta\left(\Omega_\mu\right)\left[\Theta(-\varepsilon_n)-\Theta(\varepsilon_n-\Omega_\mu)\right]\right\}. \tag{62}$$

From Eqs. (60)–(62), the expression of the region in Eq. (59) can be written as

$$\beta_2\left(0,\Omega_\mu\right) = -4\left(\pi\nu_{\mathrm{F}}\right)^2 T\Omega_\mu \frac{1}{\left(4\pi T\right)^2}\psi'\left(\frac{1}{2}+\frac{\left|\Omega_\mu\right|}{4\pi T}\right) = -\frac{\pi^2}{8}\frac{\nu_{\mathrm{F}}^2}{T}\Omega_\mu. \tag{63}$$

One can find the important relation between the two types of regions

$$\beta_1\left(0,\Omega_\mu\right) = -2\beta_2\left(0,\Omega_\mu\right). \tag{64}$$

The total AL correction can be written as

$$Q^{(\mathrm{AL})}\left(\omega_\nu\right) = a^2 g_{\mathrm{T}}\frac{1}{T}\sum_{\boldsymbol{K}}\sum_{\Omega_\mu}(1-\mathcal{F}_{\boldsymbol{K}})^2 \times \mathcal{L}_{\boldsymbol{K}}(\Omega_\mu+\omega_\nu)\mathcal{L}_{\boldsymbol{K}}(\Omega_\mu)\Omega_\mu^2. \tag{65}$$

In Eq. (65), the sum over Ω_μ reads

$$T\sum_{\Omega_\mu}\mathcal{L}_{\boldsymbol{K}}(\Omega_\mu+\omega_\nu)\mathcal{L}_{\boldsymbol{K}}(\Omega_\mu)\Omega_\mu^2 = \frac{12T_{\mathrm{c}}}{\nu_{\mathrm{F}}^2\chi_{\boldsymbol{K}}}. \tag{66}$$

From Eqs. (65) and (66), the AL correction to thermal conductivity leads to

$$\frac{\delta\kappa^{(\mathrm{AL})}}{\kappa_0} = \frac{2}{2\pi}\frac{1}{g_{\mathrm{T}}}\left(\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}\right)^2\left\langle\frac{(1-\mathcal{F}_{\boldsymbol{K}})^2}{\chi_{\boldsymbol{K}}}\right\rangle_K, \tag{67}$$

which is the first non-vanishing correction due to AL channel. Such a correction is always positive. As in the MT, this correction depends on the lattice structure factor $\mathcal{F}_{\boldsymbol{K}}$, but it does not vanish in the regime far from T_{c}. This is a good feature of the system since the dynamical contribution plays an important role when it is far from T_{c} and in this region, one has to compare such a correction with that of the DOS contribution. From Eqs. (47), (53), and (67), the total superconducting fluctuation correction to the thermal conductivity close to critical temperature can be written as

$$\frac{\delta\kappa}{\kappa_0} = \frac{3}{\pi^2}\frac{1}{g_{\mathrm{T}}}\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}} \times\left\langle\frac{(1-\mathcal{F}_{\boldsymbol{K}})\left[\frac{3\pi}{2}\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}((1-\mathcal{F}_{\boldsymbol{K}})-1)\right]}{\chi_{\boldsymbol{K}}}\right\rangle_K. \tag{68}$$

This correction has been obtained at every order in the tunneling amplitude in the ladder approximation. Its behavior is plotted in Fig. 3 as a function of the reduced temperature for the case of a two-dimensional sample and for different values of the ratio $\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}$.

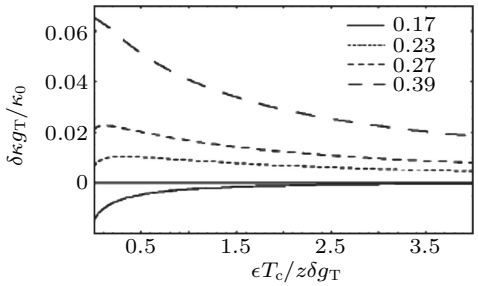


Fig. 3. Total fluctuation correction to the thermal conductivity for different values of $\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}$.

Two different regimes of temperature can be recognized: far from the critical temperature $\varepsilon \gg \frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}$ and close to the critical temperature $\varepsilon \ll \frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}$. The condition $\varepsilon \gg \frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}$ is equivalent to the condition $\tau_{\mathrm{GL}} \ll \tau_{\mathrm{dwell}} = \left(g_{\mathrm{T}}\delta\right)^{-1}$. In other words, the lifetime of a Cooper pair is shorter than the time the electrons spend in the grain before tunneling. For this reason, the tunneling is not efficient and the system behaves as an ensemble of zero-dimensional grains. As a consequence, only the DOS and the AL terms contribute significantly to the superconducting fluctuations. From Eq. (39), the correction to thermal conductivity can be written as

$$\frac{\delta\kappa}{\kappa_0} \simeq \frac{3}{\pi^2}\frac{1}{g_{\mathrm{T}}}\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}\frac{1}{\varepsilon}\left[\frac{3\pi}{2}\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}\left(1+\frac{1}{N}\right)-1\right]. \tag{69}$$

Substituting Eqs. (12) and (69) into Eq. (1), it can be demonstrated that there is a positive violation of WF law

$$\frac{\delta L}{L_0} = \frac{\delta\kappa}{\kappa_0}-\frac{\delta\sigma}{\sigma_0} \simeq \left[-\frac{3}{\pi^2}+\frac{9}{2\pi}\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}\frac{N+1}{N}+\frac{7\xi(3)}{\pi^2}\right]\frac{\delta}{T_{\mathrm{c}}}\frac{1}{\varepsilon}. \tag{70}$$

Close to $T_{\mathrm{c}}, \varepsilon \ll \frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}$, the correction to thermal conductivity is

$$\frac{\delta\kappa(\varepsilon=0)}{k_0} = \frac{3}{N\pi^2}\frac{1}{g_{\mathrm{T}}}\left(\frac{3\pi}{2}\frac{g_{\mathrm{T}}\delta}{T_{\mathrm{c}}}-1\right). \tag{71}$$

From Eqs. (12) and (71) one learns that the deviation from the WF law is much more evident in the near T_{c} region than in the far T_{c} region because of the pronounced singular behavior of the electrical conductivity which is in turn due to the increasing number of Cooper pairs close to the critical temperature.

9. Conclusion

The fluctuations' corrections of superconductivity are analytically studied by Feynman diagrams. This study is restricted to the limit of large tunneling conductance in a way that the weak localization and charging effects have been ignored in the limit. The Lorenz number has been obtained for a granular s-wave superconductor in nanometer-scale near and far from the critical temperature. It has been demonstrated that far from the critical temperature there is a positive violation of the WF law, and near to the critical temperature the deviation from the WF law is much more evident than far from the critical temperature. The relations of these corrections are derived in different temperatures for various tunneling intensities for the first time. It has been shown that the Lorenz number and therefore WF law will be violated. The theory developed gives a good description of existing experiments. In the experimental systems close to the metal–insulator–transition, the localization effects as well as Coulomb interaction can play an essential role. Although in our theory that all these effects are ignored gives reasonable values of physical quantities, it allows us to reproduce the main features of experimental curves.

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