Adaptive fuzzy synchronization for a class of fractional-order neural networks*

Heng Liu(刘恒)[1,2], Sheng-Gang Li(李生刚)[1,†], Hong-Xing Wang(王宏兴)[2], and Guan-Jun Li(李冠军)[2]

[1] College of Mathematics and Information Science, Shaanxi Normal Universtiy, Xi'an 710119, China
[2] Department of Applied Mathematics, Huainan Normal University, Huainan 232038, China

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In this paper, synchronization for a class of uncertain fractional-order neural networks with external disturbances is discussed by means of adaptive fuzzy control. Fuzzy logic systems, whose inputs are chosen as synchronization errors, are employed to approximate the unknown nonlinear functions. Based on the fractional Lyapunov stability criterion, an adaptive fuzzy synchronization controller is designed, and the stability of the closed-loop system, the convergence of the synchronization error, as well as the boundedness of all signals involved can be guaranteed. To update the fuzzy parameters, fractional-order adaptations laws are proposed. Just like the stability analysis in integer-order systems, a quadratic Lyapunov function is used in this paper. Finally, simulation examples are given to show the effectiveness of the proposed method.

Keywords: fractional-order neural network, adaptive fuzzy control, fractional-order adaptation law

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1. Introduction

Fractional calculus is a concept for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration.[1] This idea is around 300 years old, but compared with the corresponding integer-order theory, the developments in this field have been rather slow. But lately, there has been a steady growth of interest in this field because lots of actual systems show fractional-order differential dynamical behaviors.[2–8] Applications of fractional-order differential equations to different areas are investigated by many researchers, and some basic results have been obtained (see, for example, Refs. [2], [3], [6], [7], and [9] and the references therein). One can refer to Refs. [1] and [10] for more details about the fractional calculus theory. In a manner of speaking, dynamics analysis, stability analysis, and the control of fractional-order systems have become very hot and promising research topics.

One main advantage of fractional-order systems consists in that the fractional-order differential operator takes into consideration the fact that the future system states depend on both the present states and all the previous states. That is to say, the fractional-order systems are nonlocal while the integer-order ones are local. Noting that fractional calculus provides a powerful tool for describing memory and hereditary properties of the systems, some researchers have put it into integer-order neural networks to achieve fractional-order neural models, which may better describe the dynamical behavior of the neurons.[2,11–18] Up to now, various significant issues about fractional-order neural networks have been discussed. In Ref. [19], by using numerical simulations, the complex dynamics of a class of fractional-order neural networks have been addressed. In Ref. [20], the dynamics of a general class of fractional-order neural networks were investigated by means of the Krasnoselskii fixed point theorem and the contraction mapping principle. In Refs. [12] and [13], based on the Lyapunov stability criterion and linear control strategies, some effective criteria concerning stability analysis of fractional-order neural networks and memristor-based fractional-order neural networks were proposed, respectively. Besides, more and more researchers have focused on the synchronization of fractional-order neural networks, and many results have been reported (see, Refs. [13]–[18] and [21]–[27]). It should be stressed that in the aforementioned literatures considering the synchronization of fractional-order neural networks, prior knowledge of the system models is needed. However, as we know, system uncertainties and external disturbances usually exist in actual systems,[28–34] so it is advisable to design a synchronization controller for fractional-order neural networks when the system models are fully unknown.

It is well known that fuzzy systems with linguistic information or heuristic knowledge have been successfully used to control uncertain systems because the accurate mathematical models of the systems are not needed.[34–43] Based on

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†Corresponding author. E-mail: shengganglinew@126.com

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the universal approximation theorem,[44] many adaptive fuzzy control methods have been developed for uncertain integer-order nonlinear systems.[35,36,45–49] However, with respect to fractional-order systems, one can hardly check the stability of the systems by looking for the dominant roots or via algebraic methods, which are usually utilized to analyze the stability of integer-order systems. That is to say, most of the control methods for integer-order systems cannot be extended to fractional-order systems directly. To the best of our knowledge, synchronization of fractional-order neural networks by using adaptive fuzzy control is rarely reported and needs to be investigated further.

Motivated by the above discussion, we will investigate the synchronization problem of a class of fractional-order neural networks. Roughly stated, the main contributions of this paper can be summarized as follows.

Synchronization for a class of fractional-order neural networks with fully unknown system models is discussed by using adaptive fuzzy control.

The fuzzy logic systems use synchronization errors instead of system states as their inputs. Thus, the control systems will be more sensitive to the synchronization errors.

Fractional-order adaptations laws are designed to update the fuzzy system parameters online.

The remainder of this paper is organized as follows. Section 2 introduces the fuzzy logic systems and lists some preliminaries. In Section 3, an adaptive fuzzy synchronization controller is designed and the stability of the closed-loop system is discussed by using fractional Lyapunov stability criterion. Simulation studies are presented in Section 4. Finally, Section 5 gives the conclusion of this work.

2. Preliminaries

2.1. Some basic results of fractional calculus

The fractional-order integro-differential operator can be seen as an extended concept of the integer-order one. The commonly used definitions in the literature are Grünwald–Letnikov, Riemann–Liouville, and Caputo definitions. The main reason why Caputo's derivative is introduced for engineering applications is that its Laplace transform only requires integer-order derivatives for the initial conditions. On the contrary, the classical Riemann–Liouville definition needs fractional-order initial conditions which are difficult to be physically interpreted. We will use the Caputo derivative in this paper. The fractional-order integral with fractional order α can be defined as

$$ {}_0I_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \tag{1} $$

where Γ represents the Euler Gamma function. The Caputo fractional derivative is expressed by

$$ {}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau, \tag{2} $$

where α is the fractional order, and n is an integer satisfying $n-1 \le \alpha < n$.

The Laplace transform of the Caputo fractional derivative is

$$ \int_0^{\infty} \mathrm{e}^{-st} {}_0^C D_t^{\alpha} f(t) \mathrm{d}t = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \tag{3} $$

where $F(s)$ is the Laplace transform of $f(t)$, $F(s) = \mathscr{L}\{f(t)\}$. For convenience, we always assume $\alpha \in (0,1)$ in the rest of this paper. The following definitions and results are needed.

Definition 1[1] The Mittag–Leffler function with two parameters can be written as

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{4} $$

where $\alpha, \beta > 0$, and $z \in C$. Note that $E_{1,1}(z) = \mathrm{e}^z$.

The Laplace transform of the Mittag–Leffler function is[1]

$$ \mathscr{L}\{t^{\beta-1} E_{\alpha,\beta}(-at^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha}+a}. \tag{5} $$

Lemma 1[1] Suppose that $x(t) \in C^1[0,T]$, where T is a positive constant, then the following two equations hold:

$$ {}_0I_t^{-\alpha} {}_0^C D_t^{\alpha} x(t) = x(t) - x(0), \tag{6} $$

$$ {}_0^C D_t^{\alpha} {}_0I_t^{-\alpha} x(t) = x(t). \tag{7} $$

Lemma 2[1] For a complex number β and two real numbers α, μ satisfying $0 < \alpha < 2$,

$$ \frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\} \tag{8} $$

the following equality holds for all the integer $n \ge 1$:

$$ E_{\alpha,\beta}(z) = -\sum_{j=1}^{n} \frac{1}{\Gamma(\beta - \alpha j) z^j} + o\left(\frac{1}{|z|^{n+1}}\right), $$
$$ |z| \to \infty, \quad \mu \le |\arg(z)| \le \pi. \tag{9} $$

Lemma 3[1] Let $0 < \alpha < 2$, and β is an arbitrary real number. If there exists a positive constant μ such that $\pi\alpha/2 < \mu \le \min\{\pi, \pi\alpha\}$, then we have

$$ |E_{\alpha,\beta}(z)| \le \frac{l_0}{1+|z|}, \tag{10} $$

where l_0 is a positive real constant, $\mu \le |\arg(z)| \le \pi$, and $|z| \ge 0$.

Lemma 4 (Fractional-order extension of Lyapunov direct method[50]) Let $\boldsymbol{x}(t) = 0$ be an equilibrium point of the non autonomous fractional-order nonlinear system

$$ {}_0^C D_t^{\alpha} \boldsymbol{x}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)). \tag{11} $$

If there exists a Lyapunov function $V(t, \boldsymbol{x}(t))$ and class-K functions $g_i, i = 1,2,3$ such that

$$ g_1(\|\boldsymbol{x}(t)\|) \le V(t, \boldsymbol{x}(t)) \le g_2(\|\boldsymbol{x}(t)\|), \tag{12} $$

$$ {}_0^C D_t^\alpha V(t,\boldsymbol{x}(t)) \le -g_3(\|\boldsymbol{x}(t)\|), \tag{13} $$

then system (11) is asymptotically stable.

Lemma 5[4] Let $\boldsymbol{x}(t) \in \mathcal{R}^n$ be a continuous and derivable function. Then, for any $t > 0$,

$$ \frac{1}{2} {}_0^C D_t^\alpha \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{x}(t) \le \boldsymbol{x}^{\mathrm{T}}(t) {}_0^C D_t^\alpha \boldsymbol{x}(t). \tag{14} $$

Lemma 6[9] Let $\boldsymbol{x}(t) \in \mathcal{R}^n$ be a continuous and derivable function. Then, for any $t > 0$,

$$ \frac{1}{2} {}_0^C D_t^\alpha \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{B}\boldsymbol{x}(t) \le \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{B} {}_0^C D_t^\alpha \boldsymbol{x}(t), \tag{15} $$

where $\boldsymbol{B} \in \mathcal{R}^{n\times n}$ is a positive definite constant matrix.

Lemma 7 (Fractional monotonicity principle) If ${}_0^C D_t^\alpha y(t) \le 0$, then we have $y(t) \le y(0)$ for all $t > 0$, and furthermore, function $y(t)$ is monotone decreasing.

Proof There exists a nonnegative function $h(t)$ such that

$$ {}_0^C D_t^\alpha y(t) + h(t) = 0. \tag{16} $$

Using the Laplace transform to Eq. (16), we have

$$ Y(s) = \frac{y(0)}{s} - \frac{H(s)}{s^\alpha}, \tag{17} $$

where $Y(s)$ and $H(s)$ are the Laplace transforms of $y(t)$ and $h(t)$, respectively. Taking the inverse Laplace transform on Eq. (17) yields

$$ y(t) = y(0) - {}_0^C D_t^{-\alpha} h(t). \tag{18} $$

It follows from Eq. (18) that $y(t) \le y(0)$ for all $t > 0$, and thus the function $y(t)$ is monotone decreasing.

2.2. Description of a fuzzy system

A fuzzy logic system consists of four parts:[35,36,38–40,51] the knowledge base, the fuzzifier, the fuzzy inference engine working on the fuzzy rules, and the defuzzifier. The structure of a fuzzy logic system is shown in Fig. 1.

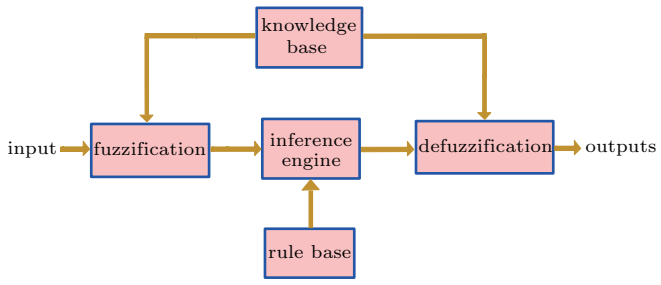


Fig. 1. (color online) The structure of a fuzzy logic system.

Usually, a fuzzy logic system is modeled by

$$ \hat{f}(\boldsymbol{x}(t)) = \frac{\sum_{j\in J} \theta_j(t)\mu_j(\boldsymbol{x}(t))}{\sum_{j\in J} \mu_j(\boldsymbol{x}(t))}, \tag{19} $$

where $\hat{f}$ (a Lipschitz continuous mapping from a compact subset $\boldsymbol{\Omega} \subseteq \mathcal{R}^n$ to the real line $\mathcal{R}$) is called the output of the fuzzy logic system, $\boldsymbol{x} = [x_1,\ldots,x_n]^{\mathrm{T}} \in C^1[\mathcal{I},\Omega]$ (the set of all continuous mappings from $\mathcal{I} = [0,+\infty) \subseteq \mathcal{R}$ to $\boldsymbol{\Omega}$ which have continuous derivatives) is called the input vector, $J = \prod_{i=1}^{n} \mathcal{F}_i$, $\mathcal{F}_i$ is consisted of N_i fuzzy sets $(1 \le i \le n)$, μ_j (a mapping from $\mathcal{R}^n$ to the closed unit interval $[0,1] \subseteq \mathcal{R}$) is called the membership function of rule j $(j \in J)$ (here $N = N_1 \times N_2 \times \cdots \times N_n$ rules defined as follows are also need to approximate the unknown function $\hat{f}(\boldsymbol{x}(t))$ $(j \in J)$: rule j $(j = (F_1,F_2,\ldots,F_n))$: if $\boldsymbol{x}(t)$ is $(F_1(t),F_2(t),\ldots,F_n(t))$, then $\hat{f}(\boldsymbol{x}(t))$ is $B^j(t))$, and θ_j (a mapping from $\mathcal{I}$ to $\mathcal{R}$) is called the centroid of the j-th consequent set $(j \in J)$; we may identify J with $\{1,2,\ldots,N\}$ for the sake of convenience. Write $\boldsymbol{\theta}(t) = [\theta_1(t),\ldots,\theta_N(t)]^{\mathrm{T}}$ and $\boldsymbol{\psi}(\boldsymbol{x}(t)) = [q_1(x(t)),q_2(x(t)),\ldots,q_N(x(t))]^{\mathrm{T}}$, where q_j (called the j-th fuzzy basis function, $j \in J$) is a continuous mapping (and thus $\psi : \Omega \longrightarrow \mathcal{R}^N$ is continuous) defined by

$$ q_j(x(t)) = \frac{\theta_j(t)}{\sum_{s\in J} \mu_s(\boldsymbol{x}(t))}. $$

Then system (19) can be rewritten as

$$ \hat{f}(\boldsymbol{x}(t)) = \boldsymbol{\theta}^{\mathrm{T}}(t)\boldsymbol{\psi}(\boldsymbol{x}(t)). \tag{20} $$

In contrast to the conventional control techniques, fuzzy logic systems are best utilized in complex ill-defined processes that can be controlled by a skilled human operator without much knowledge of their underlying dynamics. The basic idea behind fuzzy logic systems is to incorporate the "expert experience" of a human operator in the design of the controller in controlling a process whose input–output relationship is described by a collection of fuzzy control rules involving linguistic variables rather than a complicated dynamic model.

3. Adaptive fuzzy synchronization controller design and stability analysis

3.1. Problem statement

In this paper, we will consider the fractional-order neural networks with the form

$$ \begin{aligned} {}_0^C D_t^\alpha x_i(t) &= -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + I_i, \\ & i = 1,2,\ldots,n, \end{aligned} \tag{21} $$

where α is the fractional order, n is the number of units in the neural network, $x_i(t)$ represents the state of the i-th unit at time t, a_{ij} is the constat connection weight of the j-th neuron on the i-th neuron, $f_i(x)$ is a nonlinear function, c_i corresponds to the rate with which the i-th neuron will reset its potential to the resting state when disconnected from the network, and I_i corresponds to the external input.

Let us denote $\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$, $\boldsymbol{C} = \mathrm{diag}(c_1, c_2, \ldots, c_n)$, $\boldsymbol{I} = [I_1, I_2, \ldots, I_n]^{\mathrm{T}}$, $\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \ldots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, then system (21) can be rearranged to the following compact form:

$$ {}_0^C D_t^{\alpha} \boldsymbol{x}(t) = -\boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{I}. \tag{22} $$

To guarantee the existence and uniqueness of the solutions of the fractional-order neural network (22) (see Refs. [9] and [52]), we assume that the functions f are Lipschitz-continuous, i.e., for all $\boldsymbol{x}(t), \boldsymbol{y}(t) \in \mathcal{R}^n$,

$$ \|\boldsymbol{f}(\boldsymbol{x}(t)) - \boldsymbol{f}(\boldsymbol{y}(t))\| \leq l\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|, \tag{23} $$

where l is the Lipschitz constant.

The slave system is given by

$$ {}_0^C D_t^{\alpha} \boldsymbol{y}(t) = -\boldsymbol{C}\boldsymbol{y}(t) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{y}(t)) + \boldsymbol{I} + \boldsymbol{G}\boldsymbol{u}(t) + \boldsymbol{d}(t), \tag{24} $$

where $\boldsymbol{y}(t) \in \mathcal{R}^n$ is the state vector of the slave system, $\boldsymbol{G} \in \mathcal{R}^{n\times n}$ is a positive definite control gain matrix, $\boldsymbol{d}(t) = [d_1(t), d_2(t), \ldots, d_n(t)]^{\mathrm{T}} \in \mathcal{R}^n$ is an unknown external disturbance, and $\boldsymbol{u}(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}} \in \mathcal{R}^n$ is the control input.

The objective of our work is to design a robust controller such that the slave system (24) synchronizes the master system (22). To proceed, the following assumption is needed.

Assumption 1 The external disturbance is bounded, i.e., $\|d_i(t)\| \leq \bar{d}_i, i = 1, 2, \ldots, n$ where $\bar{d}_i$ is an unknown positive constant.

3.2. Controller design and stability analysis

In this subsection, an adaptive fuzzy synchronization controller will be designed. Define the synchronization error as $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t)$. From Eqs. (22) and (24), we have

$$ {}_0^C D_t^{\alpha} \boldsymbol{e}(t) = -\boldsymbol{C}\boldsymbol{e}(t) + \boldsymbol{F}(\boldsymbol{e}(t)) - \boldsymbol{G}\boldsymbol{u}(t) - \boldsymbol{d}(t), \tag{25} $$

where $\boldsymbol{F}(\boldsymbol{e}(t)) = \boldsymbol{A}(\boldsymbol{f}(\boldsymbol{x}(t)) - \boldsymbol{f}(\boldsymbol{y}(t)))$ is a fully unknown nonlinear function.

Remark 1 The nonlinear function $\boldsymbol{F}$ should be written as $\boldsymbol{F}(\boldsymbol{x}(t), \boldsymbol{y}(t), \boldsymbol{e}(t))$ in fact. In this paper, we will approximate this function by using fuzzy logic systems whose inputs are synchronization errors, so the denotation $\boldsymbol{F}(\boldsymbol{e}(t))$ is only for convenience here.

If the nonlinear functions $\boldsymbol{f}(\boldsymbol{x}(t))$, $\boldsymbol{f}(\boldsymbol{y}(t))$ and the external disturbance $\boldsymbol{d}(t)$ are known, then we can design the following ideal controller:

$$ \boldsymbol{u}_{\mathrm{eq}} = \boldsymbol{G}^{-1}\boldsymbol{F}(\boldsymbol{e}(t)) - \boldsymbol{G}^{-1}\boldsymbol{d}(t). \tag{26} $$

Thus, we have the following results.

Theorem 1 If the nonlinear functions $\boldsymbol{f}(\boldsymbol{x}(t)), \boldsymbol{f}(\boldsymbol{y}(t))$ and the external disturbance $\boldsymbol{d}(t)$ are known, and the controller input is chosen as Eq. (26), then the synchronization error will converge to the origin asymptotically and all signals in the closed-loop system will remain bounded.

Proof Substituting the ideal controller (26) into Eq. (25) gives

$$ \boldsymbol{P}\,{}_0^C D_t^{\alpha} \boldsymbol{e}(t) = -\boldsymbol{C}\boldsymbol{e}(t). \tag{27} $$

Let the quadratic Lyapunov function be

$$ V_1(t) = \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t). \tag{28} $$

By Lemma 6, the α-th derivative with respect to time of Eq. (28) is bounded as

$$ \begin{aligned} {}_0^C D_t^{\alpha} V_1(t) &= \frac{1}{2}\,{}_0^C D_t^{\alpha} \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) \leq \boldsymbol{e}^{\mathrm{T}}(t)\,{}_0^C D_t^{\alpha} \boldsymbol{e}(t) \\ &\leq -2c_{\min} V_1(t), \end{aligned} \tag{29} $$

where $c_{\min} = \min\{c_1, c_2, \ldots, c_n\}$. It follows from Lemma 4 and Eq. (29) that the synchronization error will converge to zero asymptotically. By using Lemma 7, we know that all signals will keep bounded. This completes the proof of Theorem 1.

Note that if $\boldsymbol{f}(\boldsymbol{x}(t)), \boldsymbol{f}(\boldsymbol{y}(t))$ and the external disturbance $\boldsymbol{d}(t)$ are fully unknown, the ideal control input (26) cannot be used directly. To solve this problem, let us approximate the unknown nonlinear function $F_i(\boldsymbol{e}(t))$ through the fuzzy logic system (20) as

$$ \hat{F}_i(\boldsymbol{\theta}_i(t), \boldsymbol{e}(t)) = \boldsymbol{\theta}_i^{\mathrm{T}}(t)\boldsymbol{\psi}_i(\boldsymbol{e}(t)), i = 1, 2, \ldots, n, \tag{30} $$

where $\boldsymbol{\psi}_i(\boldsymbol{x}(t))$ is the fuzzy basis function, and $\boldsymbol{\theta}_i(t)$ is an adjustable parameter of the fuzzy systems, which makes $\sup\{\left|F_i(\boldsymbol{e}(t)) - \hat{F}_i(\boldsymbol{\theta}_i^*, \boldsymbol{e}(t))\right| \mid t \in \mathcal{I}\}$ as small as possible ($i = 1, 2, \ldots, n$). For a fixed synchronization error vector, define the parameter estimation error related to $\boldsymbol{\theta}_i$ as

$$ \tilde{\boldsymbol{\theta}}_i(t) = \boldsymbol{\theta}_i(t) - \boldsymbol{\theta}_i^*, \tag{31} $$

and define the fuzzy approximation error related to $\boldsymbol{\theta}_i^*$ as

$$ \varepsilon_i(\boldsymbol{e}(t)) = F_i(\boldsymbol{e}(t)) - \hat{F}_i(\boldsymbol{\theta}_i^*, \boldsymbol{e}(t)). \tag{32} $$

As in Refs. [36] and [44]–[48], we can assume that the fuzzy systems do not violate the universal approximator property. Thus it is rational to give the following assumption.

Assumption 2 The fuzzy approximation error is bounded, i.e., there exists an unknown positive constant $\varepsilon^* > 0$ such that

$$ \sup\{|\varepsilon_i(\boldsymbol{e}(t))| \mid t \in \mathcal{I}, i = 1, 2, \cdots, n\} \leq \varepsilon^*. \tag{33} $$

Remark 2 It should be emphasized that Assumption 2 is not restrictive because the upper bound ε^* is assumed to be unknown. Note that $\varepsilon_i(\boldsymbol{e}(t))$ is a continuous function, it is

bounded on any finite interval. Therefore, such a constant ε^* always exists.

According to the above discussion, one obtains

$$\begin{aligned}
&\hat{F}_i(\boldsymbol{\theta}_i(t),\boldsymbol{e}(t))-F_i(\boldsymbol{e}(t))\\
=\ &\hat{F}_i(\boldsymbol{\theta}_i(t),\boldsymbol{e}(t))-\hat{F}_i(\boldsymbol{\theta}_i^*,\boldsymbol{e}(t))+\hat{F}_i(\boldsymbol{\theta}_i^*,\boldsymbol{e}(t))-F_i(\boldsymbol{e}(t))\\
=\ &\hat{F}_i(\boldsymbol{\theta}_i(t),\boldsymbol{e}(t))-\hat{F}_i(\boldsymbol{\theta}_i^*,\boldsymbol{e}(t))-\varepsilon_i(\boldsymbol{e}(t))\\
=\ &\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t)\boldsymbol{\psi}_i(\boldsymbol{e}(t))-\varepsilon_i(\boldsymbol{e}(t)). \qquad (34)
\end{aligned}$$

Denote

$$\begin{aligned}
&\boldsymbol{\theta}(t)\boldsymbol{\psi}(\boldsymbol{e}(t))\\
=\ &[\boldsymbol{\theta}_1^{\mathrm{T}}(t)\boldsymbol{\psi}_1(\boldsymbol{e}(t)),\boldsymbol{\theta}_2^{\mathrm{T}}(t)\boldsymbol{\psi}_2(\boldsymbol{e}(t)),\ldots,\boldsymbol{\theta}_n^{\mathrm{T}}(t)\boldsymbol{\psi}_n(\boldsymbol{e}(t))]^{\mathrm{T}}
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\boldsymbol{\theta}}(t)\boldsymbol{\psi}(\boldsymbol{e}(t))\\
=\ &[\tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}(t)\boldsymbol{\psi}_1(\boldsymbol{e}(t)),\tilde{\boldsymbol{\theta}}_2^{\mathrm{T}}(t)\boldsymbol{\psi}_2(\boldsymbol{e}(t)),\ldots,\tilde{\boldsymbol{\theta}}_n^{\mathrm{T}}(t)\boldsymbol{\psi}_n(\boldsymbol{e}(t))]^{\mathrm{T}},
\end{aligned}$$

then equation (34) can be rewritten as

$$\hat{\boldsymbol{F}}(\boldsymbol{\theta}(t),\boldsymbol{e}(t))-\boldsymbol{F}(\boldsymbol{e}(t))=\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\boldsymbol{\psi}(\boldsymbol{e}(t))-\boldsymbol{\varepsilon}(\boldsymbol{e}(t)), \quad (35)$$

where $\boldsymbol{\theta}(t)=[\boldsymbol{\theta}_1^{\mathrm{T}}(t),\boldsymbol{\theta}_2^{\mathrm{T}}(t),\ldots,\boldsymbol{\theta}_n^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathcal{R}^{N\times n}$ (N is the number of fuzzy rules).

Based on the above discussion, now we are ready to give the following results.

Theorem 2 Suppose that Assumptions 1 and 2 are satisfied. Let $\hat{\bar{d}}_i(t)$ and $\hat{\varepsilon}^*(t)$ be the estimations of the unknown constants $\bar{d}_i$ and ε^*, respectively. If the synchronization controller $\boldsymbol{u}(t)$ is designed as

$$\begin{aligned}
\boldsymbol{u}(t)\ =\ &\boldsymbol{G}^{-1}[\boldsymbol{\theta}^{\mathrm{T}}(t)\boldsymbol{\psi}(\boldsymbol{e}(t))+\hat{\varepsilon}^*(t)\mathrm{sign}(\boldsymbol{e}(t))\\
&+\mathrm{diag}(\hat{\bar{\boldsymbol{d}}}(t))\mathrm{sign}(\boldsymbol{e}(t))+\boldsymbol{K}\boldsymbol{e}(t)], \qquad (36)
\end{aligned}$$

where

$$\hat{\bar{\boldsymbol{d}}}(t)=[\hat{\bar{d}}_1(t),\hat{\bar{d}}_2(t),\ldots,\hat{\bar{d}}_n(t)]^{\mathrm{T}}$$

and $\boldsymbol{K}=\mathrm{diag}(k_1,k_2,\ldots,k_n)$ is a positive definite gain matrix to be chosen, and the control system parameters $\boldsymbol{\theta}_i(t)$, $\hat{\varepsilon}^*(t)$, and $\hat{\bar{d}}_i(t)$ are respectively updated by

$$\begin{aligned}
&{}_0^C D_t^\alpha\boldsymbol{\theta}_i(t)=\gamma_{1i}e_i(t)\boldsymbol{\psi}_i(e(t))-\gamma_{1i}\gamma_{2i}\boldsymbol{\theta}_i(t), \qquad (37)\\
&{}_0^C D_t^\alpha\hat{\varepsilon}^*(t)=\gamma_3\sum_{i=1}^n|e_i(t)|-\gamma_3\gamma_4\hat{\varepsilon}^*(t), \qquad (38)\\
&{}_0^C D_t^\alpha\hat{\bar{d}}_i(t)=\gamma_{5i}|e_i(t)|-\gamma_{5i}\gamma_{6i}\hat{\bar{d}}_i(t), \qquad (39)
\end{aligned}$$

where γ_{1i}, γ_{2i}, γ_3, γ_4, γ_{5i}, and γ_{6i} are positive design parameters, then the synchronization will converge to an arbitrary small region of the origin if appropriate design parameters are chosen.

Proof Substituting the synchronization controller (36) into Eq. (25) and using Eq. (35), we have

$$\begin{aligned}
{}_0^C D_t^\alpha\boldsymbol{e}(t)\ =\ &-\boldsymbol{C}\boldsymbol{e}(t)+\boldsymbol{F}(\boldsymbol{e}(t))-\boldsymbol{G}\boldsymbol{u}(t)-\boldsymbol{d}(t),\\
=\ &-\boldsymbol{P}\boldsymbol{e}(t)-\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\boldsymbol{\psi}(\boldsymbol{e}(t))+\varepsilon(\boldsymbol{e}(t))-\boldsymbol{d}(t)\\
&-\hat{\varepsilon}^*(t)\mathrm{sign}(\boldsymbol{e}(t))-\mathrm{diag}(\hat{\bar{\boldsymbol{d}}}(t))\mathrm{sign}(\boldsymbol{e}(t)), \qquad (40)
\end{aligned}$$

where $\boldsymbol{P}=\boldsymbol{C}+\boldsymbol{K}$. From the definition of $\boldsymbol{P}$ we can know immediately that it is a positive definite matrix.

Multiplying $\boldsymbol{e}^{\mathrm{T}}(t)$ into both sides of Eq. (40) yields

$$\begin{aligned}
&\boldsymbol{e}^{\mathrm{T}}(t){}_0^C D_t^\alpha\boldsymbol{e}(t)\\
=\ &-\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t)-\boldsymbol{e}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\boldsymbol{\psi}(\boldsymbol{e}(t))\\
&+\boldsymbol{e}^{\mathrm{T}}(t)\varepsilon(\boldsymbol{e}(t))-\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{d}(t)\\
&-\boldsymbol{e}^{\mathrm{T}}(t)\hat{\varepsilon}^*(t)\mathrm{sign}(\boldsymbol{e}(t))-\boldsymbol{e}^{\mathrm{T}}(t)\mathrm{diag}(\hat{\bar{\boldsymbol{d}}}(t))\mathrm{sign}(\boldsymbol{e}(t))\\
\le\ &-p_{\min}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)-\sum_{i=1}^n e_i(t)\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t)\boldsymbol{\psi}_i(\boldsymbol{e}(t))+\varepsilon^*\sum_{i=1}^n|e_i(t)|\\
&+\sum_{i=1}^n\bar{d}_i|e_i(t)|-\hat{\varepsilon}^*(t)\sum_{i=1}^n|e_i(t)|-\sum_{i=1}^n\hat{\bar{d}}_i(t)|e_i(t)|\\
=\ &-p_{\min}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)-\sum_{i=1}^n e_i(t)\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t)\boldsymbol{\psi}_i(\boldsymbol{e}(t))\\
&-\tilde{\varepsilon}^*(t)\sum_{i=1}^n|e_i(t)|-\sum_{i=1}^n\tilde{\bar{d}}_i(t)|e_i(t)|, \qquad (41)
\end{aligned}$$

where $p_{\min}$ is the minimum eigenvalue of $\boldsymbol{P}$, and

$$\tilde{\varepsilon}^*(t)=\hat{\varepsilon}^*(t)-\varepsilon^*, \qquad (42)$$

$$\tilde{\bar{d}}_i(t)=\hat{\bar{d}}_i(t)-\bar{d}_i \qquad (43)$$

are the estimation errors of the unknown constants ε^* and $\bar{d}_i$, respectively. Note that the α-th derivative of a constant is zero, from Eqs. (31), (42), and (43) we have ${}_0^C D_t^\alpha\tilde{\boldsymbol{\theta}}_i(t)={}_0^C D_t^\alpha\boldsymbol{\theta}_i(t)$, ${}_0^C D_t^\alpha\tilde{\varepsilon}^*(t)={}_0^C D_t^\alpha\hat{\varepsilon}^*(t)$, and ${}_0^C D_t^\alpha\tilde{\bar{d}}_i(t)={}_0^C D_t^\alpha\hat{\bar{d}}_i(t)$.

Let us consider the following Lyapunov function candidate:

$$\begin{aligned}
V_2(t)\ =\ &\frac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)+\sum_{i=1}^n\frac{1}{2\gamma_{1i}}\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}_i(t)\\
&+\frac{1}{2\gamma_3}\tilde{\varepsilon}^{*2}(t)+\sum_{i=1}^n\frac{1}{2\gamma_{5i}}\tilde{\bar{d}}_i^2(t). \qquad (44)
\end{aligned}$$

Then, by using Lemma 5, Eqs. (37)–(39) and (41), we obtain

$$\begin{aligned}
&{}_0^C D_t^\alpha V_2(t)\\
\le\ &\boldsymbol{e}^{\mathrm{T}}(t){}_0^C D_t^\alpha\boldsymbol{e}(t)+\sum_{i=1}^n\frac{1}{\gamma_{1i}}\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t){}_0^C D_t^\alpha\tilde{\boldsymbol{\theta}}_i(t)\\
&+\frac{1}{\gamma_3}\tilde{\varepsilon}^*(t){}_0^C D_t^\alpha\tilde{\varepsilon}^*(t)+\sum_{i=1}^n\frac{1}{\gamma_{5i}}\tilde{\bar{d}}_i(t){}_0^C D_t^\alpha\tilde{\bar{d}}_i(t)\\
=\ &\boldsymbol{e}^{\mathrm{T}}(t){}_0^C D_t^\alpha\boldsymbol{e}(t)+\sum_{i=1}^n\frac{1}{\gamma_{1i}}\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t){}_0^C D_t^\alpha\boldsymbol{\theta}_i(t)\\
&+\frac{1}{\gamma_3}\tilde{\varepsilon}^*(t){}_0^C D_t^\alpha\hat{\varepsilon}^*(t)+\sum_{i=1}^n\frac{1}{\gamma_{5i}}\tilde{\bar{d}}_i(t){}_0^C D_t^\alpha\hat{\bar{d}}_i(t)\\
\le\ &-p_{\min}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)-\sum_{i=1}^n e_i(t)\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t)\boldsymbol{\psi}_i(\boldsymbol{e}(t))
\end{aligned}$$

$$
\begin{aligned}
&-\tilde{\varepsilon}^*(t)\sum_{i=1}^{n}|e_i(t)| \\
&+\sum_{i=1}^{n}\frac{1}{\gamma_{1i}}\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t){}_0^C D_t^{\alpha}\boldsymbol{\theta}_i(t)+\frac{1}{\gamma_3}\tilde{\varepsilon}^*(t){}_0^C D_t^{\alpha}\hat{\varepsilon}^*(t) \\
&-\sum_{i=1}^{n}\tilde{\bar{d}}_i(t)|e_i(t)|+\sum_{i=1}^{n}\frac{1}{\gamma_{5i}}\tilde{\bar{d}}_i(t){}_0^C D_t^{\alpha}\hat{\bar{d}}_i(t) \\
=\ & -p_{\min}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)-\sum_{i=1}^{n}\gamma_{2i}\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t)\boldsymbol{\theta}_i(t)-\gamma_4\tilde{\varepsilon}^*(t)\hat{\varepsilon}^*(t) \\
&-\sum_{i=1}^{n}\gamma_{6i}\tilde{\bar{d}}_i(t)\hat{\bar{d}}_i(t) \\
=\ & -p_{\min}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)-\sum_{i=1}^{n}\gamma_{2i}\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}_i(t) \\
&-\gamma_4\tilde{\varepsilon}^*(t)\tilde{\varepsilon}^*(t)-\gamma_4\tilde{\varepsilon}^*(t)\varepsilon^*(t) \\
&-\sum_{i=1}^{n}\gamma_{2i}\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t)\boldsymbol{\theta}_i^*-\sum_{i=1}^{n}\gamma_{6i}\tilde{\bar{d}}_i(t)\tilde{\bar{d}}_i(t)-\sum_{i=1}^{n}\gamma_{6i}\tilde{\bar{d}}_i(t)\bar{d}_i \\
\le\ & -p_{\min}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)-\sum_{i=1}^{n}\frac{\gamma_{2i}}{2}\tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}_i(t) \\
&-\frac{\gamma_4}{2}\tilde{\varepsilon}^{*2}(t)-\sum_{i=1}^{n}\frac{\gamma_{6i}}{2}\tilde{\bar{d}}_i^2(t) \\
&+\sum_{i=1}^{n}\frac{\gamma_{2i}}{2}\boldsymbol{\theta}_i^{*\mathrm{T}}\boldsymbol{\theta}_i^*+\frac{\gamma_4}{2}\varepsilon^{*2}+\sum_{i=1}^{n}\frac{\gamma_{6i}}{2}\bar{d}_i^2 \\
\le\ & -\hbar_1 V_2(t)+\hbar_2, \qquad (45)
\end{aligned}
$$

where $\hbar_1=\min\{2p_{\min},\gamma_{1i}\gamma_{2i},\gamma_2\gamma_4,\gamma_{5i}\gamma_{6i}\}$ and

$$\hbar_2=\sum_{i=1}^{n}\frac{\gamma_{2i}}{2}\theta_i^{*\mathrm{T}}\theta_i^*+\frac{\gamma_4}{2}\varepsilon^{*2}+\sum_{i=1}^{n}\frac{\gamma_{6i}}{2}\bar{d}_i^2$$

are two positive constants.

It follows from Eq. (45) that there exists a nonnegative function $z(t)$ such that

$${}_0^C D_t^{\alpha}V_2(t)+z(t)=-\hbar_1 V_2(t)+\hbar_2. \qquad (46)$$

Taking the Laplace transform on Eq. (46), one obtains

$$
\begin{aligned}
V_2(s) &= \frac{s^{\alpha-1}}{s^{\alpha}+\hbar_1}V_1(0)+\frac{\hbar_2}{s(s^{\alpha}+\hbar_1)}-\frac{Z(s)}{s^{\alpha}+\hbar_1} \\
&= \frac{s^{\alpha-1}}{s^{\alpha}+\hbar_1}V_2(0)+\frac{s^{\alpha-(1+\alpha)}\hbar_2}{(s^{\alpha}+\hbar_1)}-\frac{Z(s)}{s^{\alpha}+\hbar_1}, \qquad (47)
\end{aligned}
$$

where $V_2(s)$ and $Z(s)$ are the Laplace transforms of $V_2(t)$ and $z(t)$, respectively. By using Eq. (5), equation (47) can be solved as

$$
\begin{aligned}
V_2(t) &= V_2(0)E_{\alpha,1}(-\hbar_1 t^{\alpha})+\hbar_2 t^{\alpha}E_{\alpha,1+\alpha}(-\hbar_1 t^{\alpha}) \\
&\quad -z(t)*t^{-1}E_{\alpha,0}(-\hbar_1 t^{\alpha}), \qquad (48)
\end{aligned}
$$

where $*$ denotes the convolution operator. Noting that both $t^{-1}E_{\alpha,0}(-\hbar_1 t^{\alpha})$ and $z(t)$ are nonnegative, we have $m(t)*t^{-1}E_{\alpha,0}(-\hbar_1 t^{\alpha})\ge 0$. Then, we have

$$V_2(t)\le V_2(0)E_{\alpha,1}(-\hbar_1 t^{\alpha})+\hbar_2 t^{\alpha}E_{\alpha,1+\alpha}(-\hbar_1 t^{\alpha}). \qquad (49)$$

Noting that $\arg(-\hbar_1 t^{\alpha})=-\pi$, $|-\hbar_1 t^{\alpha}|\ge 0$ for all $t\ge 0$ and $\alpha\in(0,2)$, according to Lemma 3, we know that there exists a positive constant g_0 such that

$$|E_{\alpha,1}(-\hbar_1 t^{\alpha})|\le\frac{g_0}{1+\hbar_1 t^{\alpha}}, \qquad (50)$$

i.e.,

$$\lim_{t\to\infty}V_2(0)E_{\alpha,1}(-\hbar_1 t^{\alpha})=0. \qquad (51)$$

Thus, for every $\varepsilon>0$, there exists a constant $t_1>0$ such that $t>t_1$ implies

$$V_2(0)E_{\alpha,1}(-\hbar_1 t^{\alpha})<\frac{\varepsilon}{3}. \qquad (52)$$

On the other hand, using Lemma 2, we know that for every $\varepsilon>0$, there exists a positive constant t_2 such that

$$t^{\alpha}E_{\alpha,\alpha+1}(-\hbar_1 t^{\alpha})\le\frac{\hbar_2}{\hbar_1}+\frac{\varepsilon}{3} \qquad (53)$$

for all $t>t_2$. If we can choose the controller design parameters such that $\hbar_2/\hbar_1\le\varepsilon/3$, then it follows from Eqs. (49), (52), and (53) that

$$V_2(t)<\varepsilon. \qquad (54)$$

From Eq. (54) and the definition of the Lyapunov function $V_2(t)$, we know that all signals in the closed-loop system will keep bounded and $\|\boldsymbol{e}(t)\|\le\sqrt{2\varepsilon}$ for all $t>t_2$. This ends the proof of Theorem 2.

Remark 3 Fractional-order adaptation laws are also proposed to update the design parameters in Refs. [9], [11], and [53]. It should be stressed that in the above literature, the right side of the fractional-order adaptation laws contains only a positive term (for example, the adaptation law (37) is changed as ${}_0^C D_t^{\alpha}\boldsymbol{\theta}(t)=\gamma_{1i}e_i(t)\psi_i(\boldsymbol{e}(t))$). According to Lemma 7, the estimation of these design parameters will be monotone increasing. Thus, their boundednesses are very hard to be ensured. But in this paper, we add a term (for example, the term $-\gamma_{1i}\gamma_{2i}\boldsymbol{\theta}_i(t)$ in Eq. (37) in the fractional-order adaptation laws to solve this problem. It can be seen that the proposed fractional-order adaptation laws have a very similar form to those in Eq. (45), so it is easy to know that we can guarantee the boundedness of the parameters to be updated by choosing proper design parameters.

Remark 4 To guarantee that the synchronization error eventually converges to a small enough region of the origin, we should make $\hbar_2/\hbar_1$ as small as possible, and this can be done by choosing proper controller design parameters, for example, we can choose larger k_i, γ_{1i}, γ_3, and γ_{5i} and smaller γ_{2i}, γ_4, and γ_{6i}.

Remark 5 The term $\boldsymbol{K}\boldsymbol{e}(t)$ in Eq. (36) can be removed in some conditions. We add this term here only for the purpose of theoretical analysis. In fact, if $\boldsymbol{K}=0$, we can replace the

term $p_{\min}$ in Eq. (45) by $c_{\min}$, and the stability of the closed-loop system can be guaranteed still. In the simulation studies, we will set $\boldsymbol{K} = 0$. That is to say, the proposed fuzzy logic systems have very good approximation abilities.

Remark 6 To approximate the unknown nonlinear functions, fuzzy logic systems, whose input are chosen as the synchronization errors, are employed in this paper. We know that the controlled slave fractional-order neural networks will synchronize to the master system, so the synchronization errors may be very small after a period of time. In this case, to guarantee the performance of the fuzzy logics systems, the fuzzy rules should work well when the synchronization errors vary very slowly in a small region of the origin. So, how to choose appropriate fuzzy membership functions is an important work in this paper. We will discuss this problem in Section 4.

Remark 7 Synchronization for fractional-order neural networks has been investigated in many papers, for example, in Refs. [11], [13], [15]–[18], and [21]–[27]. It should be mentioned that the adaptive fuzzy control method has rarely been used to synchronize fractional-order neural networks. In addition, the model we used is very similar to that of Refs. [11] and [27]. A comparison between our method and their results is presented as follows: 1) In Refs. [11] and [27], adaptive control and fractional-order sliding mode control methods are used respectively. In this paper, an adaptive fuzzy control approach is proposed. 2) In Refs. [11] and [27], the prior knowledge of the system model is needed. But in our method, the system model can be fully unknown. 3) Bearing the results of Refs [11] and [27] in mind, the advantages of our control methods are: prior knowledge is not needed and the proposed method has better robustness; the exact values of the parameters are not needed (in fact, their exact values are hard to obtain in practice.[8,36])

4. Simulation studies

In this section, a simulation example will be given to show the effectiveness of the proposed synchronization method.

Let $\boldsymbol{x}(t) = [x_1(t), x_2(t), x_3(t)]^{\mathrm{T}}$, $\alpha = 0.95$, $f_i(x_i(t)) = \tanh(x_i(t))$, $c_i = 1, I_i = 0$ for $i = 1, 2, 3$, $\boldsymbol{A} = \begin{bmatrix} 2.00 & -1.20 & 0 \\ 2.00 & 1.71 & 1.15 \\ -4.75 & 0 & 1.10 \end{bmatrix}$. With the above system parameters, and let the initial condition be $\boldsymbol{x}(0) = [-3,\ 4,\ 3]^{\mathrm{T}}$, the master fractional-order neural network (22) has a chaotic attractor, which is presented in Fig. 2.

The initial condition for the slave fractional-order neural networks is chosen as $\boldsymbol{y}(0) = [2, -1, -1]^{\mathrm{T}}$, and the external disturbance is defined as $\boldsymbol{d}(t) = [\sin t, \cos t, \sin t + \cos t]^{\mathrm{T}}$. The control design parameters are given as $\gamma_{11} = \gamma_{12} = \gamma_{13} = 1, \gamma_{21} = \gamma_{22} = \gamma_{23} = 0.05, \gamma_3 = 1.5, \gamma_4 = 0.03, \gamma_{51} = \gamma_{52} = \gamma_{53} = 1, \gamma_{61} = \gamma_{62} = \gamma_{63} = 0.01$, and $k_1 = k_2 = k_3 = 0$. The initial conditions for $\hat{\hat{d}}_i(t)$ and $\hat{\varepsilon}^*(t)$ are chosen as $\hat{\hat{d}}_1(0) = \hat{\hat{d}}_2(0) = \hat{\hat{d}}_3(0) = \hat{\varepsilon}^*(0) = 0.01$. To avoid the chattering phenomenon produced by the sign function in Eq. (36), the discontinuous function $\mathrm{sign}(\cdot)$ is replaced by $\arctan(10\cdot)$.

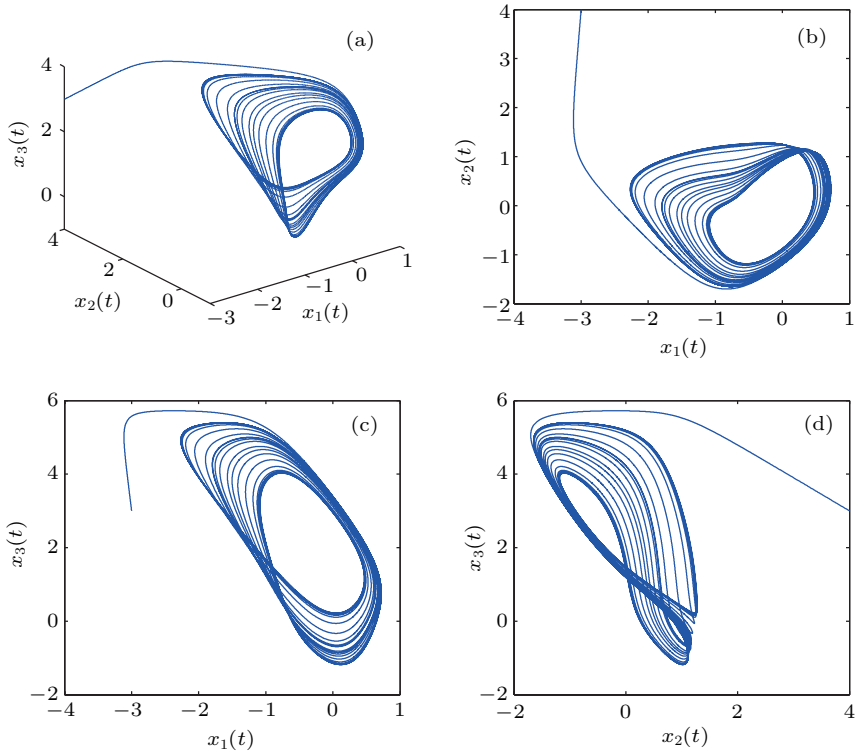


Fig. 2. (color online) Chaotic behavior of system (21) in (a) $x_1(t)$–$x_2(t)$–$x_3(t)$, (b) $x_1(t)$–$x_2(t)$ plane, (c) $x_1(t)$–$x_3(t)$ plane and (d) $x_1(t)$–$x_4(t)$ plane.

We define 7 Gaussian membership functions distributed on interval $[-4, 4]$ for each input $e_i(t)$. The synchronization errors may be very small after a period of time, so the memberships should have good performance nearby the origin. In this paper, the parameters of the proposed membership functions are listed in Table 1 (the membership functions have the form $\exp[-(e_i(t)-\rho)^2/2\delta^2]$). It is easy to know that there are $7\times7\times7=343$ fuzzy rules used. The initial conditions for $\boldsymbol{\theta}_i(t)$ are defined as $\boldsymbol{\theta}_1(0)=\boldsymbol{\theta}_2(0)=\boldsymbol{\theta}_3(0)=\underline{0}\in\mathcal{R}^{343}$.

Table 1. Parameters of the membership functions.

Membership functions	Parameter δ	Parameter ρ
Function 1	1	–3
Function 2	0.4	–1
Function 3	0.1	–0.2
Function 4	0.05	0
Function 5	0.1	0.2
Function 6	0.4	1
Function 7	1	3

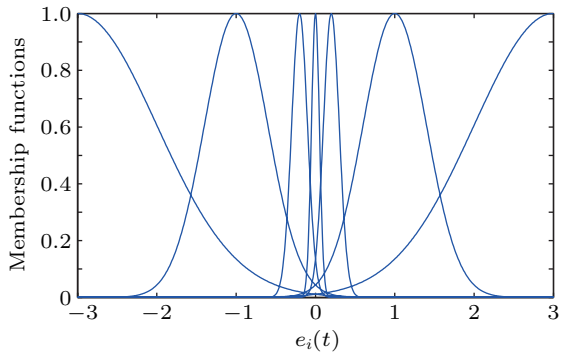


Fig. 3. (color online) Membership functions.

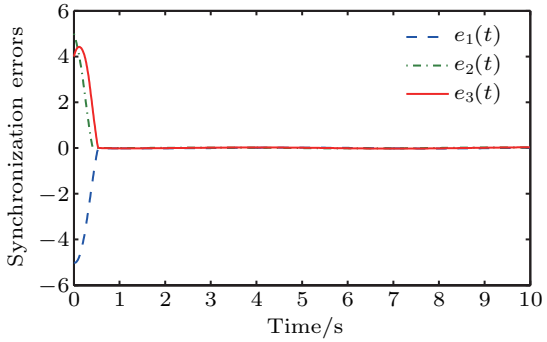


Fig. 4. (color online) Synchronization errors.

The simulation results are presented in Figs. 4–7. The time responses of the synchronization errors are included in Fig. 4. It is shown that the synchronization errors converge to a very small region of the origin after a short period of time although the controller design matrix K in Eq. (36) is chosen as 0. The varieties of the control inputs are depicted in Fig. 5. The norms of the fuzzy system parameters and the estimations of the unknown system parameters are presented in Figs. 5 and 7, respectively. It is worth mentioning that all design parameters are chosen to be small in the simulation, so the control inputs and the time responses of the parameters to be updated online vary in a very stable way. That is to say, good synchronization performance has been achieved based on the proposed control method.

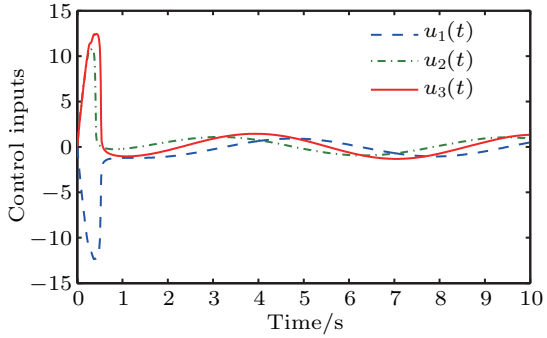


Fig. 5. (color online) Control inputs.

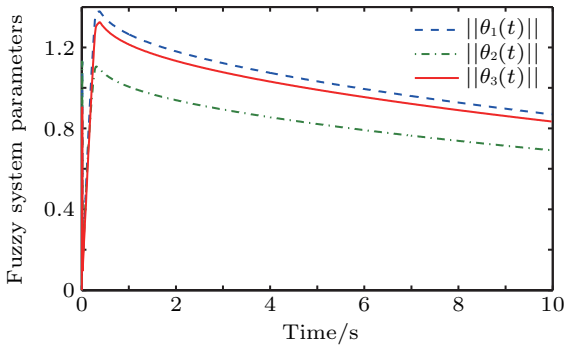


Fig. 6. (color online) Fuzzy system parameters.

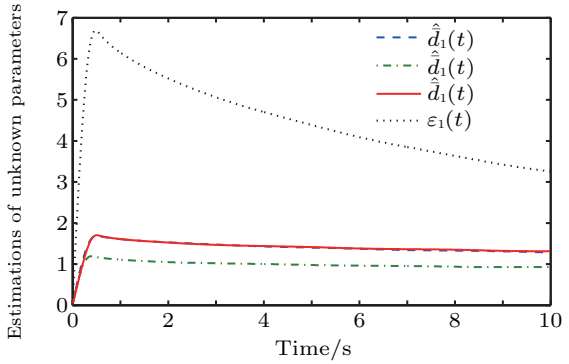


Fig. 7. (color online) Estimations of the unknown system parameters.

In the rest of this section, we will give some simulation results to show the differences between our control method and that in Ref. [11]. Our design parameters are chosen as above. In Ref. [11], the same system model is used in the simulation, and their design parameters are also chosen as those in Ref. [11] except that the initial values and the external disturbances are changed as above.

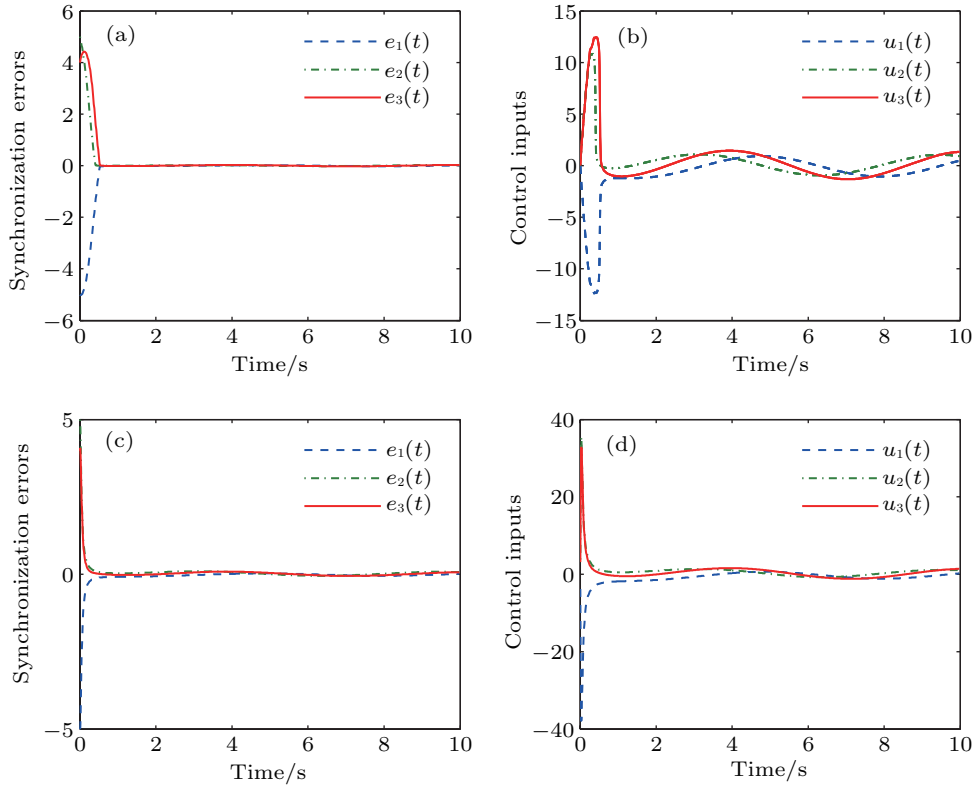


Fig. 8. (color online) Simulation results. (a) Synchronization errors and (b) control inputs of our method, (c) synchronization errors and (d) control inputs of Ref. [11].

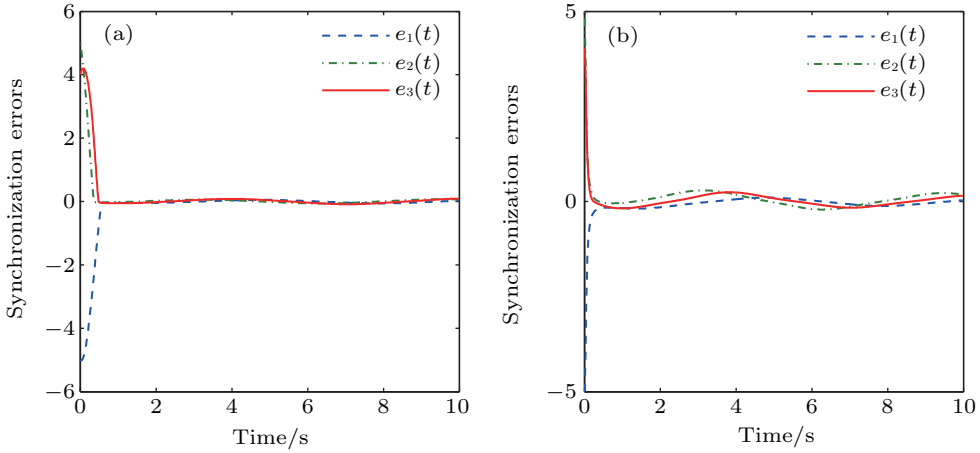


Fig. 9. (color online) Simulation results with large disturbance. Synchronization errors of (a) our method and (b) Ref. [11].

It can be concluded from Fig. 8 that the two methods have similar synchronization performance, but in our method, the synchronization will converge to the origin with smaller fluctuation. It should be noted that the comparison results are hard to be concluded because the design parameters in the two controllers are very different. To show the robustness of the two methods, we will increase the disturbances. Let $\boldsymbol{d}(t) = [3\sin t, 3\cos t, 3\sin t + 3\cos t]$, the simulation results are presented in Fig. 9, from which we can conclude that our method has better robustness compared with the synchronization method in Ref. [11].

5. Conclusion

Synchronization of two identical fractional-order neural networks by using an adaptive fuzzy control approach is discussed in this paper. We use the fuzzy logic systems to handle the case that the model of the system is fully unknown. Based on the fractional Lyapunov stability criterion, an adaptive fuzzy controller is given to guarantee that the synchronization errors converge to an arbitrary small region of the origin. It is shown that fractional-order adaptations can be designed to cancel the fuzzy approximation errors and the estimation errors of unknown parameters. By using the results of this paper,

some control methods which are proposed for integer-order nonlinear systems may be extended to control fractional-order nonlinear systems. Working towards the simplicity of the controller design and giving simpler stability analysis methods for fractional-order nonlinear systems are our further investigation directions.

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