Scaling of weighted spectral distribution in weighted small-world networks*

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Many real-world systems can be modeled by weighted small-world networks with high clustering coefficients. Recent studies for rigorously analyzing the weighted spectral distribution (*WSD*) have focused on unweighted networks with low clustering coefficients. In this paper, we rigorously analyze the *WSD* in a deterministic weighted scale-free small-world network model and find that the *WSD* grows sublinearly with increasing network order (i.e., the number of nodes) and provides a sensitive discrimination for each input of this model. This study demonstrates that the scaling feature of the *WSD* exists in the weighted network model which has high and order-independent clustering coefficients and reasonable power-law exponents.

Keywords: weighted spectral distribution, weighted small-world network, scaling

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1. Introduction

The weighted spectral distribution (*WSD*) is a metric defined on the normalized Laplacian spectrum. Although the *WSD* is first proposed[1–3] for the study of Internet topologies, our recent work[4–7] has demonstrated that the *WSD* is a good indicator for distinguishing diverse evolving physical systems. Additionally, we explained the physical meaning of the *WSD* in Internet topologies,[8] that is, the metric indicates the transformation from single-homed to multi-homed networks. Also, we designed an algorithm[9] for calculating the *WSD* which induces that the metric can be applied to networks with more than millions of nodes. However, these studies[1–9] focused on unweighted networks. To rigorously analyze the *WSD* in diachronic networks which focus on the time evolution and are similar to dynamical processes in nature,[10] we have designed a tree model[5] and a scale-free model.[6] However, the two models' clustering coefficients are strictly zero. Furthermore, the γ exponent of the degree distribution $P(k) \propto k^{-\gamma}$ of the used scale-free models[6,7] is one, whereas the γ exponent of most real evolving scale-free networks falls in a range between two and three.[11] Therefore, in this paper, we rigorously study the *WSD* on diachronic weighted small-world scale-free networks which have more reasonable γ exponents.

Synchronic random models are rigorous tools to analyze many graph metrics,[12] e.g., average path length and graph spectra. However, these models consider the statistical ensemble of graphs at a fixed time point and cannot capture the dynamical process in nature.[10] For diachronic networks, both random models[13–16] and deterministic models[17–23] have received much attention. Specifically, deterministic models are suitable for the rigorous analysis of more graph metrics, e.g., diameter, clustering coefficients, graph matrices, and graph spectra. The first deterministic scale-free model was proposed by Barabasi *et al.*[20] Another elegant model, called pseudo fractal scale-free web (PSW), was introduced by Dorogovtsev *et al.*[11] The PSW has been extended to many versions,[21–23] and the latest version proposed by Zhang *et al.*[23] is called deterministic weighted scale-free small-world network (WSSN) model. The WSSN model has highly tunable exponents for three power-law distributions of node degree, node strength, and edge weight. Additionally, all of the exponents fall in a range between 2 and 3 and the average clustering coefficient of the model is high and independent of the network order. These features of this model might lead to a better understanding of realistic networks. Therefore, in this paper, we rigorously study the *WSD* by using the WSSN model.

Currently, the scaling feature of many graph metrics has received much attention. The average receiving time is defined as the average of the mean first-passage times for a random walker to a given hub node in a graph, averaged over all source points in the graph.[24] Dai and Sun *et al.*[25–27] studied the average receiving time and the average weighted receiving time in deterministic Koch and hierarchical network models and found that the average receiving time grows sublinearly with increasing network order and the average weighted receiving time grows as power-law function of the network order. The Kemeny constant is defined as the expected time for a walker starting from a node to another node selected randomly from all nodes.[28] Zhang *et al.*[29] studied the Kemeny

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constant in deterministic scale-free fractal networks and found that the Kemeny constant scales linearly with network order. Additionally, many graph metrics (e.g., degree distribution, clustering coefficient, and spectral radius)[30–32] can be used to capture the unaltered features of evolving systems. As one number of these graph metrics with scaling features, this study on the *WSD* is helpful for a better understanding of evolving weighted and small-world systems.

2. WSSN model

The WSSN model is controlled by two parameters m and δ. Zhang *et al.*[23] denoted the network after t steps of the model by $G(t)$. Specifically, the network is constructed as follows.[23] For $t = 0$, $G(0)$ is a triangle consisting of three links with unit weight. For $t \geq 1$, $G(t)$ is obtained from $G(t-1)$: $m \cdot w$ new nodes are added for each of the links with weight w in $G(t-1)$, and each new node is connected to both ends of this link in $G(t-1)$ by two new links of unit weight; moreover, the weight of these links in $G(t-1)$ is increased by $m \cdot \delta \cdot w$. It is noteworthy that m and δ are positive integers. When $m = 1$ and $\delta = 1$, the constructed networks $G(t)$ with $t = 0$, 1, and 2 are shown in Fig. 1.

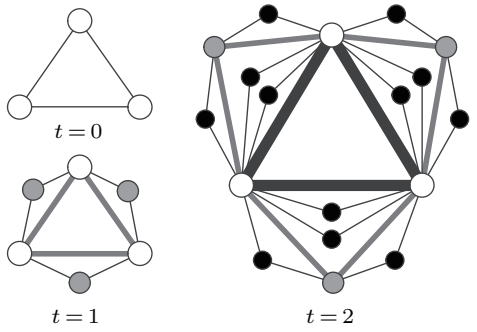


Fig. 1. $G(t)$ ($t = 0$, 1, and 2) with $m = 1$ and $\delta = 1$. The white, gray, and black nodes are respectively added at steps $t = 0$, 1, and 2, and the weights of the most thick lines, the second thick lines, and the thinnest lines are respectively 4, 2, and 1.

In $G(t)$, we represent all nodes added at step k as k and let (i, j) denote an edge linking two nodes i and j for $i \leq j$. If $i = j$, both i and j must be zero. Then, according to the study of Zhang *et al.*,[23] the strength $s_t(k)$ of node k and the weight $w_t(i, j)$ of edge (i, j) in $G(t)$ can be expressed as

$$s_t(k) = 2 \cdot (1 + m \cdot \delta + m)^{t-k}, \tag{1}$$

$$w_t(i, j) = (1 + m \cdot \delta)^{t-j}, \tag{2}$$

where the strength of a node is defined as the sum over all weights of edges that are connected to the node.

Additionally, Zhang *et al.*[23] derived the number of all nodes in $G(t)$ as

$$Num_t = \frac{3}{2+\delta} \cdot \left((1 + m \cdot \delta + 2 \cdot m)^t + \delta + 1\right). \tag{3}$$

3. *WSD* in weighted networks

To this day, the study on the *WSD* has been restricted to unweighted networks. Let $G = (V, E)$ denote a simple and undirected graph where V and E are respectively node set and edge set, and d_v denote the degree of node v in graph G. Let n denote the cardinality of node set V. If $A = (a_{ij})_{n\times n}$ is the adjacency matrix of graph G ($a_{ij} = 1$ if there is an edge between nodes v_i and v_j and $a_{ij} = 0$ otherwise) and B is a diagonal matrix with $b_{ii} = d_{v_i}$, the normalized Laplacian matrix of graph G is defined as $L(G) = I - B^{-1/2} \cdot A \cdot B^{-1/2}$ where I denotes the identity matrix.[1] Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote all eigenvalues of matrix $L(G)$, called the normalized Laplacian spectrum of graph G. Then, the *WSD* is defined as[1]

$$W(G, N) = \sum_{i=1,2,\ldots,n} (1 - \lambda_i)^N, \tag{4}$$

where 4 is the best selection of parameter N.[1–9]

Because all eigenvalues of $L(G)$ fall in a range $[0, 2]$, Fay *et al.*[1] transformed Eq. (4) into another expression as follows:

$$W(G, N) \approx \sum_{\theta \in \Omega_i} (1 - \theta)^N \cdot f(\lambda = \theta), \tag{5}$$

where $\Omega_i \in \{(2(i-1)/h, 2i/h]\}_{i=1}^{h}$ denotes equally spaced intervals in $[0, 2]$, and $f(\lambda = \theta)$ represents the number of eigenvalues falling in Ω_i. Note that θ is an arbitrary value in Ω_i. When $h \to +\infty$, equation (5) turns into Eq. (4). According to Eq. (5), the *WSD* can be expressed by using the weighted sum of the normalized Laplacian spectral distribution, so it is called the "weighted spectral distribution".[1]

Additionally, Fay *et al.*[1] derived the following equation:

$$\begin{aligned} \sum_{i=1,2,\ldots,n} (1 - \lambda_i)^N &= \mathrm{tr}\left((B^{-1/2} \cdot A \cdot B^{-1/2})^N\right) \\ &= \sum_C \frac{a_{u_1,u_2} a_{u_2,u_3} \cdots a_{u_N,u_1}}{b_{u_1,u_1} b_{u_2,u_2} \cdots b_{u_N,u_N}}, \end{aligned} \tag{6}$$

where $\mathrm{tr}(M)$ denotes the trace of matrix M, and $C = v_{u_1} v_{u_2} \ldots v_{u_N}$ denotes all N-cycles in graph G.

If G is an unweighted graph, $A = (a_{ij})_{n\times n}$ is a $0-1$ matrix and node degree $d_{v_i} = \sum_{j=1,2,\ldots,n} a_{ij}$. For this case, Fay *et al.*[1] obtained the following equation:

$$\sum_{i=1,2,\ldots,n} (1 - \lambda_i)^N = \sum_C \frac{1}{d_{v_{u_1}} d_{v_{u_2}} \cdots d_{v_{u_N}}}. \tag{7}$$

In this paper, we consider the *WSD* on weighted networks. Thus, entry a_{ij} in A should be the weight of edge between nodes v_i and v_j, and entry $b_{ii} = \sum_{j=1,2,\ldots,n} a_{ij}$ in B should be the strength of node v_i. Additionally, equation (6) holds for weighted networks. Therefore, for weighted networks, we can obtain

$$\sum_{i=1,2,\ldots,n} (1 - \lambda_i)^N = \sum_C \frac{a_{u_1,u_2} a_{u_2,u_3} \cdots a_{u_N,u_1}}{s_{v_{u_1}} s_{v_{u_2}} \cdots s_{v_{u_N}}}, \tag{8}$$

where s_v denotes the strength of node v in graph G.

4. Calculation method of WSD in weighted networks

Based on Eq. (8), for the WSD calculation, it is critical to enumerate all N-cycles (4 is the best selection of N) in weighted graph G. Specifically, all 4-cycles can be classified as four patterns as shown in Fig. 2.[6,7,9]

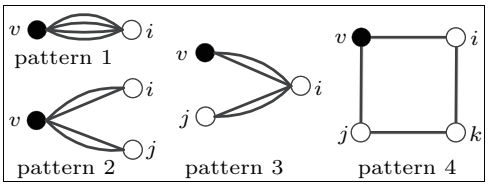


Fig. 2. All patterns of 4-cycles with start node v in simple and undirected graph G.

It is noteworthy that the multiple edges between two nodes in these patterns are related to only one edge in weighted graph G that does not have any multiple edge. Let $N(v)$ denote the set of neighbors of node v, $N(i)=\{v,i_1,i_2,\ldots,i_{s_i}\}$ represent the set of neighbors of node $i\in N(v)$, and $C(i,j)=N(i)\cap N(j)=\{v,c_1,c_2,\ldots,c_{t_{ij}}\}$ refer to the set of common neighbors of nodes i and j with $i,j\in N(v)$ and $i\neq j$.

As an extension for unweighted graphs,[6] in weighted graph G, the sum over all 4-cycles associated with patterns 1 and 3 $WSD_{13}(v)$, and the sum over all 4-cycles associated with patterns 2 and 4 $WSD_{24}(v)$ can be calculated as

$$WSD_{13}(v)=\sum_{i\in N(v)}\sum_{j\in N(i)}\frac{(a_{v,i})^2\cdot(a_{i,j})^2}{s_v\cdot(s_i)^2\cdot s_j}, \tag{9}$$

$$WSD_{24}(v)=\sum_{\substack{i\neq j\\ i,j\in N(v)}}\sum_{k\in C(i,j)}\frac{a_{v,i}\cdot a_{i,k}\cdot a_{k,j}\cdot a_{j,v}}{s_v\cdot s_i\cdot s_k\cdot s_j}, \tag{10}$$

where for each 4-cycle, the corresponding term of the sum is the product of edge weights divided by the product of node strengths.

However, it is extremely difficult to derive Eq. (10) in the WSSN model, because we cannot obtain the formula of $C(i,j)$ for $\forall i,j\in N(v)\wedge i\neq j$. In this paper, we decompose Eq. (10) into the sum over all 4-cycles associated with pattern 2 $WSD_2(v)$ and the sum over all 4-cycles associated with pattern 4 $WSD_4(v)$ as follows:

$$WSD_2(v)=\sum_{\substack{i\neq j\\ i,j\in N(v)}}\frac{(a_{v,i})^2\cdot(a_{v,j})^2}{(s_v)^2\cdot s_i\cdot s_j}, \tag{11}$$

$$WSD_4(v)=\sum_{v-i-k-j-v}\frac{a_{v,i}\cdot a_{i,k}\cdot a_{k,j}\cdot a_{j,v}}{s_v\cdot s_i\cdot s_k\cdot s_j}, \tag{12}$$

where v-i-k-j-v denotes all 4-cycles associated with pattern 4.

As is well known, the WSD in weighted graph G can be expressed as

$$WSD=\sum_{v\in V}(WSD_{13}(v)+WSD_2(v)+WSD_4(v)). \tag{13}$$

5. *WSD* in WSSN model

First, we will further classify the nodes labeled as k (i.e., added at step k) in $G(t)$ (see Section 2). Based on the construction rule of the WSSN model, each node k must be generated by an edge in $G(k-1)$ and any edge with weight w in $G(k-1)$ must generate $m\cdot w$ nodes k. Thus, we define $k_{(i,j)}$ as a node k generated by edge (i,j) where $0\le i\le j\le k$. If $i=j$, both i and j must be zero, and if $j=k$, all of i, j, and k must be zero. Let $n^k_{(i,j)}$ denote the number of all nodes $k_{(i,j)}$.

Then, we can obtain the following theorem (see Appendix A for the proof).

Theorem 1 In the model WSSN, Case A: $n^0_{(0,0)}=3$; Case B: $n^k_{(0,0)}=3m\cdot(1+m\cdot\delta)^{k-1}$ if $k\ge 1$; Case C: $n^k_{(0,j)}=6m^2\cdot(1+m\cdot\delta)^{k-j-1}\cdot(1+m+m\cdot\delta)^{j-1}$ if $1\le j<k$; and Case D: $n^k_{(i,j)}=6m^3\cdot(1+m\cdot\delta)^{k-j-1}\cdot(1+m+m\cdot\delta)^{j-i-1}\cdot(1+2m+m\cdot\delta)^{i-1}$ if $1\le i<j<k$. □

Second, we will derive the formula of $N_t(k_{(i,j)})$ which is defined as a set of neighbors of node $k_{(i,j)}$ in $G(t)$. As is well known, $N_k(k_{(i,j)})$ includes only one node i and only one node j, and for $\forall 1\le l\le t-k$, $N_{k+l}(k_{(i,j)})$ includes all nodes in $N_{k+l-1}(k_{(i,j)})$ and all nodes $(k+l)_{(i,k_{(i,j)})}$, $(k+l)_{(j,k_{(i,j)})}$ and $(k+l)_{(k_{(i,j)},s)}$ where $s>k$ and $i,j,s\in N_{k+l-1}(k_{(i,j)})$.

Thus, $N_t(k_{(i,j)})$ can be expressed as

$$N_t(k_{(i,j)})=I^k_{i,j}(i)\cup I^k_{i,j}(j)\cup\left(\bigcup_{l=1}^{t-k}I^k_{i,j}(k+l)\right), \tag{14}$$

where $I^k_{i,j}(l)$ only includes the nodes added at step l of the WSSN model.

As is well known, $\left\|I^k_{i,j}(i)\right\|=\left\|I^k_{i,j}(j)\right\|=1$, where $\|\cdot\|$ denotes the cardinality of a set.

Additionally, according to the edges that generate nodes $k+l$,

$$\begin{aligned}I^k_{i,j}(k+l)\ =\ &\left\{(k+l)_{(i,k)}\right\}_{n^l_i}\cup\left\{(k+l)_{(j,k)}\right\}_{n^l_j}\\ &\times\cup\left(\bigcup_{s=1}^{l-1}\left\{(k+l)_{(k,k+s)}\right\}_{n^l_s}\right),\end{aligned} \tag{15}$$

where $\{x\}_y$ denotes a set composed of y different nodes labeled as x.

In $G(k+l-1)$, the weight of edge $(k,k+s)$ for $\forall 1\le s\le l-1$ is $w_{k+l-1}(k,k+s)=(1+m\cdot\delta)^{l-s-1}$, and the weight of edges (i,k) and (j,k) is $w_{k+l-1}(i,k)=w_{k+l-1}(j,k)=(1+m\cdot\delta)^{l-1}$. Because the number of edges $(k,k+s)$ attached to node $k_{(i,j)}$ is equal to $\|I^k_{i,j}(k+s)\|$, we have $n^l_s=\|I^k_{i,j}(k+s)\|\cdot m\cdot w_{k+l-1}(k,k+s)$ in Eq. (15). Moreover, $n^l_j=\|I^k_{i,j}(j)\|\cdot m\cdot w_{k+l-1}(j,k)$ and $n^l_i=\|I^k_{i,j}(i)\|\cdot m\cdot w_{k+l-1}(i,k)$ in Eq. (15).

Thus, for $\forall 1\le l\le t-k$, we can obtain

$$\begin{aligned}\left\|I^k_{i,j}(k+l)\right\|\ =\ &2m\cdot(1+m\cdot\delta)^{l-1}+\textstyle\sum_{s=1}^{l-1}\left\|I^k_{i,j}(k+s)\right\|\\ &\times m\cdot(1+m\cdot\delta)^{l-s-1}.\end{aligned} \tag{16}$$

Based on Eq. (16), $\|I^k_{i,j}(k+l+1)\| - (1+m\cdot\delta)\cdot\|I^k_{i,j}(k+l)\| = m\cdot\|I^k_{i,j}(k+l)\|$. Using $\|I^k_{i,j}(k+1)\| = 2m$, we can derive the following equations:

$$\left\|I^k_{i,j}(k+l)\right\| = 2m\cdot(1+m+m\cdot\delta)^{l-1}, \tag{17}$$

$$n^l_i = n^l_j = m\cdot(1+m\cdot\delta)^{l-1}, \tag{18}$$

$$n^l_s = 2m^2\cdot(1+m\cdot\delta)^{l-s-1}\cdot(1+m+m\cdot\delta)^{s-1}. \tag{19}$$

Third, we will derive $WSD_{13} = \sum_{k_{(i,j)}\in G(t)} WSD_{13}(k_{(i,j)})$. According to Eq. (9), for each neighbor $v \in N_t(k_{(i,j)})$, we need to derive its neighbor set $N_t(v)$. Based on Eqs. (14) and (15), except for nodes in $I^k_{i,j}(i)$ and $I^k_{i,j}(j)$, the neighbor sets of all nodes in $N_t(k_{(i,j)})$ are easy to derive. Thus, it is critical to investigate the generation process of edge (i,j) which generates node $k_{(i,j)}$ with $0\le i\le j\le k$ (both i and j must be zero if $i=j$ and all of i, j, and k must be zero if $j=k$).

Category 1 When $k=0$, both nodes i and j can be expressed as $0_{(0,0)}$, and the number of nodes $k_{(i,j)}$ is $n^0_{(0,0)} = 3$ (see Theorem 1).

Category 2 When $k=1$, both nodes i and j can be expressed as $0_{(0,0)}$, and the number of nodes $k_{(i,j)}$ is $n^1_{(0,0)} = 3m$ (see Theorem 1).

When $2\le k\le t$, we analyze the following five categories:

Category 3 When $i=0$ and $j=0$, both nodes i and j can be expressed as $0_{(0,0)}$, and the number of nodes $k_{(i,j)}$ is $n^k_{(0,0)} = 3m\cdot(1+m\cdot\delta)^{k-1}$ (see Theorem 1).

Category 4 When $i=0\wedge j\in\{1,2,\ldots,k-1\}$, node i can be expressed as $0_{(0,0)}$, node j must be generated by an edge (i,v) where $v\in N_{j-1}(i_{(0,0)})$, and the number of nodes $i_{(0,0)}$ is $n^0_{(0,0)} = 3$. Additionally, each edge (i,v) must generate $m\cdot w_{j-1}(i,v) = m\cdot(1+m\cdot\delta)^{j-v-1}$ nodes j, and each edge (i,j) must generate $m\cdot w_{k-1}(i,j) = m\cdot(1+m\cdot\delta)^{k-j-1}$ nodes k. Specifically, according to Eqs. (14) and (15), we can obtain

$$N_{j-1}(i_{(0,0)}) = I^0_{0,0}(0)\cup I^0_{0,0}(0)\cup\left(\bigcup\nolimits_{l=1}^{j-i-1} I^0_{0,0}(l)\right), \tag{20}$$

$$\left\|I^0_{0,0}(0)\right\| = 1, \quad \left\|I^0_{0,0}(l)\right\| = 2m\cdot(1+m+m\cdot\delta)^{l-1}. \tag{21}$$

Note that the intersection of two sets $I^0_{0,0}(0)$ in Eq. (20) is null, because the two sets include different nodes that are marked as the same symbol.

Categories 5, 6, and 7 When $1\le i<j\le k-1$, we assume that the edge that generates node i is (x,y) where $0\le x\le y\le i-1$ (both x and y must be zero if $x=y$). Then, node j must be generated by an edge (i,v) or (v,i) where $v\in N_{j-1}(i_{(x,y)})$. Additionally, each edge (i,v) or (v,i) must generate $m\cdot w_{j-1}(\min(i,v),\max(i,v)) = m\cdot(1+m\cdot\delta)^{j-\max(i,v)-1}$ nodes j, and each edge (i,j) must generate $m\cdot w_{k-1}(i,j) = m\cdot(1+m\cdot\delta)^{k-j-1}$ nodes k.

Specifically, we can obtain

$$N_{j-1}(i_{(x,y)}) = I^i_{x,y}(x)\cup I^i_{x,y}(y)\cup\left(\bigcup\nolimits_{l=1}^{j-i-1} I^i_{x,y}(i+l)\right), \tag{22}$$

$$\left\|I^i_{x,y}(x)\right\| = 1, \quad \left\|I^i_{x,y}(y)\right\| = 1,$$

$$\left\|I^i_{x,y}(i+l)\right\| = 2m\cdot(1+m+m\cdot\delta)^{l-1}. \tag{23}$$

Now, we should further analyze the following three categories for nodes $i_{(x,y)}$.

Category 8 When node i is generated by edge $(0,0)$, the number of nodes $i_{(0,0)}$ is $n^i_{(0,0)} = 3m\cdot(1+m\cdot\delta)^{i-1}$ (see Theorem 1).

Category 9 When node i is generated by edge $(0,y)$ where $y\in\{1,2,\ldots,i-1\}$, the number of nodes $i_{(0,y)}$ is $n^i_{(0,y)} = 6m^2\cdot(1+m\cdot\delta)^{i-y-1}\cdot(1+m+m\cdot\delta)^{y-1}$ (see Theorem 1).

Category 10 When node i is generated by edge (x,y) where $x\in\{1,2,\ldots,i-2\}$ and $y\in\{x+1,x+2,\ldots,i-1\}$, the number of nodes $i_{(x,y)}$ is $n^i_{(x,y)} = 6m^3\cdot(1+m\cdot\delta)^{i-y-1}\cdot(1+m+m\cdot\delta)^{y-x-1}\cdot(1+2m+m\cdot\delta)^{x-1}$ (see Theorem 1).

According to the above analysis, we can determine the edges which generate the nodes in $I^k_{i,j}(i)$ and $I^k_{i,j}(j)$. Thus, the neighbor set of each node in $I^k_{i,j}(i)\cup I^k_{i,j}(j)\subseteq N_t(k_{(i,j)})$ can be derived. Assume that node $k_{(i,j)}$ is generated by edge $\left(i_{(x,y)}, j_{(u,v)}\right)$, i.e., nodes i and j are respectively generated by edges (x,y) and (u,v).

Then, based on Eqs. (1), (2), (9), (14), and (15), in weighted graph $G(t)$, we can derive the following equation:

$$\begin{aligned}
&WSD_{13}(k_{(i_{(x,y)},j_{(u,v)})})\\
&= \frac{(1+m\cdot\delta)^{4t-2k-2i}}{2^4(1+m+m\cdot\delta)^{3t-2i-k}}\left(\frac{1}{(1+m+m\cdot\delta)^{t-x}}+\frac{1}{(1+m+m\cdot\delta)^{t-y}}\right.\\
&\quad\left.+\frac{2(1+m+m\cdot\delta)^{t-i+1}}{(2+m+2m\cdot\delta)\cdot(1+m\cdot\delta)^{2t-2i}}-\frac{2(1+m+m\cdot\delta)^{i-t+1}}{2+m+2m\cdot\delta}\right)+\frac{(1+m\cdot\delta)^{4t-2k-2j}}{2^4(1+m+m\cdot\delta)^{3t-2j-k}}\left(\frac{1}{(1+m+m\cdot\delta)^{t-u}}\right.\\
&\quad\left.+\frac{1}{(1+m+m\cdot\delta)^{t-v}}+\frac{2(1+m+m\cdot\delta)^{t-j+1}}{(2+m+2m\cdot\delta)\cdot(1+m\cdot\delta)^{2t-2j}}-\frac{2(1+m+m\cdot\delta)^{j-t+1}}{2+m+2m\cdot\delta}\right)\\
&\quad+\sum_{l=1}^{t-k}\left[\frac{m\cdot(1+m\cdot\delta)^{4t-4k-3l-1}}{2^4(1+m+m\cdot\delta)^{3t-3k-2l}}\left(\frac{1}{(1+m+m\cdot\delta)^{t-i}}+\frac{2}{(1+m+m\cdot\delta)^{t-k}}+\frac{1}{(1+m+m\cdot\delta)^{t-j}}\right.\right.\\
&\quad\left.\left.+\frac{2^2(1+m+m\cdot\delta)^{t-k-l+1}}{(2+m+2m\cdot\delta)\cdot(1+m\cdot\delta)^{2t-2k-2l}}-\frac{2^2(1+m+m\cdot\delta)^{k+l-t+1}}{2+m+2m\cdot\delta}\right)+\sum_{s=1}^{l-1}\frac{m^2(1+m\cdot\delta)^{4t-4k-3l-s-1}}{2^3(1+m+m\cdot\delta)^{3t-3k-2l-s+1}}
\end{aligned}$$

$$\times\left(\frac{1}{(1+m+m\cdot\delta)^{t-k}}+\frac{1}{(1+m+m\cdot\delta)^{t-k-s}}+\frac{2(1+m+m\cdot\delta)^{t-k-l+1}}{(2+m+2m\cdot\delta)\cdot(1+m\cdot\delta)^{2t-2k-2l}}-\frac{2(1+m+m\cdot\delta)^{k+l-t+1}}{2+m+m\cdot\delta}\right)\Bigg]. \quad (24)$$

The above analysis has classified nodes $k_{(i,j)}$ as seven categories: 1. $k=0$; 2. $k=1$; 3. $2\le k\le t$ and $i=0\wedge j=0$; 4. $2\le k\le t$ and $i=0\wedge j\in\{1,2,\ldots,k-1\}$; 5. $2\le k\le t$ and $1\le i<j\le k-1\wedge x=0\wedge y=0$; 6. $2\le k\le t$ and $1\le i<j\le k-1\wedge x=0\wedge y\in\{1,2,\ldots,i-1\}$; 7. $2\le k\le t$ and $1\le i<j\le k-1\wedge x\in\{1,2,\ldots,i-2\}\wedge y\in\{x+1,x+2,\ldots,i-1\}$. For each category above, the edges (x,y) and (u,v) that respectively generate nodes i and j can be determined.

Thus, we expand $WSD_{13}=\sum_{k_{(i,j)}\in G(t)}WSD_{13}(k_{(i,j)})$ as follows:

$$WSD_{13}=\sum_{l=1}^{7}Wkij(l), \quad (25)$$

where $Wkij(l)$ $(l=1,2,\ldots,7)$ are the sums of $WSD_{13}(k_{(i,j)})$ and respectively related to the above seven categories 1–7.

Specifically, we can obtain

$$Wkij(1)=n^0_{(0,0)}\cdot WSD_{13}(0_{(0_{(0,0)},0_{(0,0)})}), \quad (26)$$

$$Wkij(2)=n^1_{(0,0)}\cdot WSD_{13}(1_{(0_{(0,0)},0_{(0,0)})}), \quad (27)$$

$$Wkij(3)=\sum_{k=2}^{t}n^k_{(0,0)}\cdot D_{13}(k_{(0_{(0,0)},0_{(0,0)})}), \quad (28)$$

$$\begin{aligned}Wkij(4)=&\sum_{k=2}^{t}\sum_{j=1}^{k-1}n^0_{(0,0)}\cdot\big(2m\cdot(1+m\cdot\delta)^{j-1}\cdot WSD_{13}(k_{(0_{(0,0)},j_{(0,0)})})\\&+\sum_{l=1}^{j-1}2m^2\cdot(1+m\cdot\delta)^{j-l-1}\big)\cdot(1+m+m\cdot\delta)^{l-1}\cdot WSD_{13}(k_{(0_{(0,0)},j_{(0,l)})})\cdot m\cdot(1+m\cdot\delta)^{k-j-1}, \quad (29)\\ Wkij(5)=&\sum_{k=3}^{t}\sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}n^i_{(0,0)}\cdot\big(2m\cdot(1+m\cdot\delta)^{j-i-1}\cdot WSD_{13}(k_{(i_{(0,0)},j_{(0,i)})})\\&+\sum_{l=1}^{j-i-1}2m^2\cdot(1+m\cdot\delta)^{j-i-l-1}\cdot(1+m+m\cdot\delta)^{l-1}\cdot WSD_{13}(k_{(i_{(0,0)},j_{(i,i+l)})})\big)\cdot m\cdot(1+m\cdot\delta)^{k-j-1}, \quad (30)\\ Wkij(6)=&\sum_{k=4}^{t}\sum_{i=2}^{k-2}\sum_{j=i+1}^{k-1}\sum_{y=1}^{i-1}n^i_{(0,y)}\cdot\big(m\cdot(1+m\cdot\delta)^{j-i-1}\cdot WSD_{13}(k_{(i_{(0,y)},j_{(0,i)})})+m\cdot(1+m\cdot\delta)^{j-i-1}\cdot WSD_{13}(k_{(i_{(0,y)},j_{(y,i)})})\\&+\sum_{l=1}^{j-i-1}2m^2\cdot(1+m\cdot\delta)^{j-i-l-1}\cdot(1+m+m\cdot\delta)^{l-1}\cdot WSD_{13}(k_{(i_{(0,y)},j_{(i,i+l)})})\big)\cdot m\cdot(1+m\cdot\delta)^{k-j-1}, \quad (31)\\ Wkij(7)=&\sum_{k=5}^{t}\sum_{i=3}^{k-2}\sum_{j=i+1}^{k-1}\sum_{x=1}^{i-2}\sum_{y=x+1}^{i-1}n^i_{(x,y)}\cdot\big(m\cdot(1+m\cdot\delta)^{j-i-1}\cdot WSD_{13}(k_{(i_{(x,y)},j_{(x,i)})})\\&+m\cdot(1+m\cdot\delta)^{j-i-1}\cdot WSD_{13}(k_{(i_{(x,y)},j_{(y,i)})})\\&+\sum_{l=1}^{j-i-1}2m^2\cdot(1+m\cdot\delta)^{j-i-l-1}\cdot(1+m+m\cdot\delta)^{l-1}\cdot WSD_{13}(k_{(i_{(x,y)},j_{(i,i+l)})})\big)\cdot m\cdot(1+m\cdot\delta)^{k-j-1}. \quad (32)\end{aligned}$$

Fourth, we will derive $WSD_2=\sum_{k_{(i,j)}\in G(t)}WSD_2(k_{(i,j)})$. According to Eq. (11), it is critical to extract all pairs of nodes in $N_t(k_{(i,j)})$ for the calculation of $WSD_2(k_{(i,j)})$. Based on Eq. (14), the node pairs u and v can be classified as three cases as follows.

Case 1 For $u\in I^k_{i,j}(i)$ and $\forall v\in N_t(k_{(i,j)})/\{u\}$, the sum over 4-cycles associated with $k_{(i,j)}-u-k_{(i,j)}-v-k_{(i,j)}$ can be expressed as

$$\begin{aligned}&WSD^i_2(k_{(i,j)})\\&=\frac{1}{2^3(2+m+2m\cdot\delta)}\left(\frac{(1+m\cdot\delta)^{2t-2k}}{(1+m+m\cdot\delta)^{2t-i-k-1}}\right.\\&\left.-\frac{(1+m\cdot\delta)^{4t-4k}}{(1+m+m\cdot\delta)^{4t-i-3k-1}}\right)\\&+\frac{(1+m\cdot\delta)^{4t-4k}}{2^4(1+m+m\cdot\delta)^{4t-i-j-2k}}, \quad (33)\end{aligned}$$

where for each 4-cycle, the corresponding term of the sum is the product of edge weights divided by the product of node strengths.

Case 2 For $u\in I^k_{i,j}(j)$ and $\forall v\in N_t(k_{(i,j)})/\{u\}$, the sum over 4-cycles associated with $k_{(i,j)}-u-k_{(i,j)}-v-k_{(i,j)}$ can be expressed as

$$\begin{aligned}&WSD^j_2(k_{(i,j)})\\&=\frac{1}{2^3(2+m+2m\cdot\delta)}\left(\frac{(1+m\cdot\delta)^{2t-2k}}{(1+m+m\cdot\delta)^{2t-j-k-1}}\right.\\&\left.-\frac{(1+m\cdot\delta)^{4t-4k}}{(1+m+m\cdot\delta)^{4t-j-3k-1}}\right)\end{aligned}$$

$$+\frac{(1+m\cdot\delta)^{4t-4k}}{2^4(1+m+m\cdot\delta)^{4t-i-j-2k}}. \tag{34}$$

Case 3 For$u\in I^k_{i,j}(k+l)$ and $\forall v\in N_t(k_{(i,j)})/\{u\}$ where $l\in\{1,2,\ldots,t-k\}$, the sum over 4-cycles associated with $k_{(i,j)}-u-k_{(i,j)}-v-k_{(i,j)}$ can be expressed as

$$\begin{aligned}
&WSD_2^{k+l}(k_{(i,j)})\\
&=\frac{1}{2^3(2+m+2m\cdot\delta)}\left(\frac{(1+m\cdot\delta)^{2t-2k-2l}}{(1+m+m\cdot\delta)^{2t-2k-l-1}}\right.\\
&\quad\left.-\frac{(1+m\cdot\delta)^{4t-4k-2l}}{(1+m+m\cdot\delta)^{4t-4k-l-1}}\right)+\frac{(1+m\cdot\delta)^{4t-4k-2l}}{2^4(1+m+m\cdot\delta)^{4t-i-l-3k}}\\
&\quad+\frac{(1+m\cdot\delta)^{4t-4k-2l}}{2^4(1+m+m\cdot\delta)^{4t-j-l-3k}}-\frac{(1+m\cdot\delta)^{4t-4k-4l}}{2^4(1+m+m\cdot\delta)^{4t-4k-2l}}.
\end{aligned} \tag{35}$$

Therefore, we can obtain

$$\begin{aligned}
&WSD_2(k_{(i,j)})\\
&=WSD_2^i(k_{(i,j)})+WSD_2^j(k_{(i,j)})\\
&\quad+\sum_{l=1}^{t-k}\left\|I^k_{i,j}(k+l)\right\|\cdot WSD_2^{k+l}(k_{(i,j)}),
\end{aligned} \tag{36}$$

$$\begin{aligned}
WSD_2&=\sum_{k_{(i,j)}\in G(t)}WSD_2(k_{(i,j)})\\
&=n^0_{(0,0)}\cdot WSD_2(0_{(0,0)})+\sum_{k=1}^{t}n^k_{(0,0)}\cdot WSD_2(k_{(0,0)})\\
&\quad+\sum_{j=1}^{t-1}\sum_{k=j+1}^{t}n^k_{(0,j)}\cdot WSD_2(k_{(0,j)})\\
&\quad+\sum_{i=1}^{t-2}\sum_{j=i+1}^{t-1}\sum_{k=j+1}^{t}n^k_{(i,j)}\cdot WSD_2(k_{(i,j)}).
\end{aligned} \tag{37}$$

Fifth, we will derive $WSD_4=\sum_{k_{(i,j)}\in G(t)}WSD_4(k_{(i,j)})$. According to Eq. (12), we need to determine all 4-cycles of pattern 4 with start node $k_{(i,j)}$ for calculating $WSD_4(k_{(i,j)})$. Using the schematic illustration shown in Fig. 3, we present seven cases for these 4-cycles as follows.

Case 1 Assume that node j is generated by edge (u,i) if $u=x\vee u=y$ where edge (x,y) generates node i (or generated by edge (i,u) if $u>i\wedge u\in N_{j-1}(i_{(x,y)})$), $k_{(i,j)}-i-u-j-k_{(i,j)}$ and $k_{(i,j)}-j-u-i-k_{(i,j)}$ are among these cases.

An example of Case 1 in Fig. 3 is $A-B-F-C-A$.

Case 2 For any node $p\geq 1$ that is generated by edge (i,j) and different from the start node $k_{(i,j)}$, $k_{(i,j)}-i-p-j-k_{(i,j)}$ and $k_{(i,j)}-j-p-i-k_{(i,j)}$ are among these cases. An example of case 2 in Fig. 3 is $A-B-L-C-A$.

Case 3 Assume node $p\in N_t(k_{(i,j)})$, for any pair of nodes l and $q(l\leq q)$ that are generated by edge $(k_{(i,j)},p)$ if $k<p$ (or edge $(p,k_{(i,j)})$ if $k>p$), $k_{(i,j)}-l-p-q-k_{(i,j)}$ and $k_{(i,j)}-q-p-l-k_{(i,j)}$ are among these cases.

An example of Case 3 in Fig. 3 is $A-H-C-M-A$.

Case 4 Assume that node $l\in N_t(k_{(i,j)})$ and p is a node generated by edge $(k_{(i,j)},l)$ if $k<l$ (or edge $(l,k_{(i,j)})$ if $k>l$), for any node q generated by edge $(k_{(i,j)},p)$, $k_{(i,j)}-l-p-q-k_{(i,j)}$ and $k_{(i,j)}-q-p-l-k_{(i,j)}$ are among these cases.

An example of Case 4 in Fig. 3 is $A-C-Q-P-A$.

Case 5 Assume that node $l\in N_t(k_{(i,j)})$ and q is a node generated by edge $(k_{(i,j)},l)$ if $k<l$ (or edge $(l,k_{(i,j)})$ if $k>l$), for any node p generated by edge (l,q), $k_{(i,j)}-l-p-q-k_{(i,j)}$ and $k_{(i,j)}-q-p-l-k_{(i,j)}$ are among these cases.

An example of Case 5 in Fig. 3 is $A-C-J-Q-A$.

Case 6 For any node q that is generated by edge $(j,k_{(i,j)})$, $k_{(i,j)}-i-j-q-k_{(i,j)}$ and $k_{(i,j)}-q-j-i-k_{(i,j)}$ are among these cases.

An example of Case 6 in Fig. 3 is $A-B-C-H-A$.

Case 7 For any node q that is generated by edge $(i,k_{(i,j)})$, $k_{(i,j)}-j-i-q-k_{(i,j)}$ and $k_{(i,j)}-q-i-j-k_{(i,j)}$ are among these cases.

An example of Case 7 in Fig. 3 is $A-C-B-G-A$.

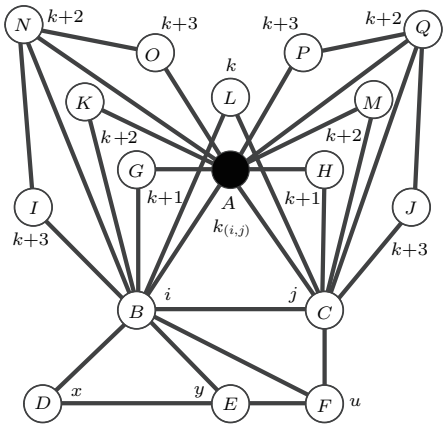


Fig. 3. A subgraph of $G(t)$ that is used to illustrate all 4-cycles of pattern 4 with start node $k_{(i,j)}$. In this subgraph, edge (x,y) generates node i; and if $u=x$ or y, edge (u,i) generates node j, otherwise edge (i,u) generates node j.

Theorem 2 All 4-cycles of pattern 4 with start node $k_{(i,j)}$ must belong to one of the above seven cases 1–7 (see Appendix B for the proof).

For start node $k_{(i,j)}$, the above cases 2–7 do not need to determine which edges generate nodes i and j where edge (i,j) generates the start node. Thus, it is easy to calculate $WSD_4(k_{(i,j)})$ for these cases.

For Case 2, edge (i,j) generates $m\cdot w_{p-1}(i,j)=m\cdot(1+m\cdot\delta)^{p-j-1}$ nodes p for $\forall j+1\leq p\leq t\wedge p\neq k$, and $m\cdot(1+m\cdot\delta)^{k-j-1}-1$ nodes p for $p=k$. Thus, the corresponding $WSD_4(k_{(i,j)})$ of this case can be expressed as

$$\begin{aligned}
&WSD_4^2(k_{(i,j)})\\
&=2\left(\sum_{p=j+1}^{t}\frac{w_t(i,k)\cdot w_t(i,p)\cdot w_t(j,p)\cdot w_t(j,k)}{s_t(k)\cdot s_t(i)\cdot s_t(p)\cdot s_t(j)}\right.\\
&\quad\left.-\frac{(w_t(i,k))^2\cdot(w_t(j,k))^2}{(s_t(k))^2\cdot s_t(i)\cdot s_t(j)}\right),
\end{aligned} \tag{38}$$

where node strength $s_t(\cdot)$ and edge weight $w_t(\cdot)$ are defined in Eqs. (1) and (2).

For Case 3, based on Eq. (14), if $p \in I^k_{i,j}(i) \cup I^k_{i,j}(j)$, the set of all nodes generated by edge $(p,k_{(i,j)})$ is $\bigcup_{x=k+1}^{t}\{x\}_{m\cdot w_{x-1}(p,k)}$ where $\{x\}_{m\cdot w_{x-1}(p,k)}$ denotes a set composed of $m\cdot w_{x-1}(p,k)$ nodes that are added at step x. If $p \in I^k_{i,j}(k+l)$ where $l=1,2,\ldots,t-k$, the set of all nodes generated by edge $(k_{(i,j)},p)$ is $\bigcup_{x=p+1}^{t}\{x\}_{m\cdot w_{x-1}(k,p)}$. Thus, the corresponding $WSD_4(k_{(i,j)})$ of this case can be expressed as follows:

$$\begin{aligned}WSD_4^3(k_{(i,j)}) = &\sum_{p=i,j}\Bigg[2\sum_{l=k+1}^{t-1}\sum_{q=l+1}^{t} m^2\cdot w_{l-1}(p,k)\cdot w_{q-1}(p,k)\cdot\frac{w_t(k,l)\cdot w_t(p,l)\cdot w_t(p,q)\cdot w_t(k,q)}{s_t(k)\cdot s_t(l)\cdot s_t(p)\cdot s_t(q)}\\ &+\sum_{l=k+1}^{t} m\cdot w_{l-1}(p,k)\cdot(m\cdot w_{l-1}(p,k)-1)\cdot\frac{w_t(k,l)\cdot(w_t(p,l))^2\cdot w_t(k,l)}{s_t(k)\cdot(s_t(l))^2\cdot s_t(p)}\Bigg]\\ &+\sum_{p=k+1}^{t}\Bigg[2\sum_{l=p+1}^{t-1}\sum_{q=l+1}^{t} m^2\cdot w_{l-1}(k,p)\cdot w_{q-1}(k,p)\cdot\frac{w_t(k,l)\cdot w_t(p,l)\cdot w_t(p,q)\cdot w_t(k,q)}{s_t(k)\cdot s_t(l)\cdot s_t(p)\cdot s_t(q)}\\ &+\sum_{l=p+1}^{t} m\cdot w_{l-1}(k,p)\cdot(m\cdot w_{l-1}(k,p)-1)\cdot\frac{w_t(k,l)\cdot(w_t(p,l))^2\cdot w_t(k,l)}{s_t(k)\cdot(s_t(l))^2\cdot s_t(p)}\Bigg].\end{aligned} \tag{39}$$

Like the analyses of Cases 2 and 3, we can respectively derive the corresponding $WSD_4(k_{(i,j)})$ of Cases 4–7 as follows:

$$\begin{aligned}WSD_4^4(k_{(i,j)}) = &\frac{m^2}{2^3}\sum_{p=k+1}^{t}\sum_{q=p+1}^{t}\left(\frac{(1+m\cdot\delta)^{4t-2k-p-q-2}}{(1+m+m\cdot\delta)^{4t-k-p-q-i}}+\frac{(1+m\cdot\delta)^{4t-2k-p-q-2}}{(1+m+m\cdot\delta)^{4t-k-p-q-j}}\right)\\ &+\frac{m^3}{2^2}\sum_{l=1}^{t-k}\sum_{p=k+l+1}^{t}\sum_{q=p+1}^{t}\frac{(1+m\cdot\delta)^{4t-2k-2l-p-q-2}}{(1+m+m\cdot\delta)^{4t-p-q-2k-2l+1}},\end{aligned} \tag{40}$$

$$\begin{aligned}WSD_4^5(k_{(i,j)}) = &\frac{m^2}{2^3}\sum_{q=k+1}^{t}\sum_{p=q+1}^{t}\left(\frac{(1+m\cdot\delta)^{4t-2k-p-q-2}}{(1+m+m\cdot\delta)^{4t-k-p-q-i}}+\frac{(1+m\cdot\delta)^{4t-2k-p-q-2}}{(1+m+m\cdot\delta)^{4t-k-p-q-j}}\right)\\ &+\frac{m^3}{2^2}\sum_{l=1}^{t-k}\sum_{q=k+l+1}^{t}\sum_{p=q+1}^{t}\frac{(1+m\cdot\delta)^{4t-2k-2l-p-q-2}}{(1+m+m\cdot\delta)^{4t-p-q-2k-2l+1}},\end{aligned} \tag{41}$$

$$WSD_4^6(k_{(i,j)}) = \frac{m}{2^3}\sum_{q=k+1}^{t}\frac{(1+m\cdot\delta)^{4t-2k-q-j-1}}{(1+m+m\cdot\delta)^{4t-i-j-k-q}}, \tag{42}$$

$$WSD_4^7(k_{(i,j)}) = \frac{m}{2^3}\sum_{q=k+1}^{t}\frac{(1+m\cdot\delta)^{4t-2k-q-j-1}}{(1+m+m\cdot\delta)^{4t-i-j-k-q}}. \tag{43}$$

It is worth noting that $s_t(\cdot)$ and $w_t(\cdot)$ in Eqs. (40)–(43) have been respectively substituted by Eqs. (1) and (2). Thus, $WSD_4 = \sum_{k_{(i,j)}\in G(t)} WSD_4(k_{(i,j)})$ associated with Cases 2–7 can be expressed as follows:

$$\begin{aligned}&WSD_4^{2-7}\\ &= n^0_{(0,0)}\cdot WSD_4^{2-7}(0_{(0,0)}) + \sum_{k=1}^{t} n^k_{(0,0)}\cdot WSD_4^{2-7}(k_{(0,0)})\\ &\quad+\sum_{k=2}^{t}\sum_{j=1}^{k-1} n^k_{(0,j)}\cdot WSD_4^{2-7}(k_{(0,j)})\\ &\quad+\sum_{k=3}^{t}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} n^k_{(i,j)}\cdot WSD_4^{2-7}(k_{(i,j)}),\end{aligned} \tag{44}$$

where

$$WSD_4^{2-7}(k_{(i,j)}) = \sum_{l=2}^{7} WSD_4^l(k_{(i,j)}). \tag{45}$$

For start node $k_{(i,j)}$, Case 1 needs to determine which edges generate nodes i and j. Thus, we need to classify nodes $k_{(i,j)}$ as seven categories: 1. $k=0$; 2. $k=1$; 3. $2\le k\le t$ and $i=0\wedge j=0$; 4. $2\le k\le t$ and $i=0\wedge j\in\{1,2,\ldots,k-1\}$; 5. $2\le k\le t$ and $1\le i<j\le k-1\wedge x=0\wedge y=0$; 6. $2\le k\le t$ and $1\le i<j\le k-1\wedge x=0\wedge y\in\{1,2,\ldots,i-1\}$; 7. $2\le k\le t$ and $1\le i<j\le k-1\wedge x\in\{1,2,\ldots,i-2\}\wedge y\in\{x+1,x+2,\ldots,i-1\}$. This classification of nodes $k_{(i,j)}$ has been analyzed in detail for the calculation of $WSD_{13} = \sum_{k_{(i,j)}\in G(t)} WSD_{13}(k_{(i,j)})$. Let $WSD_4^1(k_{(i,j)})$ denote the sum of $WSD_4(k_{(i,j)})$ associated with Case 1, then, using a similar method for calculating WSD_{13}, we will be able to expand $WSD_4^1 = \sum_{k_{(i,j)}\in G(t)} WSD_4^1(k_{(i,j)})$ as follows:

$$WSD_4^1 = \sum_{l=1}^{7} Wkij_4^1(l), \tag{46}$$

where $Wkij_4^1(l)$ $(l=1,2,\ldots,7)$ are the sums of $WSD_4^1(k_{(i,j)})$ and respectively related to the above seven categories 1–7.

Specifically, we can obtain

$$Wkij_4^1(1) = 0, \tag{47}$$

$$Wkij_4^1(2) = n^1_{(0,0)}\cdot\frac{(1+m\cdot\delta)^{4t-2}}{2^3(1+m+m\cdot\delta)^{4t-1}}, \tag{48}$$

$$Wkij_4^1(3) = \sum_{k=2}^{t} n^k_{(0,0)}\cdot\frac{(1+m\cdot\delta)^{4t-2k}}{2^3(1+m+m\cdot\delta)^{4t-k}}, \tag{49}$$

$$Wkij_4^1(4) = \frac{m}{2^2}\sum_{k=2}^{t}\sum_{j=1}^{k-1} n^0_{(0,0)} \cdot \left[\frac{(1+m\cdot\delta)^{4t-2k-1}}{(1+m+m\cdot\delta)^{4t-k-j}} + \sum_{l=1}^{j-1}\frac{m\cdot(1+m\cdot\delta)^{4t-2k-2l-1}}{(1+m+m\cdot\delta)^{4t-k-2l-j+1}}\right], \quad (50)$$

$$Wkij_4^1(5) = \frac{m}{2^2}\sum_{k=3}^{t}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} n^i_{(0,0)} \times \left[\frac{(1+m\cdot\delta)^{4t-2k-2i-1}}{(1+m+m\cdot\delta)^{4t-k-i-j}} \times \sum_{l=1}^{j-i-1}\frac{m\cdot(1+m\cdot\delta)^{4t-2k-2i-2l-1}}{(1+m+m\cdot\delta)^{4t-k-2i-2l-j+1}}\right], \quad (51)$$

$$Wkij_4^1(6) = 2\sum_{k=4}^{t}\sum_{j=3}^{k-1}\sum_{i=2}^{j-1}\sum_{y=1}^{i-1} n^i_{(0,y)} \times \left[\frac{m\cdot(1+m\cdot\delta)^{4t-2k-2i-1}}{2^4(1+m+m\cdot\delta)^{4t-k-i-j}} + \frac{m\cdot(1+m\cdot\delta)^{4t-2k-2i-1}}{2^4(1+m+m\cdot\delta)^{4t-k-i-j-y}} + \sum_{l=1}^{j-i-1}\frac{m^2\cdot(1+m\cdot\delta)^{4t-2k-2i-2l-1}}{2^3(1+m+m\cdot\delta)^{4t-k-2i-2l-j+1}}\right], \quad (52)$$

$$Wkij_4^1(7) = 2\sum_{k=5}^{t}\sum_{j=4}^{k-1}\sum_{i=3}^{j-1}\sum_{y=2}^{i-1}\sum_{x=1}^{y-1} n^i_{(x,y)} \times \left[\frac{m\cdot(1+m\cdot\delta)^{4t-2k-2i-1}}{2^4(1+m+m\cdot\delta)^{4t-k-i-j-x}} + \frac{m\cdot(1+m\cdot\delta)^{4t-2k-2i-1}}{2^4(1+m+m\cdot\delta)^{4t-k-i-j-y}} + \sum_{l=1}^{j-i-1}\frac{m^2\cdot(1+m\cdot\delta)^{4t-2k-2i-2l-1}}{2^3(1+m+m\cdot\delta)^{4t-k-2i-2l-j+1}}\right]. \quad (53)$$

Thus, according to Eqs. (44) and (46), we can obtain

$$WSD_4 = WSD_4^1 + WSD_4^{2-7}. \quad (54)$$

Finally, according to Eqs. (25), (37), and (54), we can derive the WSD in $G(t)$ as follows:

$$WSD = WSD_{13} + WSD_2 + WSD_4. \quad (55)$$

It is extremely complicated to manually express the WSD in $G(t)$ with three parameters m, δ, and t. Fortunately, the Symbolic Math Toolbox of Matlab can help us to simplify the expression of the WSD. However, it is not intelligent enough for the toolbox to finish the work alone. For example, if we assume that $x > 1$ and $x > y > 0$, for functions $\varphi = (t * y \wedge t + x \wedge t)/(x \wedge t + 1)$ and $\psi = (t * (y/x) \wedge t + 1)/(1 + 1/x \wedge t)$, Matlab R2015a code limit (φ, t, inf) returns to limit $((t * y \wedge t + x \wedge t)/(x \wedge t + 1), t = \text{inf})$ which is an invalid result, but limit (ψ, t, inf) can return to the expected result 1. Thus, we need to compile extra codes to deal with the transformation from φ to ψ.

Because the expression of the WSD is very long, according to Eq. (3), we only present the limit of the ratio of the WSD to the network order with $t \to +\infty$ as follows:

$$\lim_{t\to+\infty}(WSD/Num_t) = (1+m+m\cdot\delta)^2 \cdot f/(4m\cdot g_1\cdot g_2\cdot g_3\cdot g_4), \quad (56)$$

where

$$\begin{aligned} f = {} & (2\delta^{11}+28\delta^{10}+176\delta^9+656\delta^8+1612\delta^7 \\ & +2744\delta^6+3304\delta^5+2816\delta^4+1666\delta^3 \\ & +652\delta^2+152\delta+16)\cdot m^{10} \\ & +(26\delta^{10}+324\delta^9+1796\delta^8+5838\delta^7+12338\delta^6 \\ & +17736\delta^5+17584\delta^4+11886\delta^3+5248\delta^2 \\ & +1368\delta+160)\cdot m^9+(142\delta^9+1615\delta^8 \\ & +7952\delta^7+22414\delta^6+40106\delta^5 \\ & +47461\delta^4+37262\delta^3+18754\delta^2+5498\delta+716)\cdot m^8 \\ & +(436\delta^8+4604\delta^7+20132\delta^6+48588\delta^5+71854\delta^4 \\ & +67350\delta^3+39318\delta^2+13114\delta+1916)\cdot m^7 \\ & +(840\delta^7+8335\delta^6+32193\delta^5+65140\delta^4 \\ & +76696\delta^3+53484\delta^2+20666\delta+3430)\cdot m^6 \\ & +(1064\delta^6+9998\delta^5+33761\delta^4+55366\delta^3 \\ & +48806\delta^2+22546\delta+4330)\cdot m^5 \\ & +(896\delta^5+7999\delta^4+23226\delta^3 \\ & +29158\delta^2+17154\delta+3943)\cdot m^4+ \\ & (492\delta^4+4160\delta^3+10102\delta^2+8704\delta+2570)\cdot m^3 \\ & +(166\delta^3+1309\delta^2+2517\delta+1128)\cdot m^2 \\ & +(30\delta^2+210\delta+273)\cdot m+2\delta+10, \end{aligned} \quad (57)$$

$$\begin{aligned} g_1 = {} & (\delta^3+4\delta^2+5\delta+2)\cdot m^2 \\ & +(2\delta^2+8\delta+5)\cdot m+\delta+4, \end{aligned} \quad (58)$$

$$\begin{aligned} g_2 = {} & (\delta^4+5\delta^3+9\delta^2+7\delta+2)\cdot m^3 \\ & +(3\delta^3+15\delta^2+18\delta+7)\cdot m^2 \\ & +(3\delta^2+15\delta+9)\cdot m+\delta+5, \end{aligned} \quad (59)$$

$$\begin{aligned} g_3 = {} & (\delta^4+5\delta^3+9\delta^2+7\delta+2)\cdot m^3 \\ & +(4\delta^3+15\delta^2+18\delta+7)\cdot m^2 \\ & +(5\delta^2+15\delta+9)\cdot m+2\delta+5, \end{aligned} \quad (60)$$

$$\begin{aligned} g_4 = {} & (\delta^5+6\delta^4+14\delta^3+16\delta^2+9\delta+2)\cdot m^4 \\ & +(4\delta^4+24\delta^3+42\delta^2+32\delta+9)\cdot m^3 \\ & +(6\delta^3+36\delta^2+42\delta+16)\cdot m^2 \\ & +(4\delta^2+24\delta+14)\cdot m+\delta+6. \end{aligned} \quad (61)$$

Therefore, the WSD grows sublinearly as the network order of the WSSN model increases. Additionally, it is easy to verify that both the partial derivatives of Eq. (56) with respect to m and δ are smaller than 0 for $\forall m > 0 \wedge \delta > 0$ (because the expressions of the two partial derivatives are very long, we do not present the expressions in this paper, but we can use the "diff" function of Matlab to demonstrate these results). Thus, the limit of the ratio of the WSD to the network order monotonically decreases with increasing m and δ, i.e., the limit pro-

vides a sensitive discrimination for each input of the WSSN model.

6. Applications in synthetic weighted networks

The mathematical derivation process of Section 5 demonstrates that the *WSD* is a good indicator to distinguish different weighted evolving systems. In this section, we choose three evolving weighted systems derived by classical Barrat's model [33] to verify the applicability of the *WSD*. The model generates weighted networks based on growth and weights' dynamics: at each time step, the model adds a new node and connects it to m previously existing nodes for the growth; also, for each of the m existing nodes, the model updates the weights of edges connected to the node and makes the node strength increased by δ. According to the study of Barrat *et al.*,[33] δ/w_0 reflects the influence of newly added edges on the weight distribution where $w_0 = 1$ denotes the initial weights of the newly added edges. As shown in Fig. 4, the *WSD* grows sublinearly as the network order of a certain weighted evolving system increases, that is, the ratio of the *WSD* to the network order remains stable for these networks with different orders. Furthermore, figure 4 shows that the ratio of the *WSD* to the network order can effectively distinguish different weighted evolving systems. Therefore, this study is important for promoting the *WSD* applicability in weighted evolving networks.

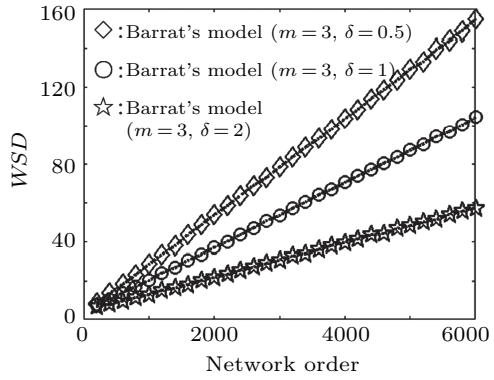


Fig. 4. *WSD* versus network order of three weighted evolving systems derived by classical Barrat's model. Note that each quantity represents the average of 10 realizations.

7. Conclusions

Weighted, high and order-independent clustering coefficient and restricted power-law exponent falling in $(2,3)$ are features existing in most realistic evolving networks. In this paper, we rigorously investigate the *WSD* in the WSSN model which captures all of the mentioned features. In contrast to the unweighted model with zero clustering coefficient, the derivation of the *WSD* in the WSSN model is very complicated. However, with the help of the Symbolic Math Toolbox of Matlab, we demonstrate that the ratio of the *WSD* to the network order is asymptotically independent of the network order of the WSSN model and monotonically decreases as each input of the model increases. Furthermore, the rigorous derivation process of the *WSD* studied in this paper will provide a helpful insight into a understanding of the *WSD* in weighted network systems.

Appendix A: The proof of Theorem 1

As is well known, the number of nodes $0_{(0,0)}$ is 3. Thus, $n^0_{(0,0)} = 3$ and Case A is correct.

Based on Eq. (2), the weight of each edge $(0,0)$ in $G(k-1)$ is $(1+m\cdot\delta)^{k-1}$. Thus, $m\cdot(1+m\cdot\delta)^{k-1}$ nodes k are generated by edge $(0,0)$ in $G(k)$ when $k \geq 1$. There are 3 edges $(0,0)$ which induce that $n^k_{(0,0)} = 3m\cdot(1+m\cdot\delta)^{k-1}$ if $k \geq 1$. Thus Case B is correct.

Using an induction method to prove Case C: when $k = 2$, j must be 1, and $n^k_{(0,j)} = n^2_{(0,1)} = 6m^2$ (i.e., Case C holds for $k = 2$ and $\forall 1 \leq j < k$).

Assume that Case C holds for $k \leq k_0$ $(k_0 \geq 2)$ and $\forall 1 \leq j < k$. When $k = k_0 + 1$, $n^k_{(0,j)}$ is determined by the number of all edges $(0,j)$ in $G(k-1)$ (i.e., $\|(0,j)\|$) and the weight of edge $(0,j)$ in $G(k-1)$ (i.e., $w_{k-1}(0,j) = (1+m\cdot\delta)^{k-j-1}$ derived by Eq. (2)).

Edge $(0,j)$ is generated if and only if nodes $j_{(0,l)}(0 \leq l \leq j-1)$ are added at step j of the WSSN model. Specifically, $j_{(0,l)}(1 \leq l \leq j-1)$ generates only one edge $(0,j)$ and $j_{(0,0)}$ generates two edges $(0,j)$. Thus, we can obtain

$$\|(0,j)\| = \sum_{l=1}^{j-1} n^j_{(0,l)} + 2n^j_{(0,0)}. \tag{A1}$$

And according to the construction rule of the WSSN model,

$$n^k_{(0,j)} = \|(0,j)\|\cdot m\cdot w_{k-1}(0,j). \tag{A2}$$

According to the above assumption and Case B, we can derive the formulas of $n^j_{(0,l)}(1 \leq l < j)$ and $n^j_{(0,0)}$. Furthermore, we can obtain

$$n^k_{(0,j)} = 6m^2\cdot(1+m\cdot\delta)^{k-j-1}\cdot(1+m+m\cdot\delta)^{j-1}. \tag{A3}$$

Therefore, for $k = k_0 + 1$ and $\forall 1 \leq j < k$, Case C remains correct.

Using an induction method to prove Case D: when $k = 3$, i and j must be 1 and 2, respectively, and $n^k_{(i,j)} = n^3_{(1,2)} = 6m^3$ (i.e., Case D holds for $k = 3$ and $\forall 1 \leq i < j < k$).

Assume that Case D holds for $k \leq k_0$ $(k_0 \geq 3)$ and $\forall 1 \leq i < j < k$. When $k = k_0 + 1$, $n^k_{(i,j)}$ is determined by the number of all edges (i,j) in $G(k-1)$ (i.e., $\|(i,j)\|$) and the weight of edge (i,j) in $G(k-1)$ (i.e., $w_{k-1}(i,j) = (1+m\cdot\delta)^{k-j-1}$ derived by Eq. (2)).

Edge (i,j) is generated if and only if nodes $j_{(l,i)}(0 \leq l \leq i-1)$, $j_{(i,l)}(i+1 \leq l \leq j-1)$ are added at step j of the WSSN model. Specifically, each node $j_{(l,i)}$ or $j_{(i,l)}(0 \leq l \leq j-1, l \neq i)$ generates only one edge (i,j). Thus, we can obtain

$$\|(i,j)\| = \sum_{l=0}^{i-1} n^j_{(l,i)} + \sum_{l=i+1}^{j-1} n^j_{(i,l)}. \tag{A4}$$

And according to the construction rule of the WSSN model, we have

$$n^k_{(i,j)} = \|(i,j)\| \cdot m \cdot w_{k-1}(i,j). \tag{A5}$$

According to the above assumption and Case C, we can derive the formulas of $n^j_{(l,i)}(1 \le l \le i-1)$, $n^j_{(i,l)}(i+1 \le l \le j-1)$ and $n^j_{(0,i)}$. Furthermore, we can obtain

$$\begin{aligned} n^k_{(i,j)} = {} & 6m^3 \cdot (1+m\cdot\delta)^{k-j-1} \cdot (1+m+m\cdot\delta)^{j-i-1} \\ & \times (1+2m+m\cdot\delta)^{i-1}. \end{aligned} \tag{A6}$$

Therefore, for $k = k_0 + 1$ and $\forall 1 \le i < j < k$, Case D remains correct.

Appendix B: The proof of Theorem 2

First of all, we extract two properties of weighted graph $G(t)$ as follows.

Property 1 For any node k (i.e., a node added at step k) with $k \ge 1$, if two different nodes i and j are connected with node k and $i \le j < k$, (i,j) must be the only one edge that generates node k and edge $(i,j) \in G(t)$.

Property 2 For any node k with $k \ge 1$, if node i is attached to node k in $G(t)$, we can determine that $i \ne k$.

The above two properties can be easily confirmed by Eq. (14).

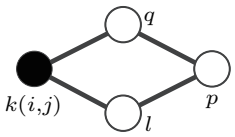


Fig. A1. A 4-cycle of pattern 4 with start node $k_{(i,j)}$.

Proof We analyze the 4-cycle as shown in Fig. A1.

Without loss of generality, we assume $l \le q$.

If $q = k$, then $l = q = k = 0$ due to Property 2. Because $G(t)$ includes only three nodes 0, we can determine $p \ge 1$. Thus, p is a node generated by edge (l,q) due to property 1, i.e., the 4-cycle belongs to Case 2.

If $q \ne k \wedge l = k$, then $l = k = 0 < 1 \le q$ and $p \ne q$ due to Property 2. If $p > q$, then $p > q \ge l$ due to the assumption $l \le q$, and edge $(l,q) \in G(t)$ generates node p due to Property 1. Because $(l,q) \in G(t)$, node q is generated by edge $(k_{(i,j)}, l)$ due to Property 1, i.e., the 4-cycle belongs to Case 5.

When $q \ne k \wedge l \ne k$:

1) if $l \le q < k$, then edge $(l,q) \in G(t)$ generates node $k_{(i,j)}$ due to Property 1. If $l \le q < p$, then node p is generated by edge (l,q), i.e., the 4-cycle belongs to Case 2. If $l \le q = p$, then $l = q = p = 0$ due to Property 2. Thus, node q is generated by edge (l,p), i.e., the 4-cycle belongs to Case 1. Note that Case 2 requires $p \ge 1$. If $q > p$, then $q \ge 1$. Because $(l,q) \in G(t)$ and $q \ge l$, node q is generated by edge $(\min(l,p), \max(l,p))$, i.e., the 4-cycle belongs to Case 1.

2) if $q > k \wedge p > q$, $p > q \ge l$. Thus, edge $(l,q) \in G(t)$ generates node p due to Property 1. Because $(l,q) \in G(t)$ and $q > k \wedge q \ge l$, $q \ge 1$ and node q is generated by edge $(\min(l,k), \max(l,k))$, i.e., the 4-cycle belongs to Case 5.

If $q > k$, then $q \ge 1$ and $p \ne q$ due to Property 2.

3) if $q > k \wedge p < q$, then edge $(\min(k,p), \max(k,p)) \in G(t)$ generates node q due to Property 1. If $\max(k,p,l) = p$, then $p \ge 1$ (otherwise $l = k = 0$ that is contradictive to the precondition $q \ne k \wedge l \ne k$). Because edge $(\min(k,p), \max(k,p)) \in G(t)$, $p > k$ due to Property 2 and node p is generated by edge $(\min(k,l), \max(k,l))$ due to Property 1, i.e., the 4-cycle belongs to Case 4. If $\max(k,p,l) = l$, then $l \ge 1$ and node l is generated by edge $(\min(k,p), \max(k,p))$ due to Property 1, i.e., the 4-cycle belongs to Case 3. If $\max(k,p,l) = k$, then $k \ge 1$ and $\{p,l\} \cap \{i,j\} = \{i,j\}$ for node $k_{(i,j)}$ due to Property 1. Then, the 4-cycle belongs to Case 6 if $l = i \wedge p = j$, or belongs to Case 7 if $p = i \wedge l = j$.

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