Negative linear compressibility of generic rotating rigid triangles*

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The compressibility properties of systems consisting of generic rotating rigid triangles are analyzed and discussed. It is shown that these systems which are usually associated with auxeticity can exhibit strongly anisotropic properties for certain conformations, which may give rise to the anomalous property of negative linear compressibility (NLC), that is, the system with particular geometry will expand along one direction when loaded hydrostatically. It is also shown that through carefully choosing the geometric features (i.e. the dimensions and the alignment of the rotating triangles as well as the angles between them) and the direction along which the linear compressibility is measured, one may control the magnitude and range of the NLC. All this provides a novel but effective method of manufacturing the systems which can be tailored to achieve particular values of NLC to fit particular practical applications.

Keywords: negative linear compressibility, rotating rigid triangles, analytical methods, microstructure

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1. Introduction

When materials are compressed hydrostatically, they usually contract in all directions. However, the theory of elasticity still brings us some surprises as evidenced by occasional reports of materials[1–11] that negative linear compressibility (NLC) can actually be exhibited, meaning that some materials can expand along one direction when subjected to hydrostatic stress. Such materials are predicted to have a number of applications including the development of artificial muscles, next-generation actuators and optical sensors.[1]

Over the past few decades, several naturally existing materials with NLC have been discovered. The ones identified so far include methanol monohydrate,[2] lanthanum niobate,[3] silver (I) hexacyanocobaltate (III)[4] and paratellurite.[5] In all of these cases, negative compressibility behaviour can be explained in terms of particular geometric features in the materials' nano or microstructures (geometry) and the way they deform when subjected to a hydrostatic pressure (the deformation mechanism). In an attempt to attain a better insight into the mechanisms that result in NLC, a number of geometry-based structures have been proposed not only to explain the occurrence of negative compressibility, but also to act as blueprints for the designing and manufacturing of man-made materials. The simplest of these models is a wine-rack-like structure which deforms through hinging[1] and it can be used to explain the experimentally measured or predicted NLC in systems including methanol monohydrate,[2] silver (I) hexacyanocobaltate (III)[4] and MIL-53.[6] Grima *et al.*[7] have shown that NLC can be observed when honeycomb-like system deforms primarily through changes in angles and the wine-rack-like structure can also be viewed as its special case. The NLC has also been reported to arise from body-centred or face-centred tetragonal structures constructed from networks of beams.[8] By using a combination of finite element simulations and analytical derivations, it is shown that the magnitude and direction of NLC depend on the cross geometry of the networks. Some other systems which can exhibit negative compressibility consist of two different materials and these two materials can either be used as separate trusses to form triangular shapes[9] or used as bi-materials strips.[10,11] Recently the mechanical and thermal properties of potassium beryllium fluoroborate were studied by Jiang *et al.* and some potential applications of negative compressibility including smart strain converters and deep-ultraviolet acoustic–optic devices have also been presented.[12] Moreover, Cairns and Goodwin[13] reviewed the phenomenology of negative compressibility in multifarious materials and presented a mechanistic understanding of negative compressibility. Miller *et al.*[14] identified several relatively common materials which display NLC and presented several potential applications. Here we need to mention the research conducted by Zhang *et al.*[15] and in their research the physical properties of tetragonal crystals were analyzed and the general expressions for these properties were given. Recently the density functional theory which was used by Wang *et al.*[16] to investigate the properties of MgAgSb has become a routine method of studying condensed matter theory. Also the same method was used by Hu *et al.*[17] to investigate the electronic, optical, and mechanical properties of γ-$Bi_2Sn_2O_7$.

In recent years, models based on rotating rigid units have drawn considerable attention in view of their ability to explain the observed auxeticity in various materials ranging from ze-

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olites to foams,[18–24] and they hold a lot more potential than others as further research is carried out. A recent study[25] shows that the metamaterials consisting of rotating rigid triangles can exhibit NLC in certain conditions if the triangles are connected in such a manner that the sides of the same length make up the perimeter of resulting pores. Furthermore, Attard *et al.*[26] have shown that some rotating rigid rectangles or rhombi with particular geometric features and connectivities can also exhibit NLC. Here we need to mention the great contributions made by Grima and Evan,[19] and Grima *et al.*[20] where they extended the previous highly symmetric rotating triangle models and proposed a more realistic and generic model built from scalene triangles, which can well predict the behaviours of real auxetic materials.[22] It has been found that the model can exhibit a large range of Poisson ratios from negative values to highly positive values, which provides the possibility that, if the Poisson ratio is high enough, the systems consisting of scalene triangles may become candidates for NLC,[7] something which would widen the search for NLC materials due to the fact that various materials can be described through this model[27–32] and the significance of analyzing the compressibility of this model may also be brought out by considering the apparent rarity of NLC.

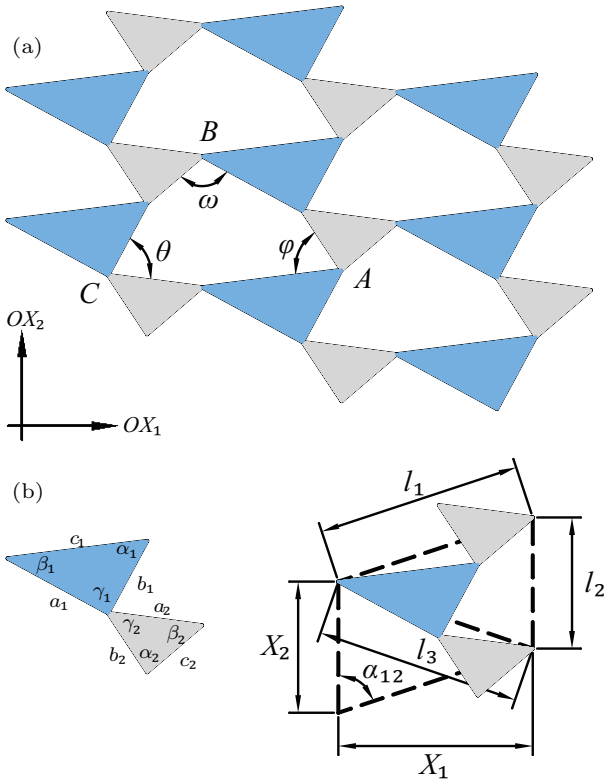


Fig. 1. (color online) (a) Model consisting of non-equivalent rigid scalene triangles, (b) parallelogramic unit cell.

In view of this, in this paper, the model composed of tessellates of two non-equivalent scalene triangles is analyzed and its potential to exhibit NLC is also specified with the aim of providing a more accurate estimate of how the negative compressibility will arise in particular applications. According to Grima *et al.*'s previous work,[22] we put more emphasis on the variations of compressibility and we also discuss the arising of NLC behaviour from mathematical aspects in an attempt to better explain this negative behaviour. All of these are detailed in the following sections, and the rest of the paper is organized as follows. In Section 2, the model is introduced and the equations for the compressibility properties are given. In Section 3, the results are presented and discussed. Finally some conclusions are drawn from the present study in Section 4.

2. Analytical modelling of linear compressibilities

It is well known that the linear compressibilities of a model are related to the Young moduli and the Poisson ratios. To be more specific, when a model is subjected to a hydrostatic pressure at its boundaries, its on-axis linear compressibility in the OX_i $(i = 1,2)$ direction can be given by:

$$\beta_{ii} = \frac{1}{E_i} - \frac{\nu_{ji}}{E_j}. \tag{1}$$

This expression clearly indicates that the on-axis linear compressibility can have negative values in the case where the Poisson ratio is highly positive so that $\nu_{ji} > E_j/E_i$. And this condition is more likely to be satisfied as $E_i \to +\infty$, $\nu_{ji} \to +\infty$, and or $E_j \to 0$. From Ref. [22] we can find that the Poisson ratio of the model consisting of rigid scalene triangles can have a highly positive value, more than that in a certain condition where the Young moduli can also tend to be infinite. All of these clearly indicate that this model has the potential to exhibit NLC, so we can now turn our attention to the model compressibility with the aim of finding out the conditions required for this special property.

As shown in Fig. 1, the model consists of two non-equivalent rigid scalene triangles with sides a_1, b_1, c_1, and a_2, b_2, c_2, respectively, with corresponding interior angles α_1, β_1, γ_1, and α_2, β_2, γ_2. These triangles are connected through simple flexure hinges, i.e., angle φ at point A, angle ω at point B, and angle θ at point C. For the convenience of reference, this system is denoted by $[a_1 \times b_1 \times c_1, a_2 \times b_2 \times c_2]$.

Using the equations derived for the Young moduli and Poisson ratios from Ref. [22] we can obtain the on-axis linear compressibilities for this model as follows:

$$\begin{aligned}\beta_{11} = {} & \frac{z}{24k_h l_1^3 l_2^5 \sin^3\alpha_{12}} \\ & \cdot \begin{pmatrix} a_1 a_2 \left(l_1^2 + l_2^2 - l_3^2\right) \sin(\theta + \gamma_1) \\ + b_1 b_2 \left(l_1^2 - l_2^2 + l_3^2\right) \sin(\theta + \gamma_2) \\ + c_1 c_2 \left(l_1^2 - l_2^2 - l_3^2\right) \sin(\theta - \alpha_1 - \beta_2) \end{pmatrix} \\ & \cdot \begin{pmatrix} 2a_1 a_2 l_2^2 \left(l_1^2 + l_2^2 - l_3^2\right) \sin(\theta + \gamma_1) \\ + b_1 b_2 \left(l_1^2 + l_2^2 - l_3^2\right) \left(l_1^2 - l_2^2 - l_3^2\right) \sin(\theta + \gamma_2) \\ + 2c_1 c_2 l_2^2 \left(l_1^2 - l_2^2 - l_3^2\right) \sin(\theta - \alpha_1 - \beta_2) \end{pmatrix}, \end{aligned} \tag{2}$$

$$\beta_{22} = \frac{zb_1b_2\sin(\theta+\gamma_2)}{6k_hl_1l_2^3\sin\alpha_{12}} \cdot \begin{pmatrix} a_1a_2\left(l_1^2+l_2^2-l_3^2\right)\sin(\theta+\gamma_1) \\ +b_1b_2\left(l_1^2-l_2^2+l_3^2\right)\sin(\theta+\gamma_2) \\ +c_1c_2\left(l_1^2-l_2^2-l_3^2\right)\sin(\theta-\alpha_1-\beta_2) \end{pmatrix}. \quad (3)$$

Then these equations can be used to obtain the equation for area compressibility, and because the process of simplification is only based on a mathematic method, to avoid wasting space, we give the simplified equation for β_A directly as follows:

$$\beta_A = \beta_{11}+\beta_{22} = \frac{z}{12k_hl_1^3l_2^3\sin^3\alpha_{12}} \cdot \begin{pmatrix} a_1a_2\left(l_1^2+l_2^2-l_3^2\right)\sin(\theta+\gamma_1) \\ +b_1b_2\left(l_1^2-l_2^2+l_3^2\right)\sin(\theta+\gamma_2) \\ +c_1c_2\left(l_1^2-l_2^2-l_3^2\right)\sin(\theta-\alpha_1-\beta_2) \end{pmatrix}^2, \quad (4)$$

where z is the thickness of the model, l_1 and l_2 are the side lengths of a parallelogramic unit cell, α_{12} is the included angle of the unit cell, l_3 is the diagonal of the unit cell opposite to angle α_{12} as illustrated in Fig. 1(b) and k_h is the stiffness constant of the flexure hinge.

Then we can derive the expression for the off-axis linear compressibility. The expression for β_L is defined as[1]

$$\beta_L = -\frac{1}{L}\left(\frac{\partial L}{\partial P}\right), \quad (5)$$

which can be re-written in terms of $\mathrm{d}\varepsilon_L$, i.e., the strain in the direction of L, as

$$\beta_L = -\frac{\mathrm{d}\varepsilon_L}{\mathrm{d}P}. \quad (6)$$

So the off-axis linear compressibility of this model when measured at an angle ξ with respect to the OX_1 axis can be obtained by using the standard axis transformation technique as follows:[33]

$$\beta_L(\xi) = -\left[\frac{\mathrm{d}\varepsilon_1}{\mathrm{d}P}\cos^2\xi + \frac{\mathrm{d}\gamma}{\mathrm{d}P}\sin\xi\cos\xi + \frac{\mathrm{d}\varepsilon_2}{\mathrm{d}P}\sin^2\xi\right], \quad (7)$$

i.e.

$$\beta_L(\xi) = \beta_{11}\cos^2\xi - \frac{\mathrm{d}\gamma}{\mathrm{d}P}\sin\xi\cos\xi + \beta_{22}\sin^2\xi, \quad (8)$$

where the term $\mathrm{d}\gamma$ is the shear strain and the $\mathrm{d}\gamma/\mathrm{d}P$ can also be given by

$$\frac{\mathrm{d}\gamma}{\mathrm{d}P} = \frac{z}{6k_hl_1^2l_2^4\sin^2\alpha_{12}} \cdot \begin{pmatrix} a_1a_2\left(l_1^2+l_2^2-l_3^2\right)\sin(\theta+\gamma_1) \\ +b_1b_2\left(l_1^2-l_2^2+l_3^2\right)\sin(\theta+\gamma_2) \\ +c_1c_2\left(l_1^2-l_2^2-l_3^2\right)\sin(\theta-\alpha_1-\beta_2) \end{pmatrix} \cdot \begin{pmatrix} a_1a_2l_2^2\sin(\theta+\gamma_1) \\ +b_1b_2\left(l_1^2-l_3^2\right)\sin(\theta+\gamma_2) \\ +c_1c_2l_2^2\sin(\theta-\alpha_1-\beta_2) \end{pmatrix}. \quad (9)$$

From Ref. [22] the equations for l_1, l_2, l_3, α_{12}, $\mathrm{d}\varepsilon_1$, $\mathrm{d}\varepsilon_2$, and $\mathrm{d}\gamma$ can be obtained directly as follows:

$$l_1 = \sqrt{c_1^2+c_2^2+2c_1c_2\cos(\theta-\alpha_1-\beta_2)}, \quad (10)$$

$$l_2 = \sqrt{b_1^2+b_2^2-2b_1b_2\cos(\theta+\gamma_2)}, \quad (11)$$

$$l_3 = \sqrt{a_1^2+a_2^2-2a_1a_2\cos(\theta+\gamma_1)}, \quad (12)$$

$$\alpha_{12} = \cos^{-1}\left(\frac{l_1^2+l_2^2-l_3^2}{2l_1l_2}\right), \quad (13)$$

$$\mathrm{d}\varepsilon_1 = \frac{1}{4l_1^2l_2^4\sin^2\alpha_{12}} \times \begin{pmatrix} 2a_1a_2l_2^2\left(l_1^2+l_2^2-l_3^2\right)\sin(\theta+\gamma_1) \\ +b_1b_2\left(l_1^2+l_2^2-l_3^2\right)\left(l_1^2-l_2^2-l_3^2\right)\sin(\theta+\gamma_2) \\ +2c_1c_2l_2^2\left(l_1^2-l_2^2-l_3^2\right)\sin(\theta-\alpha_1-\beta_2) \end{pmatrix}\mathrm{d}\theta,$$

$$\mathrm{d}\varepsilon_2 = \frac{b_1b_2}{l_2^2}\sin(\theta+\gamma_2)\,\mathrm{d}\theta, \quad (14)$$

$$\mathrm{d}\gamma = -\frac{1}{l_1l_2^3\sin\alpha_{12}} \begin{pmatrix} a_1a_2l_2^2\sin(\theta+\gamma_1) \\ +b_1b_2\left(l_1^2-l_3^2\right)\sin(\theta+\gamma_2) \\ +c_1c_2l_2^2\sin(\theta-\alpha_1-\beta_2) \end{pmatrix}\mathrm{d}\theta. \quad (15)$$

Here it should be noted that for a given set of rigid triangles the parameters a_1, b_1, c_1, a_2, b_2, c_2, α_1, β_1, γ_1, α_2, β_2, and γ_2 are all kept constant with the result that the compressibility properties of the model are only dependent on the variable θ and the angle ξ. It should also be noted that for physically realistic systems, the triangles can never overlap so that for any given parameters, the angle θ is restricted to have values in a range between $\max\{0^\circ, \beta_2-\gamma_1, \alpha_1-\gamma_2\} \le \theta \le \min\{360^\circ-\gamma_1-\gamma_2, 180^\circ+\alpha_1, 180^\circ+\beta_2\}$.

3. Discussion

Equations (2), (3), and (8) and figures 2–6 suggest that in general the values of the linear compressibility for these systems can be positive, zero or negative, with the actual sign and magnitude being dependent on the geometry of the system (i.e., the alignment and dimensions of the rotating triangles as well as the angles between them) and the actual direction along which the linear compressibility is measured. It is very significant due to not only the fact that NLC is a very unusual but highly useful property, but also the fact that there are various materials,[27–32] which can be described through the model analyzed here, being potential candidates for NLC materials.

Plots of Young's moduli, Poisson's ratios and compressibilities corresponding to a system having dimensions $[a_1\times b_1\times c_1, a_2\times b_2\times c_2] = [3\times 8\times 7, 9\times 6\times 5]$ are shown in Fig. 2. It is clearly shown that the system having an irregular structure can indeed exhibit negative linear compressibility. From a mathematical perspective, we can find that the linear compressibility in the OX_1 direction, β_{11} becomes negative through a continuous transition when $\theta \approx 66.55^\circ$,

a value that makes Eq. (2) equal to zero and then changes sign again when $\theta \approx 103.98^\circ$, which corresponds to the other root of $\beta_{11} = 0$. As illustrated in Fig. 2, in a range where $66.55^\circ < \theta < 103.98^\circ$, the value of E_1 is highly positive while the value of E_2 is quite small, which is beneficial to the arising of NLC in view of the fact that NLC can be exhibited in the OX_1 direction when $\nu_{21} > E_2/E_1$ is fulfilled. In a range where $103.98^\circ < \theta < 148.41^\circ$, the value of E_2 increases rapidly and approaches to infinity when $\theta \approx 148.41^\circ$ while the value of E_1 approaches to zero in an asymptotic manner so that NLC can also be observed in the OX_2 direction over this range. Note also that the ranges of NLC in different on-axis directions have an identical bounding angle and this can be easily inferred by looking at the expressions for β_{11} and β_{22}, which share the same numerator. It should also be noted that in this case if one solves the equation $\beta_A = 0$, one would find that the expression has a double root when $\theta \approx 103.98^\circ$, that is, the area compressibility is always positive except when the earlier-mentioned equation is satisfied. In order to better understand the deformation of this model, we can analyze it in a visualized way. As clearly shown in Fig. 3, systems with different values of θ may deform in different manners and result in different compressibility properties. It should be noted that in systems where θ is less than 103.98° (i.e., the angle corresponding to $\beta_{11} = \beta_{22} = 0$), hydrostatic pressure results in a decrease in angle θ while systems with θ being greater are on the contrary. Furthermore, as illustrated in Figs. 3(a)–3(b), the structure with $\theta < 66.55^\circ$ may become closer when the angle decreases, however, when θ is in a region where $66.55^\circ < \theta < 103.98^\circ$, the decrease in angle would result in the expansion along the OX_1 direction, thus NLC may arise. Similarly, in the cases where $103.98^\circ < \theta < 148.41^\circ$, NLC can be exhibited along the OX_2 direction and the on-axis linear compressibility will become positive again once θ is greater than 148.41°.

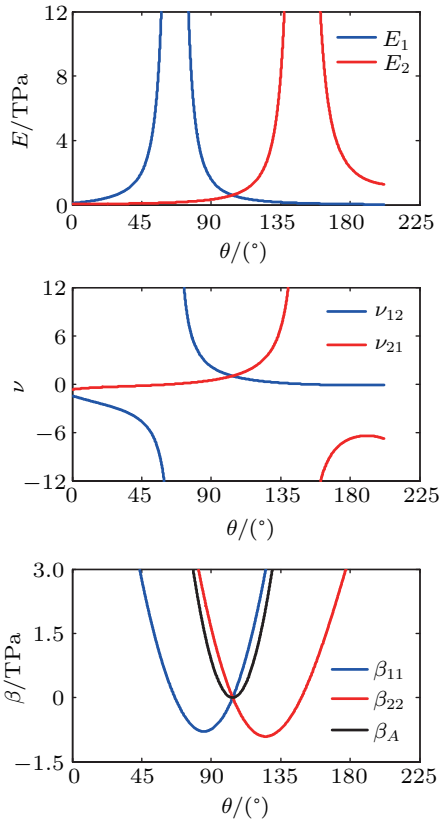


Fig. 2. (color online) Plots of Young's moduli, Poisson's ratios, and compressibilities for system denoted by $[a_1 \times b_1 \times c_1, a_2 \times b_2 \times c_2] = [3 \times 8 \times 7, 9 \times 6 \times 5]$ with $k_h = 1$ kJ·rad^{-2} and $z = 1$ mm. The other parameters (including a_1, b_1, c_1, a_2, b_2, and c_2) are all in units of mm.

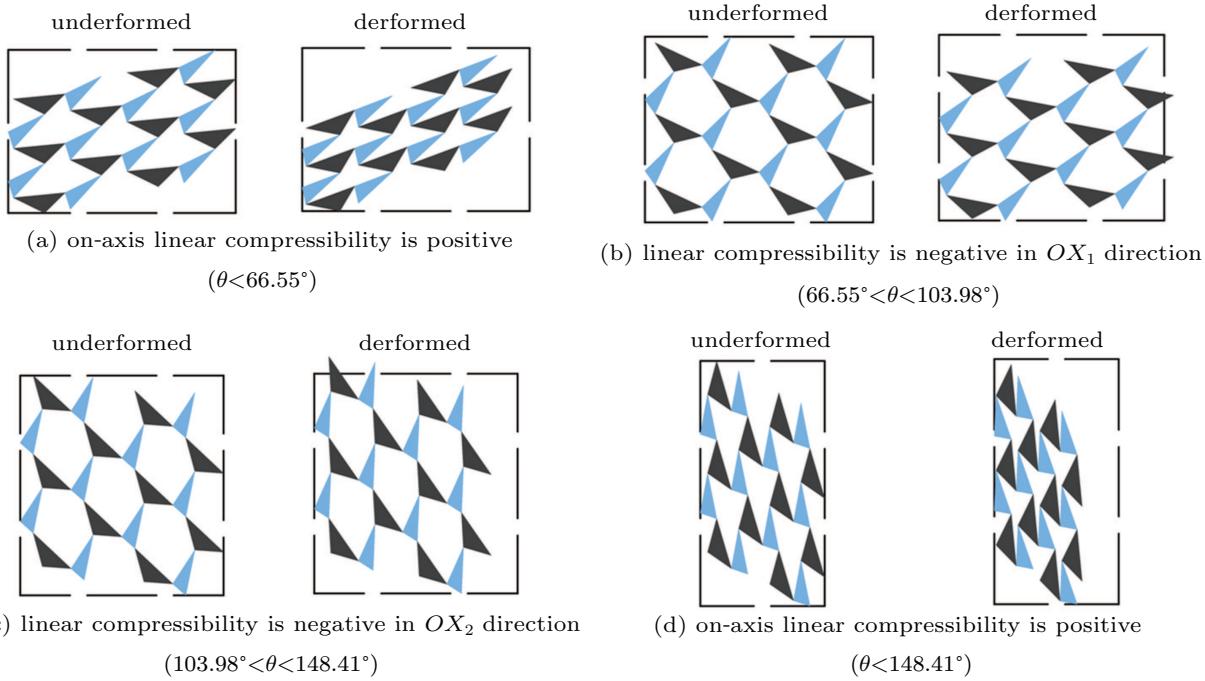


Fig. 3. (color online) Different deformations of system $[3 \times 8 \times 7, 9 \times 6 \times 5]$ with different values of θ when subjected to a hydrostatic pressure.

Although the behaviours illustrated in Figs. 2 and 3 are typical for systems where $a_1 \neq b_1 \neq c_1 \neq a_2 \neq b_2 \neq c_2$, not all systems behave in this manner. If we analyze the simplified system consisting of two congruent equilateral triangles, we can find that the Poisson ratio is equal to -1 for all values of angle θ so that the on-axis linear compressibility can never have negative values. Based on the previous work by Grima *et al.*[22], we can note that systems consisting of two similar equilateral triangles or even two similar scalene triangles denoted by $[a\times b\times c,\ ka\times kb\times kc]$ have similar properties that these isotropic systems can always exhibit a negative Poisson ratio of -1 and on-axis linear compressibilities of $2/E$ so that NLC can never be exhibited in the OX_1 or OX_2 direction. All this is clearly shown in Table 1. Here it should be highlighted that the mechanical properties including Young's moduli, Poisson's ratios and linear compressibilities are dependent on not only the dimensions but also the arrangements of the triangles, and thus, for example, a system consisting of two similar triangles denoted by $[a\times b\times c,\ kb\times ka\times kc]$ may have entirely different properties from the system $[a\times b\times c,\ ka\times kb\times kc]$.

Table 1. The Young moduli, Poisson ratios, and linear compressibilities.

System	Poisson ratios	Young moduli	Compressibilities
$[a\times a\times a, a\times a\times a]$	$\nu_{12}=\nu_{21}=-1$	$E_1=E_2=\dfrac{4\sqrt{3}k_h}{a^2z[1+\cos(\theta+60^\circ)]}$	$\beta_{11}=\beta_{22}=\dfrac{a^2z[1+\cos(\theta+60^\circ)]}{2\sqrt{3}k_h}$
$[a\times a\times a, ka\times ka\times ka]$	$\nu_{12}=\nu_{21}=-1$	$E_1=E_2=\dfrac{2\sqrt{3}k_h[1+k^2-2k\cos(\theta+60^\circ)]}{k^2a^2z\sin^2(\theta+60^\circ)}$	$\beta_{11}=\beta_{22}=\dfrac{k^2a^2z\sin^2(\theta+60^\circ)}{\sqrt{3}k_h[1+k^2-2k\cos(\theta+60^\circ)]}$
$[a\times b\times c, ka\times kb\times kc]$	$\nu_{12}=\nu_{21}=-1$	$E_1=E_2=\begin{pmatrix}\dfrac{6k_h[1+k^2-2k\cos(\theta+\gamma_1)]}{k^2z\sin^2(\theta+\gamma_1)}\\ \cdot(a-b+c)^{-1/2}\cdot(a-b+c)^{-1/2}\\ \cdot(a-b+c)^{-1/2}\cdot(a-b+c)^{-1/2}\end{pmatrix}$	$\beta_{11}=\beta_{22}=\begin{pmatrix}\dfrac{k^2z\sin^2(\theta+\gamma_1)}{3k_h[1+k^2-2k\cos(\theta+\gamma_1)]}\\ \cdot(a-b+c)^{1/2}\cdot(a+b-c)^{1/2}\\ \cdot(a+b+c)^{1/2}\cdot(-a+b+c)^{1/2}\end{pmatrix}$

Also of interest are systems consisting of isosceles triangles, and according to the results presented above we have no need to discuss the isotropic cases including systems $[a\times a\times b,\ a\times a\times b]$, $[a\times b\times a,\ a\times b\times a]$ and $[b\times a\times a,\ b\times a\times a]$. So we consider the following three cases: $[a\times a\times b,\ a\times b\times a]$, $[a\times a\times b,\ b\times a\times a]$ and $[a\times b\times a,\ b\times a\times a]$. In order to figure out the influence of relative magnitudes of the side lengths on the on-axis linear compressibilities, we introduce the parameter r, which is equal to b/a. It should be noted that the ratio r of the side lengths must be in a range of $0<r<2$, since otherwise the side lengths would not form a triangle.

Plots of on-axis linear compressibilities against θ for various values of r are shown in Fig. 4. It is evident that on changing the shape of the triangles (i.e., values of r), the on-axis linear compressibilities change accordingly. More specifically, the effect of negative linear compressibility can be maximized (i.e., increasing the magnitude of NLC and widening the range of θ where NLC is exhibited) by increasing the value of $|r-1|$ and particularly when $r=1$, which corresponds to an equilateral triangle geometry, negative on-axis linear compressibility can never be observed. As discussed above, NLC is more likely to be observed when $E_i\to+\infty$, $\nu_{ji}\to+\infty$, and or $E_j\to 0$, which may reveal a fact that a large anisotropy of the system is helpful in generating the NLC. So the behaviours as clearly illustrated in Fig. 4 can be explained by the fact that increasing the divergence of r from 1 can lead to a greater anisotropy of the system so that the effect of NLC is enhanced. However, for system $[a\times a\times b,\ a\times b\times a]$ or $[a\times b\times a,\ b\times a\times a]$, although the effect of NLC can increase as $r\to 0$, it is not so obvious as $r\to 2$. But for system $[a\times a\times b,\ b\times a\times a]$ both increasing and reducing the ratio r from 1 are both very effective methods to enhance this effect. In particular, this system will become more similar to the wine-rack-like structure as r becomes closer to 0, thus in the limit that $r\to 0$ (i.e., $b\ll a$), the expressions for on-axis linear compressibilities given by Eqs. (2) and (3) become

$$\beta_{11}=\frac{za^2}{6k_h}\cos(\theta-90^\circ)\tan\left(\frac{\theta-90^\circ}{2}\right), \tag{16}$$

$$\beta_{22}=-\frac{za^2}{6k_h}\cos(\theta-90^\circ)\cot\left(\frac{\theta-90^\circ}{2}\right), \tag{17}$$

which are equations that correspond to those of the wine-rack-like structure after recognising the differences in the definitions of the stiffness constants, angles and length parameters. A look at the plots shown in Fig. 4 may also reveal a fact that the on-axis linear compressibilities afforded by a system are highly dependent on the form in which the system exists and arranging the isosceles triangles in the same manner as system $[a\times a\times b, b\times a\times a]$ may be much more conductive to NLC. Here we should highlight that this kind of system consisting of congruent isoscele triangles is a very comprehensive model not only due to the fact that NLC can always be observed in the OX_i $(i=1,2)$ directions except when $r=1$, but also for its versatility which has the potential to exhibit

tailored negative linear compressibilities when designed in a specific geometric conformation. Also, it is evident from the plots in Fig. 4 that for the whole range of θ-values these systems exhibit NLC along the OX_1 or OX_2 direction but never in both directions simultaneously. This ensures that area compressibility is never negative so that although one dimension may increase in size, the area of the whole system always decreases when subjected to a hydrostatic pressure. On further examination of the plots, it can be observed that system $[a\times a\times b, a\times b\times a]$ has the same linear compressibility properties as system $[a\times b\times a, b\times a\times a]$. In order to explain this phenomenon, we can analyze it in a visualized way. As shown in Fig. 5, the two structures will become chiral when $\theta=\theta'$ so that the on-axis linear compressibilities of the two systems are always equal.

Also it is possible to show from Eqs. (8) and (9) that maximal and minimal linear compressibilities for these systems do not always occur on-axis so it is necessary to consider the off-axis linear compressibility. The off-axis plots that correspond to the system illustrated in Fig. 2 are shown in Fig. 6. It is clearly shown that NLC can indeed be exhibited in the off-axis direction. As illustrated in Fig. 2, this system does not exhibit any on-axis NLC when $\theta=150^\circ$, but from Fig. 6 we can know that the linear compressibility can be negative when it is measured in a proper direction, for example when $\xi=101.3^\circ$. This means that some systems with positive linear compressibilities in the OX_i $(i=1,2)$ directions may also have the possibility to exhibit negative off-axis linear compressibility just like the above-mentioned case. For systems with NLC in the OX_i $(i=1,2)$ directions, the effect of this special property can also be maximized by finding a better angle. From Eq. (7) we can obtain the expression for $\xi_{\max/\min}$, the angle where $\beta_L(\xi)$ is at a maximum/minimum:

$$\xi_{\max/\min}=\frac{1}{2}\tan^{-1}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\varepsilon_1-\mathrm{d}\varepsilon_2}\right). \tag{18}$$

It gives us a useful inspiration that if a system did not exhibit any NLC properties at $\xi_{\min}$, then it would not exhibit NLC in any other directions. From this perspective, it should be highlighted that the linear compressibility depends not only on the geometry of the system but also on the actual direction along which the linear compressibility is measured. Particularly, the shear strain (i.e., $\mathrm{d}\gamma$) of a system consisting of two similar equilateral triangles or even two similar scalene triangles denoted by $[a\times b\times c, ka\times kb\times kc]$ is always equal to zero, which means that the maximal and minimal linear compressibilities for these systems always occur on axis. As we discussed above, none of these systems exhibits NLC in the OX_i $(i=1,2)$ directions, so we know that the linear compressibility of the system denoted by $[a\times b\times c, ka\times kb\times kc]$ is always positive.

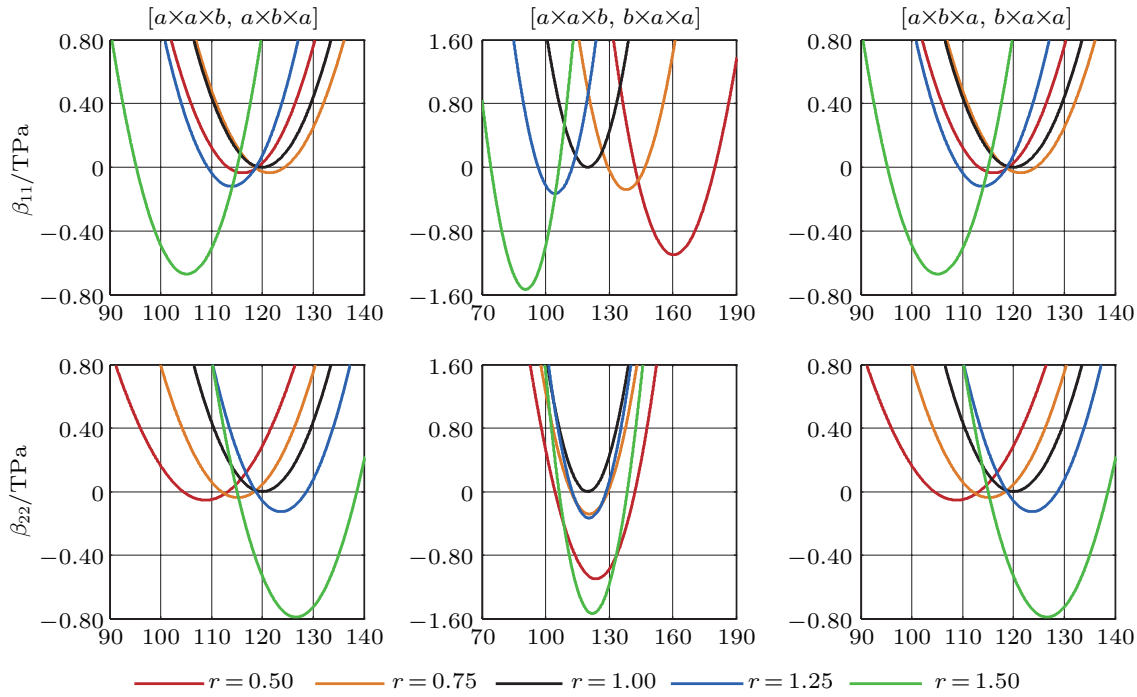


Fig. 4. (color online) Plots of on-axis linear compressibilities against θ for various values of r with $k_h=1$ kJ·rad^{-2}, $a=10$ mm, and $z=1$ mm.

Before concluding, it is necessary to highlight some important aspects of the model presented here. First of all, the model analyzed here is not regarded as an asymmetric system tacitly, and this highly generic model, which is built from scalene triangles and analyzed with the aim of making the formulas derived from this model, can be widely adaptive. It means that the structures can be symmetric or asymmetric with geometric parameters (including a_1, b_1, c_1, a_2, b_2, and

c_2) taking different values, but in both cases the accuracy of these theoretical formulas cannot be affected. From Eqs. (2) and (3), it can be inferred that both symmetric and asymmetric structures can exhibit large or tiny on-axis NLC by taking different values of θ. Also it should be noted that the mechanism responsible for NLC working, requires that the pressure is exerted on the system from outside just like the case in the methanol monohydrate system studied by Fortes *et al.*[2] Furthermore, the model works on the assumption that the triangles are perfectly rigid in the deformation process. However, we can note that such an idealized behaviour may be unlikely to be observed in real conditions and the rotation of the triangle would probably be accompanied with a change in their shape, something in real scenarios may diminish the effect of NLC. Also, it is important to highlight that the compressibility properties discussed above are only valid for a small change of pressure. Finally, it should be noted that although two-dimensional models have their limitations when compared with three-dimensional models, their superiorities lie in the fact that they are simpler to analyze and also they are adequate enough to predict the behaviour of particular projections of a complex 3D model where the linear compressibility is measured. Although the results presented here are subject to some limited conditions, it should be emphasised that they provide us with a new idea that NLC can also be achieved through models consisting of rigid scalene triangles which are usually closely associated with auxeticity.

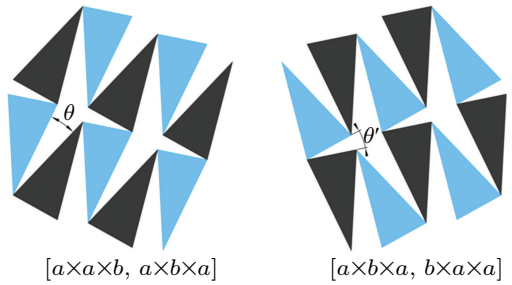


Fig. 5. (color online) Structures of two systems denoted by $[a\times a\times b, a\times b\times a]$, $[a\times b\times a, b\times a\times a]$.

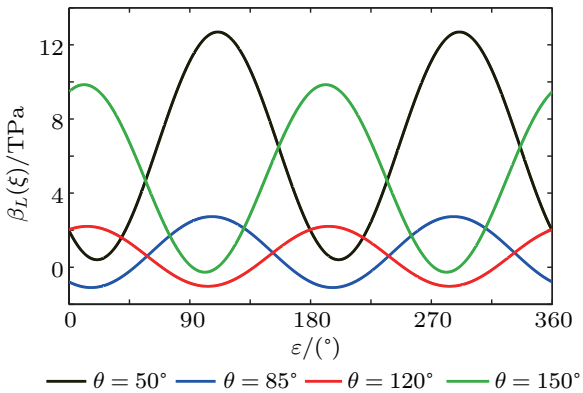


Fig. 6. (color online) Plots of off-axis linear compressibility for system $[3\times 8\times 7, 9\times 6\times 5]$ illustrated in Fig. 2 where $k_h = 1$ kJ·rad^{-2} and $z = 1$ mm. The other parameters (including a_1, b_1, c_1, a_2, b_2, and c_2) are in units of mm.

4. Conclusions

In this paper, based on Grima *et al.*'s work, the compressibility properties of the systems consisting of rigid scalene triangles are analyzed and discussed. It is shown that these systems which are usually associated with auxeticity can exhibit negative linear compressibilities in the case where the systems are highly anisotropic and have appropriate geometric features. In particular, through analytically modelling a general case and some particular cases including the systems consisting of equilateral triangles, similar triangles and isosceles triangles, we find that the magnitude and range of on-axis NLC can be affected by the dimension of the triangle and the alignment of these rotating units as well as the angle included between them. More specifically, we find that the system consisting of equilateral triangles or similar scalene triangles denoted by $[a\times b\times c, ka\times kb\times kc]$ can never exhibit NLC because of its isotropic geometry, while the system consisting of isosceles triangles can exhibit this special property and the effect of on-axis NLC can be effectively enhanced by increasing the value of $|r-1|$ which can lead to a greater anisotropy. All of these clearly suggest that a high degree of anisotropy is very conducive to the generation of NLC. Also off-axis linear compressibility has been analyzed and discussed and we find that the linear compressibility depends not only on the geometry of the system but also on the actual direction along which the linear compressibility is measured.

Given the importance of this work, it is hoped that the findings obtained here will serve as a blueprint that can be used to either design new materials or widen the search for materials which exhibit NLC.

References

[1] Baughman R H, Stafstrom S, Cui C and Pantas S O 1998 *Science* **279** 1522
[2] Fortes A D, Suard E and Knight K S 2011 *Science* **331** 742
[3] Mariathasan J W E, Finger L W and Hazen R M 1985 *Acta Cryst. B* **41** 179
[4] Goodwin A L, Keen D A and Tucker M G 2008 *Proc. Natl. Acad. Sci. USA* **105** 18708
[5] Skelton E F, Feldman J L, Liu C Y and Spain I L 1976 *Phys. Rev. B* **13** 2605
[6] Millange F, Serre C and Férey G 2002 *Chem. Commun.* 822
[7] Grima J N, Attard D, Caruana-Gauci R and Gatt R 2011 *Scr. Mater.* **65** 565
[8] Barnes D L, Miller W, Evans K E and Marmier A 2012 *Mech. Mater.* **46** 123
[9] Grima J N, Attard D and Gatt R 2008 *Phys. Stat. Sol.* **245** 2405
[10] Gatt R and Grima J N 2008 *Phys. Stat. Sol. (RRL)* **2** 236
[11] Gatt R, Attard D andGrima J N 2009 *Scr. Mater.* **60** 65
[12] Jiang X X, Luo S Y, Kang L, Gong P F, Yao W J, Huang H W, Lo W, Huang R J, Wang W, Li Y C, Li X D, Wu X, Lu P X, Li L F, Chen C T and Lin Z S 2015 *Adv. Mater.* **27** 4851
[13] Cairns A B and Goodwin A L 2015 *Phys. Chem. Chem. Phys.* **17** 20449
[14] Miller W, Evans K E and Marmier A 2015 *Appl. Phys. Lett.* **106** 231903
[15] Zhang J M, Zhang Y, Xu K W and Ji V 2008 *Chin. Phys. B* **17** 1565
[16] Wang J F, Fu X N, Zhang X D, Wang J T, Li X D and Jiang Z Y 2016 *Chin. Phys. B* **25** 086302

[17] Hu C H, Yin X H, Wang D H, Zhong Y, Zhou H Y and Rao G H 2016 *Chin. Phys. B* **25** 067801
[18] Grima J N, Gatt R, Alderson A and Evans K E 2005 *J. Phys. Soc.* **74** 2866
[19] Grima J N and Evans K E 2006 *J. Mater. Sci.* **41** 3193
[20] Grima J N, Gatt R, Ellul B and Chetcuti R 2010 *J. Non-Cryst. Sol.* **356** 1980
[21] Grima J N and Evans K E 2000 *J. Mater. Sci. Lett.* **19** 1563
[22] Grima J N, Chetcuti E, Manicaro E, Attard D, Camilleri M, Gatt R and Evans K E 2012 *Proc. R. Soc. A* **468** 810
[23] Milton G W and Cherkaev A V 1995 *J. Eng. Mater. Technol.* **117** 483.
[24] Milton G W 2013 *J. Mech. Phys. Solids* **61** 1543
[25] Dudek K K, Attard D, Caruana-Gauci R, Wojciechowski K W and Grima J N 2016 *Smart Mater. Struct.* **25** 025009
[26] Attard D, Caruana-Gauci R, Gatt R, and Grima J N 2016 *Phys. Stat. Sol.* **253** 1410
[27] Bianchi M, Scarpa F and Smith C W 2010 *Acta Mater.* **58** 858
[28] Bianchi M, Scarpa F and Smith C W 2011 *Acta Mater.* **59** 686
[29] Grima J N, Jackson R, Alderson A and Evans K E 2000 *Adv. Mater.* **12** 1912
[30] Grima J N, Gatt R, Ravirala N, Alderson A and Evans K E 2006 *Mat. Sci. Eng. A* **423** 214
[31] Grima J N, Attard D, Gatt R and Cassar R N 2009 *Adv. Eng. Mater.* **11** 533
[32] Lakes R S 1987 *Science* **235** 1038
[33] Nye J F 1957 *Physical Properties of Crystals* (New York: Oxford University Press) pp. 33–49