Studies on convergence and scaling law of Thomson backscattering spectra in strong fields*

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With the saddle point analysis method for the Bessel function structure and property, the convergence problem and the scaling laws of Thomson backscattering spectra are solved and studied in both cases that are for the plane wave laser field without and with applied external constant magnetic field. Some unclear points appeared in previous work are clarified. The extension of the method to a general situation for the laser field with an arbitrary polarization is discussed. We also make a simple analysis and discussion about the optimal spectra dependence of field parameters and its implication to practical applications.

Keywords: radiation by moving charges, Thomson scattering spectra, frequency conversion and harmonic generation, scaling law

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1. Introduction

In recent decades, a wide range of applications and the continuous development of ultra-high intensity lasers have made great success in the new field of strong field physics.[1–3] Many regimes that were considered far from now gradually became much closer to reality. The principle and technology are employed in many ways, such as table-top electron and proton accelerators,[4–6] advanced x-ray sources,[7,8] medical isotope production, ultrafast imaging, laser surgery, and materials treatment.[9,10] For example, for electron acceleration, the bubble acceleration mechanism has been researched[11–14] and has found a potential valuable application in the production of femtosecond x-rays from laser-plasma accelerators.[15] Upon the proton/ion acceleration it is found that the scaling laws of maximum ion energy have been achieved and the dependence of the scaling coefficients on laser intensity, pulse duration, and target thickness have been obtained.[16] Recently, moreover, the particle-in-cell simulation of x-ray wakefield acceleration and betatron radiation in nanotubes are performed[17] and the γ-ray flash generation in near-critical-density target irradiated by four symmetrical colliding laser pulses is also studied.[18]

Among many various generations of the high-order harmonics radiation, the Thomson scattering has attracted many researchers attention because it is a useful way to produce x-ray and/or THz radiation source.[19–34] Note that a review article[19] has summarized theoretical as well as the experimental aspects on x-ray Thomson scattering in high energy density plasmas until to 2009. In Thomson scattering, an electron initially acquires relativistic velocities in the fields of high-intensity laser and through the relativistic motion the electron emits radiation, so Thomson scattering spectra are produced. With regard to the spectra, they have been studied extensively in both cases of without and with an applied external magnetic field.

For the case of without external magnetic field, the Thomson scattering for linearly and/or circularly polarized laser field has been analyzed and discussed in Refs. [25], [26], and [27]. In Ref. [25] He *et al*. studied a case of linearly polarized and found that the phase depended on the result of radiation spectra. Additionally, they presented the solution of the backscatter spectrum for arbitrary laser intensity and arbitrary electron energy. For the case of with external magnetic field, the corresponding Thomson scattering problems have been also studied by many researchers.[28–33] For example, for circularly polarized laser field with magnetic field, it has been analyzed and discussed in our previous work.[33] It is found that there exists scale invariance for the radiation spectra in terms of harmonic orders. Meanwhile the scaling law of backscattered spectra exists remarkably when the electron–cyclotron frequency approaches the laser frequency. The results indicate that the magnetic resonance parameter plays an important role in emission intensity. Recently we also extended our study to the situation of magnetic field and laser field with an arbitrary polarization. The rich features of Thomson backscattering spectra may be useful to the tunability of the radiation

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source and even to gain the THz one.[33–35]

By analyzing the Thomson spectra generated by an electron moving in a strong laser field with or without a constant magnetic field, many researchers have performed a number of analytical treatments and numerical calculations. But for the convergence of spectra, i.e., whether the amplitude of the m-th harmonics in spectra approaches to zero when m tends to infinity, is still an open problem until now. People know that from the publications, it seems to be convergent for the presentations of extensive digital results, see Refs. [25]–[29], however, a serious research and discussion on it is necessary and conducive to a better understanding of spectra. Moreover, this consideration allows one to obtain the optimal spectra characteristics, as well as the scaling laws between the spectral intensity and applied field parameters.

Therefore, in this paper the saddle point analysis method is used to solve the structure and properties of the Bessel function, namely, to solve the convergence of Thomson backscattering spectra in the strong laser field in the presence and/or absence of an external constant magnetic field. We extended the idea and method to a general situation in the laser field with an arbitrary polarization. We have made a simple analysis and discussion on the optimal spectra and obtained scaling laws of emission spectra on laser intensity and the magnetic field strength.

This paper is organized as follows. In Section 2, we give the basic idea of solving the convergence problem by choosing a typical case of a linearly polarized laser field without external constant magnetic field. An extension to a circular and/or elliptically polarized laser field with external constant magnetic field is described in Section 3. We have a brief but important discussion on the optimal spectra and the scaling laws in Section 4. In the final Section 5 the main conclusions are summarized and some perspectives on the possible applications of the Thomson scattering are proposed.

2. Spectra convergence for linear polarized field

The laser electric field is measured by the dimensionless parameter, $a = eE_0/m_e\omega_0 c$, where E_0 is the electric field amplitude, ω_0 is the laser frequency, and m_e is the electron rest mass. Usually the radiation power depends on electron kinetic energy so that we denote $\gamma = (1-\beta_x^2-\beta_y^2-\beta_z^2)^{-1/2}$ as the relativistic mass factor with the normalized velocity of the electron as $\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)$ (in units of light speed). The electron orbit is subject to the following general initial conditions at time $t = 0$, $\beta_x = \beta_{x0}$, $\beta_y = \beta_{y0}$, $\beta_z = \beta_{z0}$. For the special case $\beta_{x0} = 0$, $\beta_{y0} = 0$, one has $\gamma_0 = (1-\beta_{z0}^2)^{-1/2}$. In the backscattering direction of the laser, when taking the standard derivative of radiation spectra, we obtain the power p_m, radiated at the harmonic frequency $\omega = m\omega_1$ ($m \in Z^+$, Z^+ are positive integers, ω_1 is the base frequency of the radiation spectrum which is associated to the period motion of electrons) per unit solid angle is as follows, refer to Eq. (5) in Ref. [25]

$$p_m = \frac{A}{\gamma_0^2(1-\beta_{z0})^2}\left(\frac{\omega_1}{\omega_0}\right)^4 s_m, \tag{1}$$

where $A = e^2\omega_0^2/4\pi^2 c$, $s_m = 0$ for $m = 2,4,\ldots$, and for $m = 1,3,5,\ldots$, they are

$$s_m = (a\pi)^2 m^2 [\mathrm{J}_{(m-1)/2}(m\xi) - \mathrm{J}_{(m+1)/2}(m\xi)]^2, \tag{2}$$

where $\xi = a^2/2(a^2+2)$.

Before studying the convergence of Eq. (2) we point out that in Ref. [25], the spectra are obtained by twice integrating the delta function of the frequency ω. However, if the fundamental frequency ω_1 is integrated, then the power index of m in s_m should be 1. Therefore we first consider this case and then the case of Eq. (2).

For the sake of convenience, we set $m = 2l+1$ ($l \in N$, N are natural numbers), then

$$s_{2l+1} \propto (2l+1)[\mathrm{J}_l(x) - \mathrm{J}_{l+1}(x)]^2, \tag{3}$$

where $x = (2l+1)\xi = (l+1/2)2\xi$. In Eq. (3), $\mathrm{J}_l(x)$ is the Bessel function of the first kind of order n. However, if we use the asymptotic expression of the Bessel function

$$\mathrm{J}_l(x) \sim \sqrt{2/\pi x}\cos(x - l\pi/2 - \pi/4)$$

when $x \to \infty$, we cannot conclude the convergence of spectra since s_m approaches to a form of like $\cos^2[\cdots]$ with a certain nonzero amplitude when $m \to \infty$.

In the following, some other valid methods are used to figure out whether s_m is convergent or not. To discuss the convergence of s_{2l+1}, we consider two cases to prove $\lim\limits_{l\to\infty} s_{2l+1} \to 0$.

Case 1: For $a \gg 1$, we have $\xi = 1/2$, $x = l+1/2$, and an integral expression of $\mathrm{J}_l(x)$ is given by,

$$\mathrm{J}_l(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(x\sin\theta - l\theta)}\,\mathrm{d}\theta. \tag{4}$$

Now by a famous saddle point analysis method for the integration of exponent function, we have

$$\mathrm{J}_l(x) - \mathrm{J}_{l+1}(x) \sim \sqrt{\frac{2}{\pi(l+1/2)}}\frac{l^{1/4}}{l^{1/2}}\cos\left[\left(l+\frac{1}{2}\right)\left(\sin\theta_0 - \theta_0\cos\theta_0 - \frac{\pi}{4} + \varphi\right)\right], \tag{5}$$

where θ_0 is determined by the saddle point equation $x\cos\theta_0 = l$, i.e., $\theta_0 = \arccos[l/(l+1/2)]$. Therefore, it is easy to obtain $[\mathrm{J}_l(x) - \mathrm{J}_{l+1}(x)]^2 \propto 1/[(l+1/2)l^{1/2}]$ and then when $l \to \infty$,

$$s_{2l+1} \propto (2l+1)[\mathrm{J}_l(x) - \mathrm{J}_{l+1}(x)]^2 \sim 1/\sqrt{l} \to 0. \tag{6}$$

We can also prove $\lim_{l\to\infty} s_{2l+1} \to 0$ with recursive formula of the Bessel function $x\mathrm{J}_l' - l\,\mathrm{J}_l = -x\,\mathrm{J}_{l+1}$, where $x = l+1/2 \approx l$. First we rewrite the recursive formula of the Bessel function as

$$\mathrm{J}_l - \mathrm{J}_{l+1} \approx \mathrm{J}_l' = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{i}\sin\theta\,\mathrm{e}^{\mathrm{i}(x\sin\theta - l\theta)}\,\mathrm{d}\theta. \tag{7}$$

Then also via saddle point analysis to this integral of Eq. (**??**), one obtains the same result as Eq. (6) when $l \to \infty$, and again one concludes the convergence of spectra. This indicates that both ways can be employed to prove the convergence and the results obtained in each way are the same.

Case 2: For any a, then $0 \le \xi \le 1/2$, we have $x = m\xi = (l+1/2)\,2\xi$. Now we divide the integral expression into two parts as follows:

$$\mathrm{J}_l(x) = \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}x(\sin\theta - \varepsilon\theta)}\,\mathrm{e}^{-\mathrm{i}(l - x\varepsilon)\theta}\,\mathrm{d}\theta, \tag{8}$$

where $\varepsilon = l/(l+1/2) \approx 1 - 1/2l$ when $l \gg 1$. Similar to the Case 1, we would obtain

$$\mathrm{J}_l(x) - \mathrm{J}_{l+1}(x) \sim \sqrt{\frac{2}{\pi(2l+1)\xi}}\frac{1}{l^{1/4}} \cdot \cos\left[\sqrt{l} + \frac{\pi}{4} - (1-2\xi)\sqrt{l}\right], \tag{9}$$

therefore, when $l \to \infty$, again we conclude

$$s_{2l+1} \propto \cos^2[2\xi\sqrt{l} + \pi/4]/\sqrt{l} \to 0.$$

Interestingly, when $\xi = 1/2$, the result of Case 2 is transformed to Case 1 so it means that Case 2 is a general situation.

Let us turn to focus on the convergence of spectra described in Eq. (2) with m^2 appearing in Ref. [25], which is gained by the integrals of the complete harmonic frequency but not the fundamental frequency. In fact, the way presented here can be extended to a more general situation in the next section, for example, the cases of circular and/or elliptic polarized laser field with external constant magnetic field. For Eq. (2), we use $m = 2l+1$ to rewrite it as

$$s_m \sim m^2\omega_1^2(\mathrm{J}_l(x) - \mathrm{J}_{l+1}(x))^2 \sim \left(l + \frac{1}{2}\right)^2[\mathrm{J}_l(x) - \mathrm{J}_{l+1}(x)]^2. \tag{10}$$

Then, with the same handling method as the Case 2 mentioned above, one obtains

$$\mathrm{J}_l(x) - \mathrm{J}_{l+1}(x) \sim \sqrt{\frac{2}{\pi(2l+1)\xi}}\frac{1}{l^{\alpha/4}} \cdot \cos\{[-\xi + (1-2\xi)l]l^{-\frac{\alpha}{2}} + \pi/4\}, \tag{11}$$

but here $\varepsilon = 1 - 1/2l^{\alpha}$ was used. Obviously $s_{2l+1} \to 0$ holds when $l \to \infty$ if $\alpha > 2$ is given. Therefore s_m would always be convergent no matter $p_m \propto m$ or $p_m \propto m^2$.

3. Spectra convergence for other cases of complex fields

Besides the case of without external magnetic field in Refs. [25]–[28], in our recent work,[33] we have meticulously researched the spectral of the Thomson backscattering in the combined circular polarized laser field and magnetic field. Analogously, the analytical expression of the radiation spectrum is given by

$$p_m \propto m^2\left(A_{(m,n)}^2 + B_{(m,n)}^2\right) \tag{12}$$

with

$$A_{(m,n)} = \mathrm{J}_{m-n}(z_m) - \mathrm{J}_{m-n-1}(z_m),$$
$$B_{(m,n)} = \mathrm{J}_{m+n}(z_m) - \mathrm{J}_{m+n+1}(z_m),$$

where $z_m = 2mn^3a^2/(n+2n^3a^2)$ is defined to simplify the formula of the radiation spectrum. It should be pointed out that equation (12) here comes from Eq. (23) of Ref. [33]. However, the correct index about the power of m is 2 not 4. The index 4 that the power of m appearing in the second bracket in Eq. (23) of Ref. [33] is a misprint due to carelessness.

Similar to the solving for the cases described in Section 2, it is easy to prove that when $\alpha > 2$ and $l \to \infty$, one gets $s_m = s_{2l+1} \to 0$. This is not surprising because according to the saddle point analysis, the Bessel function will contribute the power of m in the denominator with $1/2 + \alpha/4$ so that the overall power of m in the spectra is of the order of $2 - (1 + \alpha/2)$. Certainly we can always choose the $\alpha > 2$ which makes the spectra convergent when m approaches to infinity.

In fact, this idea can be extended to the more complex fields, for instance, a case of an arbitrary polarized laser field assisted with magnetic field like in Ref. [34]. Although it is very difficult to gain the concrete exact analytic expression of spectra in this general polarization case, it is reasonable to guess that the spectra contains still some integer power of order m. For example, the factor before the Bessel functions has a power I of m and the differences between two neighboring Bessel functions have a power J of m, then, the spectrum is a power of m with index $I - J(1/2 + \alpha/4)$. Again we would conclude that the spectra are always convergent if $\alpha > (4I - 2J)/J$. Certainly because $\alpha > 0$ must be satisfied, the convergence holds only when $I > J/2$. Fortunately this condition is always satisfied in all cases that we have considered in this paper.

4. Optimal harmonic number and scaling laws of spectra

Another interesting topic is about the optimum problem of the harmonic number and the scaling laws of the spectra related to laser intensity and other possible parameters, for example, the magnetic field strength when an external magnetic field is applied.

In Ref. [25] the authors have shown that when the intensity is enough, e.g., when $a \gg 1$ the optimal harmonic number is about $M \propto a^3$ and the $s_M \propto a^4$ so that the corresponding optimal-number spectra strength $p_M \propto a^{-4}$ because $p_m \propto \omega_1^4 s_m$ and $\omega_1 \propto a^{-2}$. Obviously, these harmonic number optimizations and scaling law of the laser intensity in spectra are easily figured out via the saddle point analysis. In the following we give the idea of achieving them and also an extension to the complex fields like in Refs. [33] and [34].

Firstly, the power of emitted spectra for either the linear polarized case without magnetic field[25] or the circular polarized case with the magnetic field[33] is always proportional to the $\omega_1^4 a^2 m^2 n^2 (\triangle \mathrm{J}_{m\pm n})^2$, where $\triangle \mathrm{J}_{m\pm n} = \mathrm{J}_{m\pm n\pm 1} - \mathrm{J}_{m\pm n}$. Note that the parameter n denotes the magnetic resonance degree, in particular $n = 1$ when magnetic field is absent. Through the extremum condition analysis it is found that the optimal harmonic number $M \propto -\mathrm{J}/\mathrm{J}_{,m}$, where $\mathrm{J}_{,m}$ is a derivative of Bessel function with respect to the harmonic order m. Again by using the saddle point analysis method to J and $\mathrm{J}_{,m}$ we obtain $M \propto 1/\theta_0^3 \sim a^3$, where θ_0 is the saddle point, so that $p_M \propto \omega_1^4 a^2 M^2 (\triangle \mathrm{J}_M)^2 \approx a^{-6} M \theta_0 \sim a^{-4}$ which holds in both cases of Refs. [25] and [33].

Secondly, let us pay attention to the effect of applied magnetic field on the optimal spectra. Similarly in the extremum condition we gain that $M \propto 1/\theta_0^3 \sim n^3 a^3$. Moreover, we note that $\omega_1 \propto n^{-3} a^{-2}$ so that $p_M \propto \omega_1^4 n^4 a^2 M^2 (\triangle \mathrm{J}_{M\pm n})^2 \approx n^{-8} a^{-6} M \theta_0 \sim n^{-6} a^{-4}$ for highly magnetic resonance $n \gg 1$. On the other hand, if the magnetic field is so strong that $(na)^2 \ll 1$, the scaling law would be different from the situation of $n \gg 1$. By the way, $(na)^2 \ll 1$ is possible even if $a \gg 1$ since the magnetic resonance parameter is defined as $n = 1/(B/B_0 - 1) \approx B_0/B \propto B^{-1}$ when the magnetic field strength is extremely large, where B_0 is the normalized laser magnetic field in the order of 100 MGs (1 Gs $= 10^{-4}$ T) for the typical laser wavelength 1 μm. However it should be emphasized that the present laboratory is hard to achieve such high magnetic field.

Our theoretical derivation about the scaling laws mentioned above can be checked as well as confirmed by numerical results. As an example, here we just give an illustration for the scaling law of $p_M \propto n^{-6} a^{-4}$ for highly magnetic resonance $n \gg 1$. The numerical results of the dependence of optimum spectra value p_M on the magnetic resonance parameter n, the laser intensity a and the combined parameters $n^3 a^2$ are plotted in Figs. 1(a), 1(b), and 1(c), respectively. Obviously the slopes of three curves are -6, -4, and -2, which are consistent with the theoretical predictions.

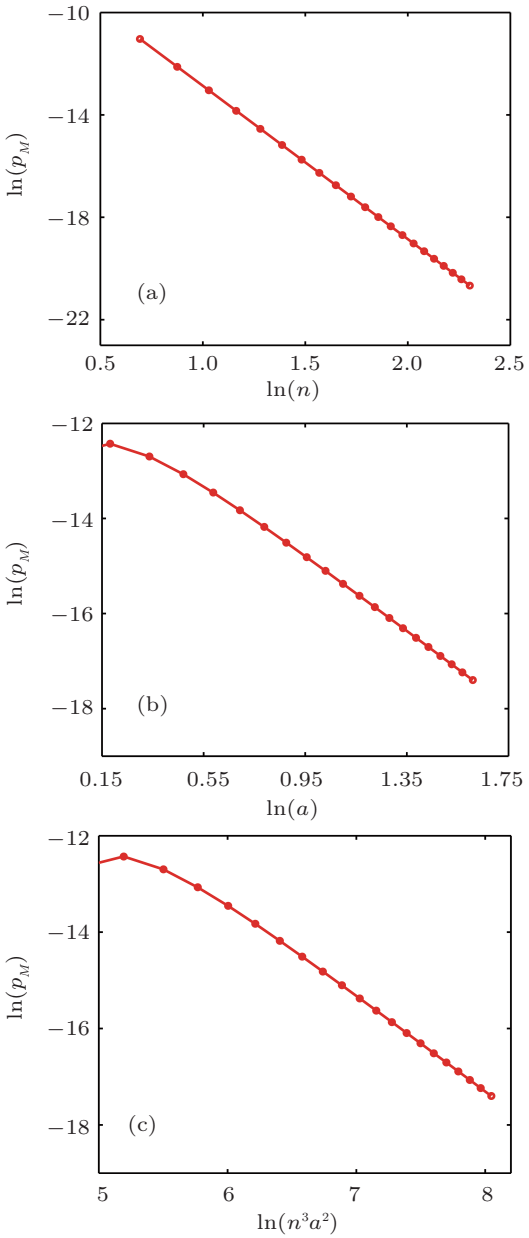


Fig. 1. (color online) The dependence of optimum spectra value p_M on the magnetic resonance parameter n in panel (a), the laser intensity a in in panel (b), and the combined parameters $n^3 a^2$ in in panel (c) in logarithmic coordinates. The fitted slopes of three curves are -6, -4, and -2, respectively.

5. Summary

In this paper we have studied two important characteristics of the Thomson backscattering spectra by using the well-known saddle point analysis method for the integral of complex exponent functions: the convergence problem of high order harmonic spectra when the order approaches to infinity and the scaling laws of the laser field and magnetic field intensity

in many different situations for the optimal spectra.

It is found that the convergence can be gained in the general polarization laser field either without or with applied constant magnetic field. About the scaling law of laser intensity, the optimal order number of harmonics with power index 3 and the corresponding spectra with index -4 obtained by He *et al*. for linearly laser field can be extended to our studied case for the circularly laser field assisted with a weak and/or highly resonance magnetic field. Meanwhile, the closer the magnetic field approaches to the resonance condition, the lower amplitude of the optimal spectra.

Our research here is expected to be helpful to deepen the understanding of the highly nonlinear characteristics of the Thomson scattering. Although the analysis and discussion are for the Thomson backscattering, it is not hard to extend to the forward scattering and even the general case with arbitrary scattering direction.

On the other hand, the concrete and precise information about the scaling law of spectra depending on the laser intensity as well as the applied magnetic field would be very useful to obtain the required x-ray source and/or THz radiation source[33–35] and also realize the conversion efficiency in future experiments.

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