Monogamy relations of quantum entanglement for partially coherently superposed states*

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Monogamy is a fundamental property of multi-partite entangled states. Recently, Kim J S [*Phys. Rev. A* **93** 032331] showed that a partially coherent superposition (PCS) of a generalized W-class state and the vacuum saturates the strong monogamy inequality proposed by Regula B *et al.* [*Phys. Rev. Lett.* **113** 110501] in terms of squared convex roof extended negativity; and this fact may present that this class of states are good candidates for studying the monogamy of entanglement. Hence in this paper, we will investigate the monogamy relations for the PCS states. We first present some properties of the PCS states that are useful for providing our main theorems. Then we present several monogamy inequalities for the PCS states in terms of some entanglement measures.

Keywords: monogamy, PCS states, entanglement

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1. introduction

Quantum entanglement[1] is an essential feature of quantum information theory, which distinguishes the quantum from classical theory. One of the fundamental differences between quantum correlation and classical relations is that a quantum system entangled with one of the other systems limits its entanglement with the remaining others.[2] This property is known as monogamy of entanglement (MoE). Monogamy relations also applies in many fields of physics, such as, characterizing the entanglement structure in multipartite quantum systems,[3–10] causing the frustration effects observed in condensed matter physics;[11] moreover, MoE is a key ingredient to make quantum cryptography secure[12,13] and MoE can apply to the blackhole information problem.

Mathematically, for a tripartite system A,B, and C, the general monogamy in terms of an entanglement measure $\mathcal{E}$ implies that the entanglement between A and BC satisfies

$$\mathcal{E}_{A|BC} \geq \mathcal{E}_{AB} + \mathcal{E}_{AC}. \tag{1}$$

This relation is first proved for qubit systems with respect to the squared concurrence C^2.[14,15] However, it is not true with respect to each entanglement measure. For example, entanglement of formation (EoF) $E_{\rm f}$ does not satisfy the inequality (1) for three qubit states. However, Bai *et al.*[16] proved that the squared EoF satisfies the monogamy inequality for n-qubit states. Moreover, Zhu and Fei[17] showed that C^{α} and $E_{\rm f}^{\beta}$ satisfy the monogamy inequalities for multiqubit systems when $\alpha \geq 2$ and $\beta \geq \sqrt{2}$ respectively. Luo *et al.*[18] proved that the squared Tsallis-q entanglement satisfies the monogamy inequality for multiqubit states when

$$q \in \left[\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}\right].$$

The inequality (1) may not be true for a qudit tripartite pure state in terms of the squared concurrence, the author in Ref. [19] presented a counterexample in a $3\otimes 3\otimes 3$ quantum system, Li and Wang showed that this counterexample satisfies the inequality (1) in terms of the squared convex roof extended negativity.[20]

Recently, the authors in Ref. [10] proposed a much sharper monogamy inequality (SMI) in an n-partite system $\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\cdots\otimes\mathcal{H}_{A_n}$, and the authors in Refs. [21]– [23] showed for the class of multiqudit generalized W-class states and the class of a partially coherent superposition (PCS) of a generalized W-class state and the vacuum, the SMI is valid with respect to the square of convex extended roof of negativity. Furthermore, these two classes saturate the SMI. Then it may be important to investigate the monogamy of these two classes of states. Recently, Geetha *et al.* analyzed the monogamous nature for the W class of symmetric N-qubit states in terms of the squared negativity and concurrence and presented

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that for these states, the concurrence tangle is zero, while the negativity tangle is non-zero,[24] the authors in Ref. [25] showed the generalized monogamy for the generalized W-class multiqubit states.

In this article, we will first present some preliminary knowledge needed in this article. Then we will provide the monogamy relations in terms of concurrence of assistance (CoA) and polygamy relation with respect to EoF for the PCS states when $d=2$. Finally, we will consider the polygamy relations in terms of Tsallis-q entanglement for the PCS states when

$$q\in\left[\frac{5+\sqrt{13}}{2},2\right]\bigcup\left[3,\frac{5-\sqrt{13}}{2}\right]$$

and $d=2$.

This paper in organized as follows. In Section 2, we will recall preliminary knowledge on the definition of concurrence, EoF and Tsallis-q entanglement for bipartite states, and then we will recall their respective monogamy inequalities for multi-qubit systems. In Section 3, we will present some properties of PCS states that are necessary for the proof of their monogamy relations. Here we also present the monogamy relations in terms of CoA and polygamy relations with respect to EoF and Tsallis-q entanglement for the PCS states. In Section 4, we will end up with a conclusion.

2. Preliminary knowledge

For a bipartite pure state $|\psi\rangle_{AB}$ in Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$, the concurrence is defined as

$$C(|\psi\rangle_{AB})=\sqrt{2\left[1-\mathrm{tr}(\rho_A^2)\right]}, \tag{2}$$

where ρ_A is the reduced density matrix by tracing over the subsystem B of $|\psi\rangle_{AB}$. For a mixed state ρ_{AB}, its concurrence is defined as

$$C(\rho_{AB})=\min_{\{p_i,|\psi_i\rangle_{AB}\}}\sum_i p_i C(|\psi_i\rangle_{AB}), \tag{3}$$

where the minimization takes over all the decompositions of ρ_{AB}. For a bipartite state ρ_{AB}, its CoA is defined as[26]

$$C_a(\rho_{AB})=\max_{\{p_i,|\psi_i\rangle_{AB}\}}\sum_i p_i C(|\psi_i\rangle_{AB}), \tag{4}$$

where the maximization takes over all the decompositions of $\rho_{AB}=\sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$. For an n-qubit state $\rho_{AB_1\cdots B_{n-1}}\in\mathcal{H}_A\otimes\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_{n-1}}$, the monogamy inequality was generalized to[17]

$$C^{\alpha}_{A|B_1\cdots B_{n-1}}\geq C^{\alpha}_{AB_1}+C^{\alpha}_{AB_2}+\cdots+C^{\alpha}_{AB_{n-1}}, \tag{5}$$

$$C^{\beta}_{A|B_1\cdots B_{n-1}}\leq C^{\beta}_{AB_1}+C^{\beta}_{AB_2}+\cdots+C^{\beta}_{AB_{n-1}}, \tag{6}$$

where $\alpha\geq 2$, $\beta\leq 0$. The inequality in terms of CoA for a pure state $|\psi\rangle_{AB_1\cdots B_{n-1}}$ was proved in Ref. [26]

$$C^2(|\psi\rangle_{A|B_1\cdots B_{n-1}})\leq C_a^2(\rho_{AB_1})+C_a^2(\rho_{AB_2})+\cdots+C_a^2(\rho_{AB_{N-1}}). \tag{7}$$

Then let us recall the definition of negativity. For a bipartite pure state $|\psi\rangle_{AB}$, its negativity is defined as

$$\mathcal{N}(|\psi\rangle_{AB})=||\psi\rangle_{AB}\langle\psi|^{T_B}||_1-1$$

where $|\psi\rangle_{AB}\langle\psi|^{T_B}$ is the partial transposition of $|\psi\rangle_{AB}$, $||\cdot||_1$ is the 1-norm. Similarly, the negativity for a mixed state ρ_{AB} is defined as

$$\mathcal{N}(\rho_{AB})=\min_{\{p_i,|\psi_i\rangle_{AB}\}}\sum_i p_i\mathcal{N}(|\psi_i\rangle_{AB}), \tag{8}$$

where the minimum is taken over all the possible pure state decompositions of $\rho_{AB}=\sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$.

Afterwards, we will recall the EoF for bipartite states. For a pure state $|\psi\rangle_{AB}\in\mathcal{H}_A\otimes\mathcal{H}_B$, the EoF of $|\psi\rangle_{AB}$ is defined as

$$E(|\psi\rangle_{AB})=S(\rho_A), \tag{9}$$

where $\rho_A=\mathrm{tr}_B|\psi\rangle_{AB}\langle\psi|$, $S(\rho)=-\mathrm{tr}\rho\log\rho$. For a mixed state ρ_{AB}, the EoF is defined as

$$E(\rho_{AB})=\min_{\{p_i,|\psi_i\rangle_{AB}\}}\sum_i p_i E(|\psi_i\rangle_{AB}),$$

where the minimization takes over all the decompositions $\rho_{AB}=\sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$ with $\sum_i p_i=1, p_i\geq 0$. It has been shown that the entanglement of formation satisfies that[17]

$$E^{\alpha}(\rho_{A|B_1B_2\cdots B_{n-1}})\geq E^{\alpha}(\rho_{AB_1})+\cdots+E^{\alpha}(\rho_{AB_{n-1}}), \tag{10}$$

where $\alpha\geq\sqrt{2}$.

At last, we will present Tsallis-q entanglement for bipartite states. Firstly, we will recall the definition of Tsallis-q entropy. The Tsallis-q entropy for a state ρ is defined as

$$T_q(\rho)=\frac{1-\mathrm{tr}\rho^q}{q-1}$$

for any $q>0$, $q\neq 1$. When $q\to 1$, the Tsallis-q entropy converges to von Neumann entropy: $\lim_{q\to 1}T_q(\rho)=-\mathrm{tr}\rho\log\rho$. For any pure state $|\psi\rangle_{AB}$, its Tsallis-q entanglement is defined as: $T_q(|\psi_{AB}\rangle)=T_q(\rho_A)$, for any $q>0$, $q\neq 1$, here $\rho_A=\mathrm{tr}_B|\psi\rangle_{AB}\langle\psi|$. For a mixed state ρ_{AB}, its Tsallis-q entanglement (TE) can be defined as:

$$T_q(\rho_{AB})=\min_{\{p_i,|\psi_i\rangle_{AB}\}}\sum_i p_i T_q(|\psi_i\rangle_{AB}) \tag{11}$$

for any $q>0$, $q\neq 1$, where the minimum is taken over all possible pure state decompositions $\{p_i,|\psi_i\rangle_{AB}\}$ of ρ_{AB}. Recently, Luo *et al.*[18] proved that for any $2\otimes d$ states ρ_{AB}, the TE of ρ_{AB} has an analytical expression

$$T_q(\rho_{AB}) = f_q(C^2(\rho_{AB})), \quad f_q(x) = \frac{1}{q-1}\left[1-\left(\frac{1+\sqrt{1-x}}{2}\right)^q - \left(1-\frac{1-\sqrt{1-x}}{2}\right)^q\right] \tag{12}$$

for

$$q \in \left[\frac{5-\sqrt{13}}{2}, 2\right] \bigcup \left[3, \frac{5+\sqrt{13}}{2}\right].$$

3. Partially coherent superposition of multiqudit generalized W-class states and vacuum

In this section, we will first recall the definition and some properties of the PCS states. The PCS states $\rho_{A_1A_2\cdots A_n}$ are defined as follows:

$$\begin{aligned}\rho_{A_1A_2\cdots A_n} = {} & p|W_n^d\rangle\langle W_n^d| + (1-p)|0\rangle^{\otimes n}\langle 0|^{\otimes n} \\ & + \lambda\sqrt{p(1-p)}(|W_n^d\rangle\langle 0|^{\otimes n} + |0\rangle^{\otimes n}\langle W_n^d|)\end{aligned} \tag{13}$$

with $p, \lambda \in [0,1]$. when $\lambda = 1$, the PCS states become the W-class states. We also find a decomposition of the PCS states:

$$\rho_{A_1A_2\cdots A_n} = |\zeta_1\rangle_{A_1A_2\cdots A_n}\langle\zeta_1| + |\zeta_2\rangle_{A_1A_2\cdots A_n}\langle\zeta_2| \tag{14}$$

with

$$\begin{aligned}|\zeta_1\rangle_{A_1A_2\cdots A_n} &= \sqrt{p}|W_n^d\rangle + \lambda\sqrt{1-p}|0\rangle^{\otimes n}, \\ |\zeta_2\rangle_{A_1A_2\cdots A_n} &= \sqrt{(1-p)(1-\lambda^2)}|0\rangle^{\otimes n}.\end{aligned}$$

Note that we assume $d = 2$ for the residual part of this article. Next we will first bring a lemma proposed in Ref. [27] about the ensemble for density matrices.

Lemma 1[27] **(HJW Theorem)**

Assume that (possibly unnormalized) states of the two sets $\{|\psi_i\rangle\}$ and $\{|\varphi_j\rangle\}$ can generate the same density matrix if and only if

$$|\psi_i\rangle = \sum_j u_{ij}|\varphi_j\rangle, \tag{15}$$

where $[u_{ij}]$ is a unitary matrix with $u_{ij} \in \mathbb{C}$. We add zero vectors such that the two sets have the same number of elements.

Then we will propose an important result in this article.

Lemma 2 For an n-qubit PCS state $\rho_{A_1A_2\cdots A_n}$, we have

$$C_a(\rho_{A_1A_i}) = C(\rho_{A_1A_i}), \tag{16}$$

where

$$\rho_{A_1A_i} = \mathrm{tr}_{A_2\cdots A_{i-1}A_{i+1}\cdots A_n}(\rho_{A_1A_2\cdots A_n}).$$

Proof It is easy to see that

$$\begin{aligned}\rho_{A_1A_i} &= p[a_1|10\rangle + a_i|01\rangle][\overline{a_1}\langle 10| + \overline{a_i}\langle 01|] + (1-p)|00\rangle\langle 00| + \lambda\sqrt{p(1-p)}[(a_1|10\rangle + a_i|01\rangle)\langle 00| + |00\rangle(\overline{a_1}\langle 10| + \overline{a_i}\langle 01|)] \\ &= |x\rangle_{A_1A_i}\langle x| + |y\rangle_{A_1A_i}\langle y|,\end{aligned} \tag{17}$$

here

$$\begin{aligned}|x\rangle_{A_1A_i} &= \sqrt{p}[a_1|10\rangle + a_i|01\rangle] + \lambda\sqrt{1-p}|00\rangle, \\ |y\rangle_{A_1A_i} &= \sqrt{(1-p)(1-\lambda^2)}|00\rangle.\end{aligned}$$

Using the HJW theorem,[27] we see that for any pure-state decomposition of $\rho_{A_1A_i} = \sum_{j=1}^r |\psi_j\rangle_{A_1A_i}\langle\psi_j|$, one has $|\psi_j\rangle_{A_1A_i} = u_{j1}|x\rangle_{A_1A_i} + u_{j2}|y\rangle_{A_1A_i}$ for some $r * r$ unitary matrix $[u_{ij}]$ for each j. Consider the normalized state

$$|\tilde{\psi}_j\rangle_{A_1A_i} = \frac{|\psi_j\rangle_{A_1A_i}}{\sqrt{p_j}}, \quad p_j = |\langle\psi_j|\psi_j\rangle_{A_1A_i}|.$$

Through computation, we find that

$$C^2(|\tilde{\psi}_j\rangle_{A_1A_i}) = 4p^2a_1^2a_i^2|u_{j1}|^4/p_j^2. \tag{18}$$

Then for the state $\rho_{A_1A_i}$, we have

$$\sum_j p_j C(|\tilde{\psi}_j\rangle_{A_1A_i}) = \sum_j 2|u_{ji}|^2 p|a_1a_i| = 2p|a_1a_i|, \tag{19}$$

$$\begin{aligned}C(\rho_{A_1A_i}) &= \min_{\{p_j, |\tilde{\psi}_j\rangle\}} \sum_j p_j C(|\tilde{\psi}_j\rangle_{A_1A_i}) \\ &= \max_{\{p_j, |\psi_j\rangle\}} \sum_j p_j C(|\tilde{\psi}_j\rangle_{A_1A_i}) \\ &= C_a(\rho_{A_1A_i}).\end{aligned} \tag{20}$$

Moreover, in Ref. [23], the author proved that

$$\mathcal{N}^2(\rho_{A_1|A_2\cdots A_n}) = \sum_{j=2}^n \mathcal{N}^2(\rho_{A_1A_j}). \tag{21}$$

□

As when $|\psi\rangle_{AB}$ is a $2\otimes d$ pure state, $\mathcal{N}^2(|\psi\rangle_{AB}) = C^2(|\psi\rangle_{AB})$, and due to the fact that $\mathcal{N}$ and C are both defined by the convex roof extended method for mixed states, then for any $2\otimes d$ mixed state ρ_{AB}, $\mathcal{N}^2(\rho_{AB}) = C^2(\rho_{AB})$. That is,

$$C^2(\rho_{A_1|A_2\cdots A_n}) = \sum_{j=2}^n C^2(\rho_{A_1A_j}). \tag{22}$$

Lemma 3[23] Assume that ρ is an n-qubit PCS state, then the reduced density matrix of ρ, obtained by tracing out some subsystems, is again a PCS state in the reduced systems.

Theorem 1 For an n-qubit PCS state $\rho_{A_1A_2\cdots A_n} \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$, the CoA of $\rho_{A_1A_2\cdots A_n}$ satisfies that

$$C_a^x(\rho_{A_1|A_{j_1}\cdots A_{j_{m-1}}}) \geq \sum_{i=1}^{m-1} C_a^x(\rho_{A_1A_{j_i}}), \tag{23}$$

where $x \geq 2$ and $\rho_{A_1|A_{j_1}\cdots A_{j_{m-1}}}$ is the m-qubit reduced density matrix of ρ with $1 \leq m \leq n$.

Proof For any PCS state $\rho_{A_1A_2\cdots A_n}$, because of the definition of C and C_a, one has

$$C_a(\rho_{A_1|A_{j_1}\cdots A_{j_{m-1}}}) \geq C(\rho_{A_1|A_{j_1}\cdots A_{j_{m-1}}}).$$

Due to Lemma 3, we have that $\rho_{A_1|A_{j_1}\cdots A_{j_{m-1}}}$ is again a PCS state. When $x \geq 2$, we have

$$\begin{aligned} C_a^x(\rho_{A_1|A_{j_1}\cdots A_{j_{m-1}}}) &\geq C^x(\rho_{A_1|A_{j_1}\cdots A_{j_{m-1}}}) \\ &\geq \sum_{i=1}^{m-1} C^x(\rho_{A_1A_{j_i}}) \\ &= \sum_{i=1}^{m-1} C_a^x(\rho_{A_1A_{j_i}}). \end{aligned} \tag{24}$$

Here we use the inequality (5) to prove the second inequality and the last equality is due to the Lemma 2. □

According to the result in Ref. [26], the CoA of an n-qubit pure state satisfies a polygamy inequality of entanglement, and here we present a monogamy relation with respect to CoA for PCS states. Note that according to inequality (24), we can give a lower bound of CoA for the reduced density matrices of a PCS state. Next we give an example.

Example 1 Here we consider a 5-qubit PCS state and assume $p = 0.3$, $\lambda = 0.7$, $a_1 = 0.4$, $a_2 = 0.3$, $a_3 = 0.5$, $a_4 = 0.6$, $a_5 = \sqrt{0.14}$, then we find a lower bound of *CoA* of $C_a(\rho_{A_1|A_2A_3})$ due to Theorem 1,

$$C_a(\rho_{A_1|A_2A_3}) \geq \sqrt[x]{0.072^x + 0.12^x} \tag{25}$$

with $x \geq 2$. The optimal lower bound can be obtained by varying with x, see Fig. 1.

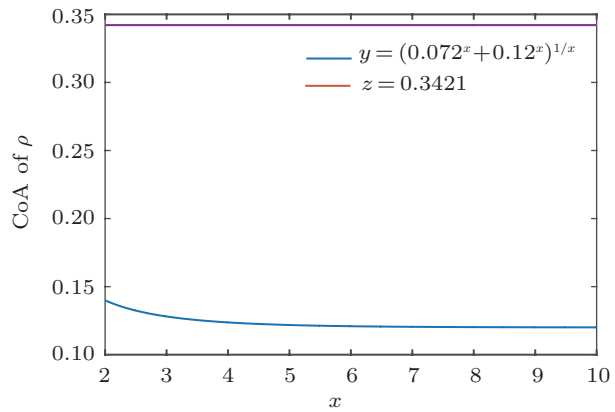


Fig. 1. (color online) This line is the lower bound of $C_a(\rho_{A_1|A_2A_3})$ as functions of $x \geq 2$.

From Fig. 1, we see the optimal bound is when $x = 2$, that is, $C_a(\rho_{A_1|A_2A_3}) \geq (0.12^2 + 0.072^2)^{1/2} \approx 0.1399$. Here we can make a comparison with the upper bound given in Ref. [28]: $C_a(\rho_{AB}) \leq \sqrt{2(1 - \mathrm{tr}\rho_A^2)} = 0.3421$.

Next we will consider the polygamy with respect to the EoF for the PCS states.

Theorem 2 For the n-qubit PCS state $\rho_{A_1A_2\cdots A_n} \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$, we have

$$E(\rho_{A_1|A_2\cdots A_n}) \leq \sum_{i=2}^{n} E(\rho_{A_1A_i}), \tag{26}$$

where $\rho_{A_1A_i}$, $(2 \leq i \leq n)$, is the 2-qubit reduced density matrix of ρ.

Proof

$$\begin{aligned} \sum_{i=2}^{n} E(\rho_{A_1A_i}) &= \sum_{i=2}^{n} f(C^2(\rho_{A_1A_i})) \\ &\geq f(\sum_{i=2}^{n} C^2(\rho_{A_1A_i})) \\ &= f(C^2(\rho_{A_1|A_2\cdots A_n})) = E(\rho_{A_1|A_2\cdots A_n}). \end{aligned} \tag{27}$$

Here we assume that

$$f = h\Big(\frac{1+\sqrt{1-x}}{2}\Big)$$

with $h(x) = -x\log_2(x) - (1-x)\log_2(1-x)$. Where the first and the last equalities are due to the Theorem 1 in Ref. [16], the first inequality is due to $f(x+y) \leq f(x) + f(y)$, the second equality is due to Eq. (22). □

Lastly, we will give the polygamy of entanglement with respect to Tsallis-q entanglement for the PCS states.

Theorem 3 For an n-qubit PCS state $\rho_{A_1A_2\cdots A_n} \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$, when

$$q \in \Big[\frac{5-\sqrt{13}}{2}, 2\Big] \cup \Big[3, \frac{5+\sqrt{13}}{2}\Big],$$

we have

$$T_q(\rho_{A_1|A_2\cdots A_n}) \leq \sum_{i=2}^{n} T_q(\rho_{A_1A_i}) \tag{28}$$

where $\rho_{A_1A_i}$, $(2 \leq i \leq n)$, is the 2-qubit reduced density matrix of $\rho_{A_1A_2\cdots A_n}$.

Proof For an n-qubit PCS state $\rho_{A_1A_2\cdots A_n}$,

$$\begin{aligned} \sum_{i=2}^{n} T_q(\rho_{A_1A_i}) &= \sum_{i=2}^{n} f_q(C^2(\rho_{A_1A_i})) \\ &\geq f_q(\sum_{i=2}^{n} C^2(\rho_{A_1A_i})) = f_q(C^2(\rho_{A_1|A_2\cdots A_n})) \\ &= T_q(\rho_{A_1|A_2\cdots A_n}), \end{aligned} \tag{29}$$

where the first and the last equalities are due to the Theorem 1 in Ref. [18], the first inequality is due to $f_q(C^2)$ being monotonic and concave when[18]

$$q \in \Big[\frac{5-\sqrt{13}}{2}, 2\Big] \bigcup \Big[3, \frac{5+\sqrt{13}}{2}\Big],$$

the second equality is due to Eq. (22). □

Combining the results obtained in Ref. [29], we know that, when $q = 2$ or $q = 3$, $T_q(\rho_{A_1|A_2\cdots A_n}) = \sum_{i=2}^{n} T_q(\rho_{A_1A_i})$ for the PCS states.

Recently, Luo *et al.*[18] proved that T_q^2 satisfies the monogamy inequality when

$$q \in \left[\frac{5-\sqrt{13}}{2}, 2\right] \bigcup \left[3, \frac{5+\sqrt{13}}{2}\right]$$

for states $\rho_{A|BC}$ in a $2 \otimes 2 \otimes 2^{n-2}$ system: $T_q^2(\rho_{A|BC}) \geq T_q^2(\rho_{AB}) + T_q^2(\rho_{AC})$. They also generalized this result to an n-qubit system, i.e., $T_q^\alpha(\rho_{A_1|A_2\cdots A_N}) \geq T_q^\alpha(\rho_{A_1A_2}) + \cdots + T_q^\alpha(\rho_{A_1A_N})$, $\alpha \geq 2$,

$$q \in \left[\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}\right].$$

Here we can also give a new general polygamy inequality for the PCS states in terms of T_q^γ for $\gamma \geq 1$.

Corollary 1 For an n-qubit PCS state ρ, when

$$q \in \left[\frac{5-\sqrt{13}}{2}, 2\right] \bigcup \left[3, \frac{5+\sqrt{13}}{2}\right],$$

$\gamma, \beta \in [1, \infty)$, $1/\gamma + 1/\beta = 1$, we have

$$T_q^\gamma(\rho_{A_1|A_2\cdots A_n}) \leq (n-1)^{\gamma/\beta} \left[\sum_{i=2}^{n} T_q^\gamma(\rho_{A_1A_i})\right]. \quad (30)$$

Proof Due to Corollary 1, for the PCS state ρ, when $\gamma, \beta \in [1, \infty)$, $1/\gamma + 1/\beta = 1$,

$$\begin{aligned} T_q(\rho_{A_1|A_2\cdots A_n}) &\leq \sum_{i=2}^{n} T_q(\rho_{A_1A_i}) \\ &\leq \left(\sum_{i=2}^{n} 1^\beta\right)^{1/\beta} \left[\sum_{i=2}^{n} T_q^\gamma(\rho_{A_1A_i})\right]^{1/\gamma} \\ &= (n-1)^{1/\beta} \left[\sum_{i=2}^{n} T_q^\gamma(\rho_{A_1A_i})\right]^{1/\gamma}. \end{aligned} \quad (31)$$

Here the first inequality is due to Corollary 1, the second inequality is due to Hölder inequality. Finally, as when $a, b \geq 0$, $x \geq 1$, $a \geq b \Longrightarrow a^x \geq b^x$, we finish the proof. □

Note that when $\beta = \gamma = 2$, Corollary 1 becomes that

$$T_q^2(\rho_{A_1|A_2\cdots A_n}) \leq (n-1) \sum_{i=2}^{n} T_q^2(\rho_{A_1A_i}),$$

and here we give a polygamy relation with respect to T_q^2 for PCS states in an n-partite system. From another point of view, we also find a new bound for the T_q^2 of the PCS states in an n-partite system.

4. Conclusion

Monogamy of entanglement is a fundamental property of multipartite entangled states. In this article, we prove that the n-qubit PCS states satisfy the monogamy inequality in terms of the x-power of CoA for its m-qubit reduced density matrices with $2 \leq m \leq n$. Then we also investigate the monogamy inequalities in terms of EoF and TE for PCS states. We believe this investigation will provide a better clarification of the usefulness of PCS states in understanding the constraints on the description of entanglement distribution. Due to the importance of multi-party entanglement, our results may be meaningful for future work on the study of multi-party quantum entanglement.

References

[1] Horodecki R, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865
[2] Terhal B M 2004 *IBM J. Res. Dev.* **48** 71
[3] Ou Y C and Fan H 2007 *Phys. Rev. A* **75** 062308
[4] Bai Y K and Wang Z D 2008 *Phys. Rev. A* **77** 032313
[5] Jung E, Park D and Son J W 2009 *Phys. Rev. A* **80** 010301
[6] Ren X J and Jiang W 2010 *Phys. Rev. A* **81** 024305
[7] Cornelio M F 2013 *Phys. Rev. A* **87** 032330
[8] Zhu X N and Fei S M 2015 *Phys. Rev. A* **92** 062345
[9] Liu F, Gao F and Wen Q Y 2015 *Sci. Rep.* **5** 16745
[10] Regula B, Di Martino S, Lee S and Adesso G 2014 *Phys. Rev. Lett.* **113** 110501
[11] Ma X S, Dakic B, Naylor W, Zeilinger A and Walther P 2011 *Nat. Phys.* **7** 399
[12] Masanes L 2009 *Phys. Rev. Lett.* **102** 140501
[13] Tomamichel M, Fehr S, Kaniewski J and Wehner S 2013 *New J. Phys.* **15** 103002
[14] Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev. A* **61** 052306
[15] Osborne T J and Verstraete F 2006 *Phys. Rev. Lett.* **96** 220503
[16] Bai Y K, Xu Y F and Wang Z D 2014 *Phys. Rev. A* **90** 062343
[17] Zhu X N and Fei S M 2014 *Phys. Rev. A* **90** 024304
[18] Luo Y, Tian T, Shao L H and Li Y M 2016 *Phys. Rev. A* **93** 062340
[19] Ou Y C 2007 *Phys. Rev. A* **75** 034305
[20] Li J J and Wang Z X 2010 *Chin. Phys. B* **19** 100310
[21] Regula B, Martino S Di, Lee S and Adesso G 2014 *Phys. Rev. Lett.* **113** 110501
[22] Choi J H and Kim J S 2015 *Phys. Rev. A* **92** 042307
[23] Kim J S 2016 *Phys. Rev. A* **93** 032331
[24] Geethaa P J, Yashodammaa K O and Sudha 2015 *Chin. Phys. B* **24** 110302
[25] Zhu X N and Fei S M 2017 *Quantum Inform. Process.* **16** 53
[26] Gour G, Bandyopadhay S and Sanders B C 2007 *J. Math. Phys.* **48** 012108
[27] Hughston L P, Jozsa R and Wootters W K 1993 *Phys. Lett. A* **183** 14
[28] Li Z G, Fei S M, Albeverio S and Li W M 2009 *Phys. Rev. A* **80** 034301
[29] Kim J S 2010 *Phys. Rev. A* **81** 062328