Analytical and numerical investigations of displaced thermal state evolutions in a laser process*

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We investigate how displaced thermal states (DTSs) evolve in a laser channel. Remarkably, the initial DTS, an example of a mixed state, still remains mixed and thermal. At long times, they finally decay to a highly classical thermal field only related to the laser parameters κ and g. The normal ordering product of density operator of the DTS in the laser channel leads to obtaining the analytical time-evolution expressions of the photon number, Wigner function, and von Neumann entropy. Also, some interesting results are presented via numerically investigating these explicit time-dependent expressions.

Keywords: displaced thermal state, laser process, infinitive operator-sum representation, photon number, Wigner function, von Neumann entropy

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1. Introduction

Recently, a displaced thermal state (DTS), which is treated as an intermediate quantum state between the thermal state and the coherent state, is introduced by Jeong and Ralph, and its integral expression is represented by[1–3]

$$\rho(V,d)=\int \mathrm{d}^2\beta P^{\mathrm{th}}(V,d;\beta)\,|\beta\rangle\langle\beta|, \tag{1}$$

where $|\beta\rangle$ is a coherent state with complex amplitude β, and

$$P^{\mathrm{th}}(V,d;\beta)=\frac{2}{\pi(V-1)}\,\mathrm{e}^{-2|\beta-d|^2/(V-1)} \tag{2}$$

refers to a Gaussian noise, V and d are respectively the mixedness and the displacement in the phase space. The mixedness V increases when the temperature T of thermal field rises, i.e., $\mathrm{e}^{\hbar\nu/k_{\mathrm{B}}T}=(V+1)/(V-1)$, $\hbar$ is the Planck's constant, ν is the frequency of thermal field, and k_{B} is Boltzmann constant. As a typical application, thermal-state superpositions are used to realize the transfer of nonclassicality.[1] Using the integration technique of the ordered bosonic operators and the operator identity $|0\rangle\langle 0|=:\mathrm{e}^{-a^\dagger a}:$ ($:\ :$ represents normal ordering, that is, all creation operators are to the left of all annihilation operators, and the creation and annihilation operators are commutative within the symbol $:\ :$), as well as the integral formula[4,5]

$$\int\frac{\mathrm{d}^2z}{\pi}\,\mathrm{e}^{-h|z|^2+sz+tz^*}=\frac{1}{h}\,\mathrm{e}^{st/h} \tag{3}$$

to carry out Eq. (1), we obtain

$$\rho(V,d)=\frac{2}{V+1}\exp\left[-\frac{2|d|^2}{V+1}\right]\exp\left[\left(\frac{2d}{V+1}\right)a^\dagger\right]\exp\left[a^\dagger a\ln\left(\frac{V-1}{V+1}\right)\right]\exp\left[\left(\frac{2d^*}{V+1}\right)a\right]. \tag{4}$$

In particular, when $V=1$, $\rho(V,d)$ becomes a pure coherent state $|d\rangle\langle d|$. For $d=0$, $\rho(V,d)$ is a thermal field $[2/(V+1)]\,\mathrm{e}^{a^\dagger a\ln[(V-1)/(V+1)]}$ with the mixedness V. In view of this, the DTS $\rho(V,d)$ is actually an intermediate state between the coherent state $|d\rangle\langle d|$ and the thermal state $[2/(V+1)]\exp\{a^\dagger a\ln[(V-1)/(V+1)]\}$. In the case of $V=1$ and $d=0$, $\rho(1,0)$ is a vacuum state.

In this work, we plan to analytically and numerically investigate the time-evolution law of the DTS when it passes through a laser process. This work is arranged as follows. In Section 2 we first obtain the infinitive operator-sum solutions (Kraus operator) of the master equation describing the laser process via the thermal entangled state approach. In Section 3 the time-evolution of the density operator of DTS in the laser process is discussed based on the infinitive operator-sum representation in Section 2. Section 3 and Section 4 respectively present the evolutions of the photon number and the Wigner function in the laser process. Finally, the von Neumann en-

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tropy evolution of the DTS is also investigated.

2. Infinitive operator-sum representation of $\rho(t)$

In the realm of optics, laser noise is a significant source causing decoherence. When a quantum state undergoes the laser process, the time-evolution density operator $\rho(t)$ yields the form[6]

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \kappa[2a\rho(t)a^{\dagger} - a^{\dagger}a\rho(t) - \rho(t)a^{\dagger}a] + g[2a^{\dagger}\rho(t)a - aa^{\dagger}\rho(t) - \rho(t)aa^{\dagger}], \tag{5}$$

where the parameters κ, g are respectively the loss and gain factors of the laser. In particular, when $g = 0$, equation (5) reduces to[6]

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \kappa\left[2a\rho(t)a^{\dagger} - a^{\dagger}a\rho(t) - \rho(t)a^{\dagger}a\right], \tag{6}$$

which describes the amplitude dissipative channel. So, it is interesting to investigate the time-evolution law of the DTS in a laser process. In order to solve the master equation (5) and obtain its explicit solutions, we introduce the thermal entangled state[7,8]

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a^{\dagger} - \eta^{*}\tilde{a}^{\dagger} + a^{\dagger}\tilde{a}^{\dagger}\right)|0\tilde{0}\rangle = D(\eta)|\eta = 0\rangle, \tag{7}$$

where $D(\eta)$ is the displacement operator with complex parameter η, $\tilde{a}^{\dagger}$ is a fictitious mode accompanying the physical photon creation operator $a^{\dagger}$. The entangled state $|\eta\rangle$ is the common eigenvector of $(a - \tilde{a}^{\dagger})$ and $(a^{\dagger} - \tilde{a})$, and the set of $|\eta\rangle$ makes up a complete quantum mechanics representation. More importantly, there are interchange relations $a \Longleftrightarrow \tilde{a}^{\dagger}$, $a^{\dagger} \Longleftrightarrow \tilde{a}$, and $a^{\dagger}a \Longleftrightarrow \tilde{a}^{\dagger}\tilde{a}$ under the state $|\eta = 0\rangle$, which provide convenience for us to find the specific solutions of the master equation (5), with details as follows. With both sides of Eq. (5) acting on the state $|\eta = 0\rangle$ and denoting $|\rho(t)\rangle = \rho(t)|\eta = 0\rangle$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t)\rangle = \{g[2a^{\dagger}\tilde{a}^{\dagger} - aa^{\dagger} - \tilde{a}\tilde{a}^{\dagger}] + \kappa[2a\tilde{a} - a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a}]\}|\rho(t)\rangle, \tag{8}$$

thus its solution is obtained as

$$|\rho(t)\rangle = \exp\left\{gt\left[2a^{\dagger}\tilde{a}^{\dagger} - aa^{\dagger} - \tilde{a}\tilde{a}^{\dagger}\right] + \kappa t\left(2a\tilde{a} - a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a}\right)\right\}|\rho(0)\rangle, \tag{9}$$

where $|\rho(0)\rangle \equiv \rho(0)|\eta = 0\rangle$, $\rho(0)$ is an initial density operator. Using the following relations $\left[a\tilde{a} - \tilde{a}^{\dagger}a^{\dagger}, (a^{\dagger} - \tilde{a})(a - \tilde{a}^{\dagger})\right] = 2\left[(a^{\dagger} - \tilde{a})(a - \tilde{a}^{\dagger})\right]$ and $\mathrm{e}^{\lambda(A+\sigma B)} = \mathrm{e}^{\lambda A}\mathrm{e}^{\sigma\left(1-\mathrm{e}^{-\lambda\varepsilon}\right)B/\varepsilon}$ ($[A, B] = \varepsilon B$), we have

$$|\rho(t)\rangle = \exp\left[\left(a\tilde{a} - \tilde{a}^{\dagger}\tilde{a}^{\dagger} + 1\right)(\kappa - g)t\right]\exp\left\{\frac{(\kappa+g)\left[1 - \mathrm{e}^{2(\kappa-g)t}\right]}{2(\kappa - g)}\left(a^{\dagger} - \tilde{a}\right)\left(a - \tilde{a}^{\dagger}\right)\right\}|\rho(0)\rangle. \tag{10}$$

Further, using the completeness relation of the state $|\eta\rangle$ and the integration technique of the ordered bosonic operators, we can rewrite Eq. (10) as

$$\begin{aligned}|\rho(t)\rangle &= T_3\,\mathrm{e}^{gT_1a^{\dagger}\tilde{a}^{\dagger}}\,\mathrm{e}^{(\tilde{a}^{\dagger}\tilde{a}+a^{\dagger}a)\ln T_2}\,\mathrm{e}^{\kappa T_1 a\tilde{a}}\rho(0)|\eta = 0\rangle \\ &= T_3\sum_{i,j=0}^{\infty}\frac{\kappa^i g^j}{i!\,j!}T_1^{i+j}a^{\dagger j}\,\mathrm{e}^{a^{\dagger}a\ln T_2}a^i\rho(0)a^{\dagger i}\,\mathrm{e}^{a^{\dagger}a\ln T_2}a^j|\eta = 0\rangle,\end{aligned} \tag{11}$$

where $T_1 = [1 - \mathrm{e}^{-2(\kappa-g)t}]/[\kappa - g\mathrm{e}^{-2(\kappa-g)t}]$, $T_2 = (\kappa - g)\,\mathrm{e}^{-(\kappa-g)t}/[\kappa - g\mathrm{e}^{-2(\kappa-g)t}]$, and $T_3 = 1 - gT_1$, thus $\rho(t)$ can be rewritten as the following expressions

$$\rho(t) = \sum_{i,j=0}^{\infty}M_{i,j}\rho(0)M_{i,j}^{\dagger},$$

$$M_{i,j} \equiv \sqrt{\frac{T_3\kappa^i g^j}{i!\,j!T_2^{2j}}T_1^{i+j}}\,a^{\dagger j}\,\mathrm{e}^{a^{\dagger}a\ln T_2}a^i, \tag{12}$$

which is actually the infinitive operator-sum representation of $\rho(t)$. For a given initial state $\rho(0)$, the density operator $\rho(t)$ can be readily calculated according to Eq. (12). Using the operator identities $\mathrm{e}^{2a^{\dagger}a\ln T_2}a^{\dagger i}\mathrm{e}^{-2a^{\dagger}a\ln T_2} = T_2^{2i}a^{\dagger i}$, $a^n a^{\dagger m} = (-\mathrm{i})^{m+n}\colon \mathrm{H}_{m,n}(\mathrm{i}a^{\dagger}, \mathrm{i}a)\colon$, as well as the relation $\mathrm{L}_m(xy) = (-1)^m\mathrm{H}_{m,m}(x, y)/m!$ with $\mathrm{L}_m(\cdot)$ and $\mathrm{H}_{m,m}(\cdot,\cdot)$ being respectively the Laguerre polynomial and two-variable Hermite polynomial, we obtain $\sum_{i,j}M_{i,j}^{\dagger}M_{i,j} = 1$, which shows that $M_{i,j}$ is a Kraus operator corresponding to the operator $\rho(t)$ in the laser theory.[8]

3. Evolution of the density operator of DTS in a laser process

When $\rho(V, d)$ is input into the laser channel, from Eq. (12) we can see

$$\rho(V, d; t) = \sum_{i,j=0}^{\infty}\frac{\kappa^i g^j T_1^{i+j}(1 - gT_1)}{i!\,j!T_2^{2j}}\int \mathrm{d}^2\beta P^{\mathrm{th}}(V, d; \beta) \times |\beta|^{2i}\,\mathrm{e}^{a^{\dagger}a\ln T_2}a^{\dagger j}|\beta\rangle\langle\beta|a^j\,\mathrm{e}^{a^{\dagger}a\ln T_2}. \tag{13}$$

Using the operator identity $\mathrm{e}^{a^\dagger a\ln T_2}a^{\dagger j}\mathrm{e}^{-a^\dagger a\ln T_2}=T_2^j a^{\dagger j}$ to show that

$$\rho(V,d;t)=T_3\sum_{i,j=0}^{\infty}\frac{\kappa^i g^j}{i!\,j!}T_1^{i+j}\int \mathrm{d}^2\beta P^{\mathrm{th}}(V,d;\beta)\times|\beta|^{2i}a^{\dagger j}\mathrm{e}^{a^\dagger a\ln T_2}|\beta\rangle\langle\beta|\,\mathrm{e}^{a^\dagger a\ln T_2}a^j. \quad (14)$$

Noticing that the operator relation[8]

$$\mathrm{e}^{\lambda a^\dagger a}=:\exp[(\mathrm{e}^\lambda-1)a^\dagger a]: \quad (15)$$

and the normally ordered expansion of the coherent state $|\beta\rangle\langle\beta|$, we have

$$\begin{aligned}\rho(V,d;t)&=T_3\int \mathrm{d}^2\beta P^{\mathrm{th}}(V,d;\beta):\exp[(\kappa T_1-1)|\beta|^2+T_2\beta a^\dagger+T_2\beta^* a-(1-gT_1)a^\dagger a]:\\&=\frac{2T_3\exp\left[-2|d|^2/(V-1)\right]}{\pi(V-1)}\int \mathrm{d}^2\beta:\exp\left[\left(\kappa T_1-\frac{V+1}{V-1}\right)|\beta|^2\right.\\&\left.+\left(T_2a^\dagger+\frac{2d^*}{V-1}\right)\beta+\left(T_2a+\frac{2d}{V-1}\right)\beta^*-(1-gT_1)a^\dagger a\right]:.\end{aligned} \quad (16)$$

Further, using the integral formula (3) to carry out the integration (16), we find that

$$\rho(V,d;t)=c_1\exp\left[\frac{4-2c(V-1)}{c(V-1)^2}|d|^2\right]\exp(c_2da^\dagger):\exp(c_3a^\dagger a):\exp(c_2d^*a), \quad (17)$$

where

$$c=\frac{V+1}{V-1}-\kappa T_1,\quad c_1=\frac{2T_3}{c(V-1)},\quad c_2=\frac{2T_2}{c(V-1)},\quad c_3=\frac{T_2^2}{c}-T_3. \quad (18)$$

Using Eq. (15) again, $\rho(V,d;t)$ can be rewritten as

$$\rho(V,d;t)=c_1\exp\left[\frac{4-2c(V-1)}{c(V-1)^2}|d|^2\right]\mathrm{e}^{c_2da^\dagger}\mathrm{e}^{a^\dagger a\ln(c_3+1)}\mathrm{e}^{c_2d^*a}. \quad (19)$$

A comparison of Eqs. (4) and (19) shows that the state $\rho(V,d;t)$ remains mixed and thermal properties in the laser process all the time, but the parameters d and V are in relation to the laser parameters κ and g after undergoing a laser process. Now, let us discuss several special cases. For instance, in the case of $g=0$, thus

$$T_1=(1-\mathrm{e}^{-2\kappa t})/\kappa,\quad T_2=\mathrm{e}^{-\kappa t},\quad T_3=1,\quad c=\frac{V+1}{V-1}-(1-\mathrm{e}^{-2\kappa t}),\quad c_1=\frac{2}{c(V-1)},\quad c_2=\frac{2\mathrm{e}^{-\kappa t}}{c(V-1)},\quad c_3=\frac{\mathrm{e}^{-2\kappa t}}{c}-1, \quad (20)$$

from Eq. (19) we easily obtain the evolution of a DTS for amplitude decay, i.e.,

$$\rho(V,d;t)=c_4\,\mathrm{e}^{-c_4|d|^2\mathrm{e}^{-2\kappa t}}\mathrm{e}^{c_4d\mathrm{e}^{-\kappa t}a^\dagger}\mathrm{e}^{a^\dagger a\ln(1-c_4)}\mathrm{e}^{c_4d^*\mathrm{e}^{-\kappa t}a}, \quad (21)$$

which is always a DTS related to the decay factor $\mathrm{e}^{-\kappa t}$, and $c_4=2/[2+(V-1)\mathrm{e}^{-2\kappa t}]$. When $V=1$ (or $d=0$), equation (19) reduces to the time-evolution of a coherent state $|d\rangle\langle d|$, or a thermal state

$$[2/(V+1)]\exp\{a^\dagger a\ln[(V-1)/(V+1)]\}$$

in the laser channel. On the other hand, at $t=0$, $\rho(V,d;t)$ just becomes the initial state $\rho(V,d)$ in Eq. (4), as expected. In the limit case ($t\to\infty$) and assuming that $\kappa>g$, noticing that $T_1\to1/\kappa$, $T_2\to0$, $T_3\to1-g/\kappa$, $c=2/(V-1)$, $c_1\to1-g/\kappa$, $c_2\to0$, and $c_3=g/\kappa-1$, thus $\rho(V,d;t)$ reduces to $[(1-g/\kappa)]\exp[a^\dagger a\ln(g/\kappa)]$, a thermal field with the average photon number $g/(\kappa-g)$. This is different from the case for amplitude damping owing to the presence of the gain factor g.

4. Evolution of the photon number

Next, we plan to derive the photon number evolution of an initial DTS $\rho(V,d)$ in the laser process. In terms of the state $\rho(V,d;t)$ in Eq. (19), the number of photons for $\rho(V,d;t)$ is expressed as

$$\begin{aligned}n(V,d;t)&=\mathrm{Tr}\left[\rho(V,d;t)a^\dagger a\right]\\&=c_1\exp\left[\frac{4-2c(V-1)}{c(V-1)^2}|d|^2\right]\\&\cdot\mathrm{Tr}\left[\mathrm{e}^{c_2da^\dagger}\mathrm{e}^{a^\dagger a\ln(c_3+1)}\mathrm{e}^{c_2d^*a}a^\dagger a\right].\end{aligned} \quad (22)$$

In order to obtain the photon number $n(V,d;t)$, we first rearrange the operator $\mathrm{e}^{c_2da^\dagger}\mathrm{e}^{a^\dagger a\ln(c_3+1)}\mathrm{e}^{c_2d^*a}a^\dagger a$. Using the operator identities

$$\mathrm{e}^{\xi a}a^\dagger\mathrm{e}^{-\xi a}=a^\dagger+\xi,\quad \mathrm{e}^{a^\dagger a\ln\xi}a^\dagger\mathrm{e}^{-a^\dagger a\ln\xi}=\xi a^\dagger, \quad (23)$$

we have

$$\begin{aligned}\mathrm{e}^{c_2 d a^\dagger}\,\mathrm{e}^{a^\dagger a \ln(c_3+1)}\,\mathrm{e}^{c_2 d^* a} a^\dagger a &= \mathrm{e}^{c_2 d a^\dagger}\,\mathrm{e}^{a^\dagger a \ln(c_3+1)}\,\mathrm{e}^{c_2 d^* a} a^\dagger\,\mathrm{e}^{-c_2 d^* a}\,\mathrm{e}^{c_2 d^* a} a = \mathrm{e}^{c_2 d a^\dagger}\,\mathrm{e}^{a^\dagger a \ln(c_3+1)}(a^\dagger + c_2 d^*)\,\mathrm{e}^{c_2 d^* a} a \\ &= \mathrm{e}^{c_2 d a^\dagger}\,\mathrm{e}^{a^\dagger a \ln(c_3+1)} a^\dagger\,\mathrm{e}^{-a^\dagger a \ln(c_3+1)}\,\mathrm{e}^{a^\dagger a \ln(c_3+1)}\,\mathrm{e}^{c_2 d^* a} a + c_2 d^*\,\mathrm{e}^{c_2 d a^\dagger}\,\mathrm{e}^{a^\dagger a \ln(c_3+1)}\,\mathrm{e}^{c_2 d^* a} a \\ &= (c_3+1)\,\mathrm{e}^{c_2 d a^\dagger} a^\dagger e^{a^\dagger a \ln(c_3+1)}\,\mathrm{e}^{c_2 d^* a} a + c_2 d^*\,\mathrm{e}^{c_2 d a^\dagger}\,\mathrm{e}^{a^\dagger a \ln(c_3+1)}\,\mathrm{e}^{c_2 d^* a} a. \end{aligned} \tag{24}$$

Thus, the photon number for $\rho(V,d;t)$ is given by

$$n(V,d;t) = c_1 \exp\left[\frac{4-2c(V-1)}{c(V-1)^2}|d|^2\right]\int \frac{\mathrm{d}^2 z}{\pi}[(c_3+1)|z|^2 + c_2 d^* z]\,\mathrm{e}^{c_2 d z^* + c_2 d^* z}\langle z|\,\mathrm{e}^{a^\dagger a \ln(c_3+1)}|z\rangle\,. \tag{25}$$

Further, using the integration formula[8]

$$\int \frac{\mathrm{d}^2 z}{\pi} z^n z^{*m} \exp(h|z|^2 + sz + \eta z^*) = \mathrm{e}^{-s\eta/h}\sum_{l=0}^{\min(m,n)} \frac{m!n!s^{m-l}\eta^{n-l}}{l!(m-l)!(n-l)!(-h)^{m+n-l+1}}, \tag{26}$$

we finally obtain the time-evolution of the photon number of the state $\rho(V,d)$ in the laser channel, i.e.,

$$\begin{aligned} n(V,d;t) = c_1 &\left(\frac{1}{c_3} + \frac{1}{c_3^2} - \frac{c_2^2}{c_3^3}|d|^2\right) \\ &\cdot \exp\left[\left(\frac{4-2c(V-1)}{c(V-1)^2} - \frac{c_2^2}{c_3}\right)|d|^2\right], \end{aligned} \tag{27}$$

when $g = 0$,

$$n(V,d;t) = \left(\frac{V-1}{2} + |d|^2\right)\mathrm{e}^{-2\kappa t},$$

corresponding to the photon number evolution of $\rho(V,d)$ for amplitude decay.

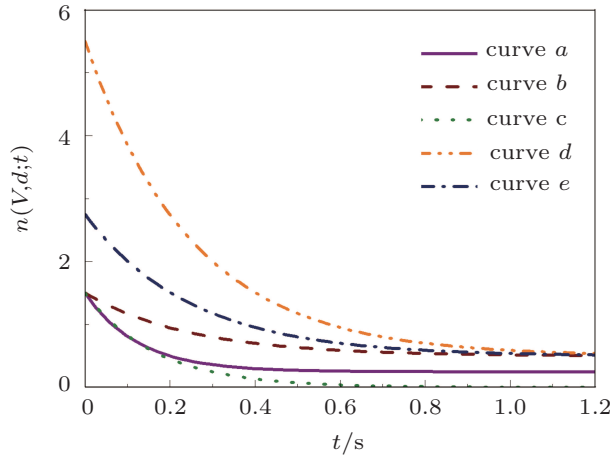


Fig. 1. (color online) Time-evolution of the photon number of the state $\rho(V,d)$ for different values (V,d,κ,g) in a laser channel: (curve *a*) $(2,1,5,1)$; (curve *b*) $(2,1,3,1)$; (curve *c*) $(2,1,3,0)$; (curve *d*) $(10,1,3,1)$; (curve *e*) $(2,1.5,3,1)$.

In Fig. 1, we plot the photon number evolution of the state $\rho(V,d)$ for different values of V, d, κ, and g in a laser channel. From the curves *a*–*c* in Fig. 1, we can see that the initial photon numbers of $\rho(V,d;t)$ are identical for the same V and d and the photon number is larger for the larger V (or larger d) (see the curves d and e in Fig. 1). For any value of V (or d), the photon number always decays monotonously with the time t. However, the larger V (or larger d) can make the decrease of the photon number with the time t faster. Besides, amplitude decay ($g = 0$) causes the faster decrease of the number of photons than laser noise ($g \neq 0$), and $n(V,d;t) \to 0$ (corresponding to the vacuum) for amplitude decay when $t \to \infty$, however it moves to a fixed value related to the parameters κ and g for the laser process.

5. Evolution of Wigner function

In the coherent state representation, the Wigner operator $\Delta(z,z^*)$ is expressed as[9–12]

$$\Delta(z,z^*) = \mathrm{e}^{2|z|^2}\int \frac{\mathrm{d}^2 z'}{\pi^2}|z'\rangle\langle -z'|\exp[2(zz'^* - z^* z')], \tag{28}$$

thus the Wigner functions for DTSs read

$$W(z;t) = \mathrm{Tr}[\rho(V,d;t)\Delta(z,z^*)]. \tag{29}$$

It is thus clear that the normal ordering formula of $\rho(V,d;t)$, i.e., Eq. (17), is useful to derive the evolution of Wigner function for an initial DTS in the laser process. Substituting Eq. (17) into Eq. (29) and using the mathematical integral formula (3), we obtain

$$\begin{aligned} W(z;t) &= c_1 \exp\left[\frac{4-2c(V-1)}{c(V-1)^2}|d|^2 + 2|z|^2\right]\int \frac{\mathrm{d}^2 z'}{\pi^2}\exp[2(zz'^* - z^* z')]\langle -z'|\exp(c_2 d a^\dagger){:}\ \exp(c_3 a^\dagger a){:}\ \exp(c_2 d^* a)|z'\rangle \\ &= \frac{c_1}{\pi(c_3+2)}\exp\left[\frac{4-2c(V-1)}{c(V-1)^2}|d|^2 + 2|z|^2 - \frac{|2z - c_2 d|^2}{c_3+2}\right], \end{aligned} \tag{30}$$

which is the evolutions of Wigner functions for DTSs in the laser process. In particular, setting $g = 0$ in Eq. (30), we obtain the Wigner function evolution of $\rho(V,d)$ for amplitude decay, i.e.,

$$W(z;t) = \frac{1}{\pi[1+(V-1)\,\mathrm{e}^{-2\kappa t}]}\exp\Big[-\frac{2}{1+(V-1)\,\mathrm{e}^{-2\kappa t}}(|z|^2+|d|^2\,\mathrm{e}^{-2\kappa t}-d^*\,\mathrm{e}^{-\kappa t}z-d\,\mathrm{e}^{-\kappa t}z^*)\Big]. \tag{31}$$

For the case of $t = 0$, equation (30) becomes

$$W(z;0) = \frac{1}{\pi V}\exp\Big[-\frac{2}{V}(|z|^2+|d|^2-d^*z-dz^*)\Big],$$

the Wigner function for a DTS. At long times ($t\to\infty$), $W(z;\infty) = (\kappa-g)/[\pi(\kappa+g)]\exp[-2(\kappa-g)\,|z|^2/(\kappa+g)]$, a Gaussian character, which corresponds to the Wigner function for the thermal state $[(1-g/\kappa)]\exp[a^\dagger a\ln(g/\kappa)]$ without the parameters V and d.

In Fig. 2, using Eq. (30), the evolution of the Wigner distribution with given V and d are depicted for several different values of κ, g, and t. It is clearly seen that the Wigner function is always positive and keeps a Gaussian distribution, but its maximum position moves with the different values of d, κ, and t. At long times, the Wigner function moves to the center of the phase space ($z = z^* = 0$) and its shape becomes a round hill corresponding to the Gaussian distribution of $W(z;\infty)$, however the shape of the Wigner function is similar to that of the Wigner function for vacuum when $\kappa \gg g$. Also, the larger κ can shorten the time needed for becoming Gaussian distribution of $W(z;\infty)$, but the larger g is quite the opposite.

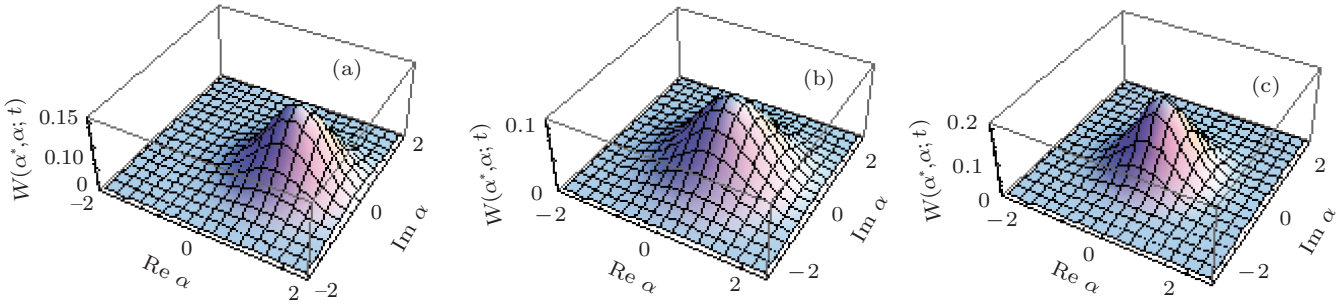


Fig. 2. (color online) Time-evolution of the Wigner function distribution of the state $\rho(V,d)$ with given V and d for different values of κ, g, and t in a laser channel: (a) $(2,1,0)$; (b) $(2,1,1)$; (c) $(5,1,1)$.

6. Entropy evolution of the DTS

In quantum information, the von Neumann entropy for a quantum state has been widely used in entanglement measures, especially for bipartite pure states. For a given density operator ρ, its von Neumann entropy is defined as the following form[13]

$$S(\rho) = -\mathrm{Tr}[\rho\ln\rho]. \tag{32}$$

In view of this definition, we see that the single natural exponential form of ρ is required for calculating its von Neumann. So, in the following section, we first derive the single natural exponential expression of the state $\rho(V,d;t)$. By applying the operator identity (23), $\rho(V,d;t)$ can be rewritten as

$$\begin{aligned}\rho(V,d;t) &= c_1\exp\Big[\frac{4-2c(V-1)}{c(V-1)^2}|d|^2\Big]\,\mathrm{e}^{c_2 d a^\dagger}\,\mathrm{e}^{a^\dagger a\ln(c_3+1)}\,\mathrm{e}^{c_2 d^* a}\\ &= c_1\exp\Big[\frac{4-2c(V-1)}{c(V-1)^2}|d|^2\Big]\,\mathrm{e}^{a^\dagger a\ln(c_3+1)}\exp\Big[\frac{c_2 d}{c_3+1}a^\dagger\Big]\,\mathrm{e}^{c_2 d^* a}.\end{aligned} \tag{33}$$

Further, using the operator theorem

$$\mathrm{e}^A\,\mathrm{e}^B = \exp\Big[A+\frac{\xi B+\eta}{1-\mathrm{e}^{-\xi}}-\frac{\eta}{\xi}\Big],\quad([A,B]=\xi B+\eta), \tag{34}$$

we can change Eq. (33) into a single natural exponential form:

$$\rho(V,d;t) = c_1\exp\left[\left(\frac{4-2c(V-1)}{c(V-1)^2}+\frac{c_2^2}{c_3}+\frac{\ln(c_3+1)}{c_3^2}\right)|d|^2\right]\exp\left[a^\dagger a\ln(c_3+1)+\frac{(c_2 d a^\dagger+c_2 d^* a)\ln(c_3+1)}{c_3}\right]. \tag{35}$$

Thus, the logarithm representation of $\rho(V,d;t)$ is

$$\ln\rho(V,d;t) = c_4+\left(a^\dagger a+\frac{c_2 d a^\dagger+c_2 d^* a}{c_3}\right)\ln(c_3+1), \tag{36}$$

where

$$c_4 = \left(\frac{4-2c(V-1)}{c(V-1)^2} + \frac{c_2^2}{c_3} + \frac{\ln(c_3+1)}{c_3^2}\right)|d|^2 + \ln c_1.$$

Therefore, the von Neumann entropy of the state $\rho(V,d;t)$ can be obtained as

$$\begin{aligned} S(\rho(V,d;t)) &= -c_4 \mathrm{Tr}\left[\rho(V,d;t)\right] - \mathrm{Tr}\left[\rho(V,d;t)\left(a^\dagger a + \frac{c_2 d a^\dagger + c_2 d^* a}{c_3}\right)\right]\ln(c_3+1) \\ &= \frac{c_1 c_4}{c_3}\exp\left[-\frac{c_2^2}{c_3}|d|^2\right] - c_1 \exp\left[\frac{4-2c(V-1)}{c(V-1)^2}|d|^2\right] \\ &\quad \times \mathrm{Tr}\left[\mathrm{e}^{c_2 d a^\dagger}\mathrm{e}^{a^\dagger a \ln(c_3+1)}\mathrm{e}^{c_2 d^* a}\left(a^\dagger a + \frac{c_2 d a^\dagger + c_2 d^* a}{c_3}\right)\right]\ln(c_3+1). \end{aligned} \tag{37}$$

Repeatedly using the operator identity (23), we obtain

$$\begin{aligned} &\mathrm{Tr}\left[\mathrm{e}^{c_2 d a^\dagger}\mathrm{e}^{a^\dagger a \ln(c_3+1)}\mathrm{e}^{c_2 d^* a}\left(a^\dagger a + \frac{c_2 d a^\dagger + c_2 d^* a}{c_3}\right)\right] \\ &= \mathrm{Tr}\left[\mathrm{e}^{c_2 d a^\dagger}\left[(c_3+1)a^\dagger + c_2 d^*\right]\mathrm{e}^{a^\dagger a \ln(c_3+1)}\mathrm{e}^{c_2 d^* a}\left(a + \frac{c_2 d}{c_3}\right)\right. \\ &\quad \left. + \mathrm{e}^{c_2 d a^\dagger}\mathrm{e}^{a^\dagger a \ln(c_3+1)}\mathrm{e}^{c_2 d^* a}\frac{c_2 d^* a}{c_3}\right] \\ &= \mathrm{Tr}\left[:\mathrm{e}^{c_2 d a^\dagger}\mathrm{e}^{a^\dagger a c_3}\mathrm{e}^{c_2 d^* a}\left[(c_3+1)\left(a^\dagger a + \frac{c_2 d}{c_3}a^\dagger + \frac{c_2 d^*}{c_3}a\right) + \frac{c_2^2|d|^2}{c_3}\right]:\right] \\ &= \int \frac{\mathrm{d}^2\alpha'}{\pi}\langle\alpha'|:\mathrm{e}^{c_2 d a^\dagger}\mathrm{e}^{a^\dagger a c_3}\mathrm{e}^{c_2 d^* a}\left[(c_3+1)\left(a^\dagger a + \frac{c_2 d}{c_3}a^\dagger + \frac{c_2 d^*}{c_3}a\right) + \frac{c_2^2|d|^2}{c_3}\right]:|\alpha'\rangle \\ &= \int \frac{\mathrm{d}^2\alpha'}{\pi}\mathrm{e}^{c_2 d\alpha'^*}\mathrm{e}^{c_3|\alpha'|^2}\mathrm{e}^{c_2 d^*\alpha'}\left[(c_3+1)\left(|\alpha'|^2 + \frac{c_2 d}{c_3}\alpha'^* + \frac{c_2 d^*}{c_3}\alpha'\right) + \frac{c_2^2|d|^2}{c_3}\right] \\ &= -\frac{1}{c_3}\left[1 + c_3 + \frac{c_2^2}{c_3^2}|d|^2\right]\exp\left(-\frac{c_2^2}{c_3}|d|^2\right). \end{aligned} \tag{38}$$

Thus, combining Eqs. (37) and (38) we finally obtain entropy evolution of the DTS in the laser process, that is,

$$\begin{aligned} S(\rho(V,d;t)) = \frac{c_1}{c_3}\exp&\left[\left(\frac{4-2c(V-1)}{c(V-1)^2} - \frac{c_2^2}{c_3}\right)|d|^2\right] \\ &\cdot\left[\left(\frac{4-2c(V-1)}{c(V-1)^2} - \frac{c_2^2}{c_3}\right)|d|^2 + \ln c_1 + \frac{(c_3+1)c_3^2 + 2c_2^2|d|^2}{c_3^2}\ln(c_3+1)\right]. \end{aligned} \tag{39}$$

When $g = 0$, $S(\rho(V,d;t))$ refers to the entropy evolution of $\rho(V,d)$ for amplitude decay, i.e.,

$$\begin{aligned} S(\rho(V,d;t)) = &-\ln\frac{2}{2+(V-1)\mathrm{e}^{-2\kappa t}} \\ &-\frac{(V-1)\mathrm{e}^{-2\kappa t}}{2}\ln\frac{(V-1)\mathrm{e}^{-2\kappa t}}{2+(V-1)\mathrm{e}^{-2\kappa t}}. \end{aligned}$$

For an initial time $t = 0$, $S(\rho(V,d;0))$ is the entropy of the DTS. For the case $t \to \infty$,

$$S(\rho(V,d;\infty)) = -\frac{g}{\kappa}\ln\frac{g}{\kappa} + \ln\left(1-\frac{g}{\kappa}\right),$$

corresponding to the entropy of the thermal state $[(1-g/\kappa)]\exp[a^\dagger a\ln(g/\kappa)]$.

Here, we can numerically compute the time-evolution entropy of $\rho(V,d)$ for some reasonable parameter values. A comparison of Fig. 1 with Fig. 3 indicates that many evolution features of the entropy are similar to those of the photon number. For instance, the initial entropy of $\rho(V,d;t)$ is also decided by V and d, the entropy always monotonously decreases with the time t, and the amplitude decay leads to $S(\rho(V,d;\infty)) \to 0$ but the laser noise cannot. On the other hand, the parameter d has a more significant effect on the initial entropy in comparison with V, which is opposite to that of the photon number. For a given time t, the entropy decreases during the early stage and increases during the late stage along with V increasing.

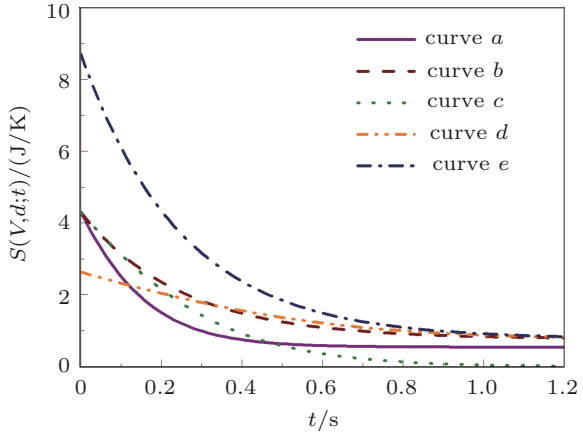


Fig. 3. (color online) Time-evolution of the von Neumann entropy of the state $\rho(V,d)$ for different values V, d, and κ, g, taking the same values as those of Fig. 1, in a laser channel.

7. Conclusions

In conclusion, in terms of the Kraus operator solutions of master equation describing the laser channel, we obtained the analytical evolution law of a DTS in a laser channel. By examining the time-dependent density operator $\rho(V,d;t)$, we found that the state $\rho(V,d;t)$ finally decays to a highly classical thermal field with the density operator $[(1-g/\kappa)]\exp[a^{\dagger}a\ln(g/\kappa)]$. We analytically and numerically investigated the evolution law of the photon number, the Wigner function, and the entropy in the laser channel. The results show that the parameter d has a more important influence on the initial entropy than V, opposite to that of the photon number, and their changes are different with V increasing for a given time t. For the Wigner function, it is always a positive and Gaussian distribution and its maximum position moves with the values of d, κ, and t. At long times, the Wigner function becomes a Gaussian distribution of $W(z;\infty)$ at the center of phase space. For any open system, the effects of noise on the system are always inevitable in the fundamental dynamics processes. In terms of Eq. (12), we can derive the analytical evolution law of any initial quantum state in a laser channel and may find some valuable ways how to effectively reduce laser transmission noise.

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