Tunneling dynamics of a few bosons with both two- and three-body interactions in a double-well potential*

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We investigate the tunneling dynamics of a few bosons with both two- and three-body interactions in a double-well potential. Uncorrelated tunneling of Rabi oscillation with the minimum period can happen only when the two- and three-body interactions satisfy a critical condition, i.e., the effective interaction energy is minimized. When the atomic interactions are slightly away from the critical condition in the weak interaction regime, the uncorrelated tunneling exhibits collapse-revival character. When the atomic interactions are strong and far away from the critical condition, the correlated tunneling with Rabi oscillation occurs. The tunneling period (the period of collapse-revival) increases (decreases) when the rate between the two-body and three-body interactions is away from the corresponding critical condition or when the number of bosons increases. Further, the tunneling properties are understood with the help of the energy spectrum of the system. Eventually, the effect of the initial configuration on the tunneling dynamics of a few bosons for both odd and even numbers of bosons is studied, which results in intriguing consequences.

Keywords: Bose–Einstein condensate, Rabi oscillation, collapse-revival

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1. Introduction

Ever since the first experimental realization of dilute degenerate atomic gases in 1995, i.e., Bose–Einstein condensates (BEC),[1–4] increasing attention has been focused on the study of dynamical properties concerning certain quantum effects, such as tunneling, at an unprecedented standard of accuracy and controllability.[5] As a prototype system to study interference and tunneling in great detail, a double well is the best candidate.[6,7] Bosons tunneling through an impassable obstacle is a classic matter in quantum mechanics.[8–10] For example, the tunneling dynamics for a few bosons has been observed to undergo Josephson oscillations[11,12] in which the bosons simply tunnel back and forth between the two wells. On the few-body levels, it is similar to the situation of repulsive atom pairs in a lattice, whose stability[13] and dynamics[14] have recently been investigated. The quantum dynamics in both strong and weak interaction regimes have been studied theoretically using the Bose–Hubbard model assuming the validity of the lowest band approximation.[15–18] These studies illuminate relevant tunneling mechanisms and resonances. A two-mode approximation can demonstrate the tunneling between different Bloch bands in an optical lattice,[19] which is valid when the energy difference between the two lowest single particle eigenstates is far smaller than that of all other energy states. Thus, one can study the crossover from a correlated strong interaction to a weak interaction regime. The crossover from the uncorrelated to the fermionization regime has been investigated for a few bosons[20,21] and reveals a transition from Rabi oscillations to fragmented pair tunneling via a highly delayed tunneling process analogous to the self-trapping for condensates.

Generally speaking, the interactions between bosons are dominated by two-body interactions,[22] whereas when the distribution of bosons is concentrated, for example, in the case of the miniaturization of the devices in the integrated atom optics and deep lattices, three-body interactions start to play an important role and can also provide a train of novel quantum phenomena.[23–29] Up to now, multi-body interactions have been honored as inelastic loss resonances.[30,31] Techniques of creating and exploiting inter-particles interactions hold.[29,32,33] With both two- and three-body interactions,[34,35] it is likely to go beyond some limitations of the Bose–Hubbard model. When both two- and three-body interactions between these bosons are included, the ground states and tunneling dynamics will be modified significantly.[23–29] The tunneling dynamics may exhibit interesting and novel effects with various interaction parameter(s)[36] and the initial configurations of the bosons.[37]

In this paper, we investigate the properties concerning tunneling dynamics of a few bosons, with both two- and three-body interactions in a double-well potential. Uncorrelated tunneling of Rabi oscillation with the minimum period can hap-

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pen only when the two- and three-body interactions satisfy a critical condition, i.e., the effective interaction energy is minimized. When the atomic interactions are slightly away from the critical condition in the weak interaction regime, the uncorrelated tunneling exhibits a collapse-revival character. When the atomic interactions are strong and far away from the critical condition, correlated tunneling with Rabi oscillation still occurs. The tunneling period (the period of collapse-revival) increases (decreases) when the rate between two-body and three-body interactions is away from the corresponding critical condition or when the number of bosons increases. The tunneling properties are understood with the help of the energy spectrum of the system. Eventually, the effect of the initial configuration on the tunneling dynamics of a few bosons for both odd and even numbers of bosons is studied,[38] which results in intriguing consequences.

The paper is organized as follows. Section 2 introduces our model and briefly reviews the computational method. In Section 3, a critical condition for Rabi oscillation is presented and the energy spectrum of the system is given to understand the tunneling dynamics for a few bosons. Section 4 gives a concise presentation of the tunneling dynamics for a few bosons in a symmetric double well by means of the percentage of bosons in the right well. In Section 5, we discuss the effect of the initial configurations on the tunneling dynamics for both odd and even numbers of bosons respectively. Finally, we present our conclusions in Section 6.

2. Model and method

When both two- and three-body interactions are considered, the tunneling dynamics of N identical bosons in a double-well potential can be described by the following two-mode Bose–Hubbard model:

$$\begin{aligned}\hat{H} &= -J(\hat{a}_{\rm l}^{\dagger}\hat{a}_{\rm r}+\hat{a}_{\rm r}^{\dagger}\hat{a}_{\rm l})+\frac{U_2}{2}[\hat{n}_{\rm l}(\hat{n}_{\rm l}-1)+\hat{n}_{\rm r}(\hat{n}_{\rm r}-1)]\\ &\quad+\frac{U_3}{6}[\hat{n}_{\rm l}(\hat{n}_{\rm l}-1)(\hat{n}_{\rm l}-2)+\hat{n}_{\rm r}(\hat{n}_{\rm r}-1)(\hat{n}_{\rm r}-2)], \quad (1)\end{aligned}$$

where l and r index the left and right wells, and $\hat{a}_j^{\dagger}$ and $\hat{a}_j$ are the creation and annihilation operators, respectively. The number operator is $n_j=\hat{a}_j^{\dagger}\hat{a}_j$ ($j=$ l, r), and the total number of bosons in the double wells is $N=\hat{a}_{\rm l}^{\dagger}\hat{a}_{\rm l}+\hat{a}_{\rm r}^{\dagger}\hat{a}_{\rm r}$, which is a conserved quantity while the counterpart of the dimension in the Hilbert space is $N+1$. J represents the tunneling rate (here, we set $J=1$) between the two wells. U_2 and U_3 are the effective two- and three-body interactions between particles in an individual well, respectively. In this paper, we mainly concentrate on the situation where the two-body interaction is repulsive ($U_2>0$) while the three-body interaction can be repulsive ($U_3>0$) or attractive ($U_3<0$). In reality, U_2 and U_3 can be controlled by the Feshbach resonance technique in an experiment.

To obtain the tunneling dynamics of the system, we consider the representation of the Hamiltonian $\hat{H}$ in the Fock space and expand the vector state of the system $|\psi(t)\rangle$ on the basis of Fock states. Set

$$|\psi(t)\rangle=\sum_{n=0}^{N}\frac{c_n(t)}{\sqrt{n!(N-n)!}}\hat{a}_{\rm l}^{\dagger n}\hat{a}_{\rm r}^{\dagger N-n}|0\rangle, \quad (2)$$

where n is the number of bosons in the left well and the rest of the $(N-n)$ bosons are in the right well, $c_n(t)$ ($n=0,1,2,\ldots,N$) is the boson occupation probability in the left well. With Eqs. (1) and (2), the dynamic evolutive equations related to c_n can be obtained from the time-dependent Schrödinger Equation $\mathrm{i}\hbar\frac{\mathrm{d}\psi(t)\rangle}{\mathrm{d}t}=\hat{H}|\psi(t)\rangle$, and read explicitly

$$\mathrm{i}\hbar\frac{\mathrm{d}c_n}{\mathrm{d}t}=-(K_nc_{n+1}+K_{n-1}c_{n-1})+(V_n+B_n)c_n, \quad (3)$$

where $K_n=J\sqrt{(n+1)(N-n)}$, $V_n=U_2[n(n-1)+(N-n)(N-n-1)]/2$, $B_n=U_3[n(n-1)(n-2)+(N-n)(N-n-1)(N-n-2)]/6$. The normalization condition $\sum_{n=0}^{N}|c_n(t)|^2=1$ holds. By integrating Eq. (3), the tunneling dynamics of different numbers of bosons can be obtained. Note that various initial conditions are configured for the system with regard to different numbers of bosons. For example, when $N=3$ (i. e., the total number is odd), the various initial conditions are denoted as (1000), (0100), (0010), and (0001) respectively, where (1000) indicates that initially all bosons are in the right well and the left well is empty, (0100) means that there is only one boson in the left well and the other two are in the right well, (0010) indicates that two bosons are in the left well while only one boson is in the right well, etc. Similarly, the possible initial configurations for the case of $N=4$ (i. e., the total number is even) are (10000), (01000), (00100), (00010), and (00001) with the corresponding implications as above. For example, (00100) denotes that there are two bosons in each well, which implies that the bosons are equally distributed in the two wells.

3. Energy spectra of the system

It is accessible to capture the effective interaction energy $E_{\rm in}$ from Eq. (3), i.e.,

$$\begin{aligned}E_{\rm in} &= \frac{U_2}{2}(N^2+2n^2-2Nn-N)+\frac{U_3}{6}[(3N-6)n^2\\ &\quad+(6N-3N^2)n+(N^3-3N^2+2N)]. \quad (4)\end{aligned}$$

The effective interaction energy $E_{\rm in}$ is minimized when U_2 and U_3 satisfy a critical condition, which can be captured analytically by the variational analysis of $E_{\rm in}$, i.e., via $\partial E_{\rm in}/\partial n=[2U_2+U_3(N-2)](n-N/2)=0$, which results in

$$U_2+(N/2-1)U_3=0, \quad (5)$$

or

$$n=N/2. \quad (6)$$

That is, no extra interaction energy cost for one particle tunneling from one well to the other and Rabi oscillation can be observed if initially, the particles are equally populated in the two wells ($n = N/2$) or if equation (5) is satisfied.

To give a better understanding of the system's tunneling dynamics, we show the variation of the total energy spectrum against U_3/U_2 in Fig. 1. The total energies E_n of the system can be obtained by means of the exact diagonalization of Eq. (3). The specific E_n for three bosons ($N = 3$) has been computed analytically both in the strong ($U_2 = 6$) and the weak ($U_2 = 0.05$) interaction regimes respectively. Here $N+1 = 4$ bands hold (see Figs. 1(a) and 1(b)),

$$\begin{aligned} E_0 &= \frac{1}{2}(2+4U_2+U_3) - M_1, \\ E_1 &= \frac{1}{2}(2+4U_2-U_3) + M_1, \\ E_2 &= \frac{1}{2}(-2+4U_2+U_3) - M_2, \\ E_3 &= \frac{1}{2}(-2+4U_2+U_3) + M_2, \end{aligned} \tag{7}$$

where $M_1 = \sqrt{16-8U_2+4U_2^2-4U_3+4U_2U_3+U_3^2}$, $M_2 = \sqrt{16+8U_2+4U_2^2+4U_3+4U_2U_3+U_3^2}$. However, it becomes difficult to diagonalize for more bosons ($N \geq 4$) analytically, then the corresponding E_n could be obtained numerically. For example, the spectrum for $N = 4$ both in the strong ($U_2 = 6$) and the weak ($U_2 = 0.05$) interaction regimes are exhibited in Figs. 1(c) and 1(d) respectively.

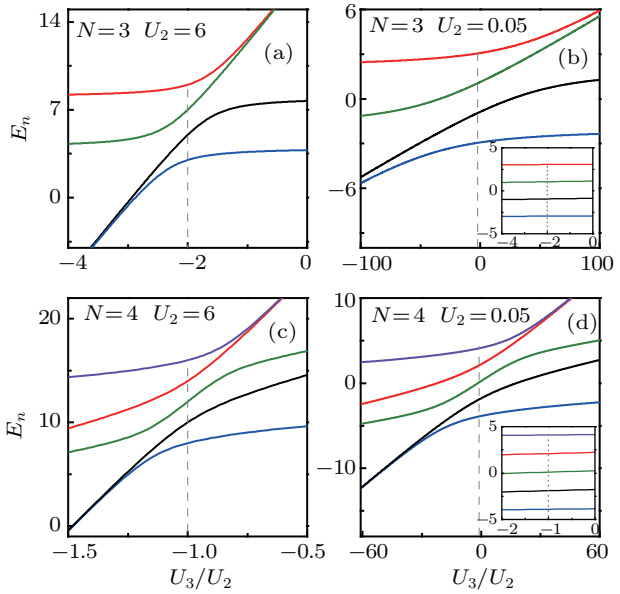


Fig. 1. (color online) The energy band diagrams with different U_3/U_2 on the basis of $U_2 = 6$ (0.05) for (a), (b) $N = 3$ and (c), (d) $N = 4$. The dotted line represents the corresponding critical condition of Eq. (5), i.e., $U_3/U_2 = -2$ for $N = 3$ while $U_3/U_2 = -1$ for $N = 4$. Note that E_0, E_1, E_2, E_3, and E_4 are denoted by blue, black, green, red, and purple lines, respectively.

When the atomic interactions are weak, the low-lying energies of the spectrum are obtained by distributing N bosons over the lowest symmetric single-boson orbitals. This yields $N+1$ energy levels and tunneling is always observed in the double well with frequency $\omega_{mn} = E_m - E_n$, which is dominated by the two closest energy bands for almost complete initial imbalance. When equation (5) is satisfied (indicated by the dotted gray lines in Fig. 1), the levels are equidistant, and the system should do the Rabi oscillation with a single frequency $\omega = \omega_{01} = \omega_{12} = \omega_{23}$. As U_3/U_2 turns away from the critical point, the corresponding equidistance should be slightly broken, and we obtain a superposition of $N+1$ very close frequencies ω_{mn}, which would result in the formation of the beat pattern in the tunneling dynamics. However, as U_3/U_2 becomes far away from the critical point, the two upper levels (for $U_3/U_2 > 0$) or the two lower levels (for $U_3/U_2 < 0$) virtually glue to one another to form a doublet, which means self-trapping occurs. According to $T = 2\pi/\omega$, the Rabi oscillation period T_R can be denoted as

$$T_R = 2\pi/(\Delta E)_{\min}, \tag{8}$$

which is suitable for the certain number of bosons when U_3/U_2 is definite. Furthermore, the evolution of T_R with U_3/U_2 will be demonstrated afterwards.

4. Tunneling dynamics

Here we are primarily interested in the coupled effect of two-body repulsive interactions U_2 and three-body interactions U_3 on the quantum tunneling dynamics for a few bosons in a symmetric double-well potential. We focus on the tunneling dynamics by monitoring the percentage of bosons in the right well with the progress in time, i.e., $P_R(t)$, which is experimentally measurable and expressed as[39]

$$P_R(t) = \sum_{n=0}^{N} \frac{N-n}{N} |c_n(t)|^2. \tag{9}$$

At the beginning ($t = 0$), the system is prepared in a definite state (all bosons are in the right well), for example, (1000) for $N = 3$.

In the strong interaction regime ($U_2 = 6$), the typical evolutions of $P_R(t)$ for different U_3/U_2 are shown in Fig. 2(a). Rabi oscillation with the minimum oscillation period can be observed when U_2 and U_3 satisfy the condition of Eq. (5) (see Figs. 2(a-2) and 2(d-2)). Figure 1 indicates that, when $U_2 = 6$, the correlated tunneling of three bosons ($N = 3$) occurs when the condition of Eq. (5) is not satisfied (see the first, the third, and the fourth columns in Fig. 2), i. e., three bosons tunnel together with the same frequency back and forth between the two wells, which is clearly presented by the evolution of the percentage of the bosons in the right well $P_R(t)$ (Fig. 2(a)) and further demonstrated by the occupation probabilities in the left well $N|c_n(t)|^2$ (Figs. 2(b) and 2(d)). We find that, when the condition of Eq. (5) is not satisfied (see the first, the third, and the fourth columns), $|c_0(t)|^2$ and $|c_3(t)|^2$ are

back and forth almost between 0 and 1, however $|c_1(t)|^2$ and $|c_2(t)|^2$ remain around zero, which could be neglected. This means the three bosons tunnel together between the two wells in these cases (see Figs. 2(d-1), 2(d-3), and 2(d-4)). Furthermore, the tunneling dynamics is proved by the frequency spectra of $P_R(t)$ in Fig. 2(c). When U_3/U_2 does not satisfy the condition of Eq. (5), the oscillation frequency (period) decreases (increases) with the increasing distance to the critical value of Eq. (5). We find that the frequency spectrum has only one peak for any U_3/U_2. As a result, when U_3/U_2 does not satisfy the critical condition of Eq. (5) ($U_3/U_2 \neq -2$ for $N = 3$), the tunneling is correlated, otherwise, it is uncorrelated, which is further exhibited in Fig. 2(d). This agrees with the energy band structure shown in Fig. 1(a) for $N = 3$ with $U_2 = 6$, where the tunneling is dominated by the two upper levels with frequency $\omega = \omega_{23} = E_3 - E_2$.

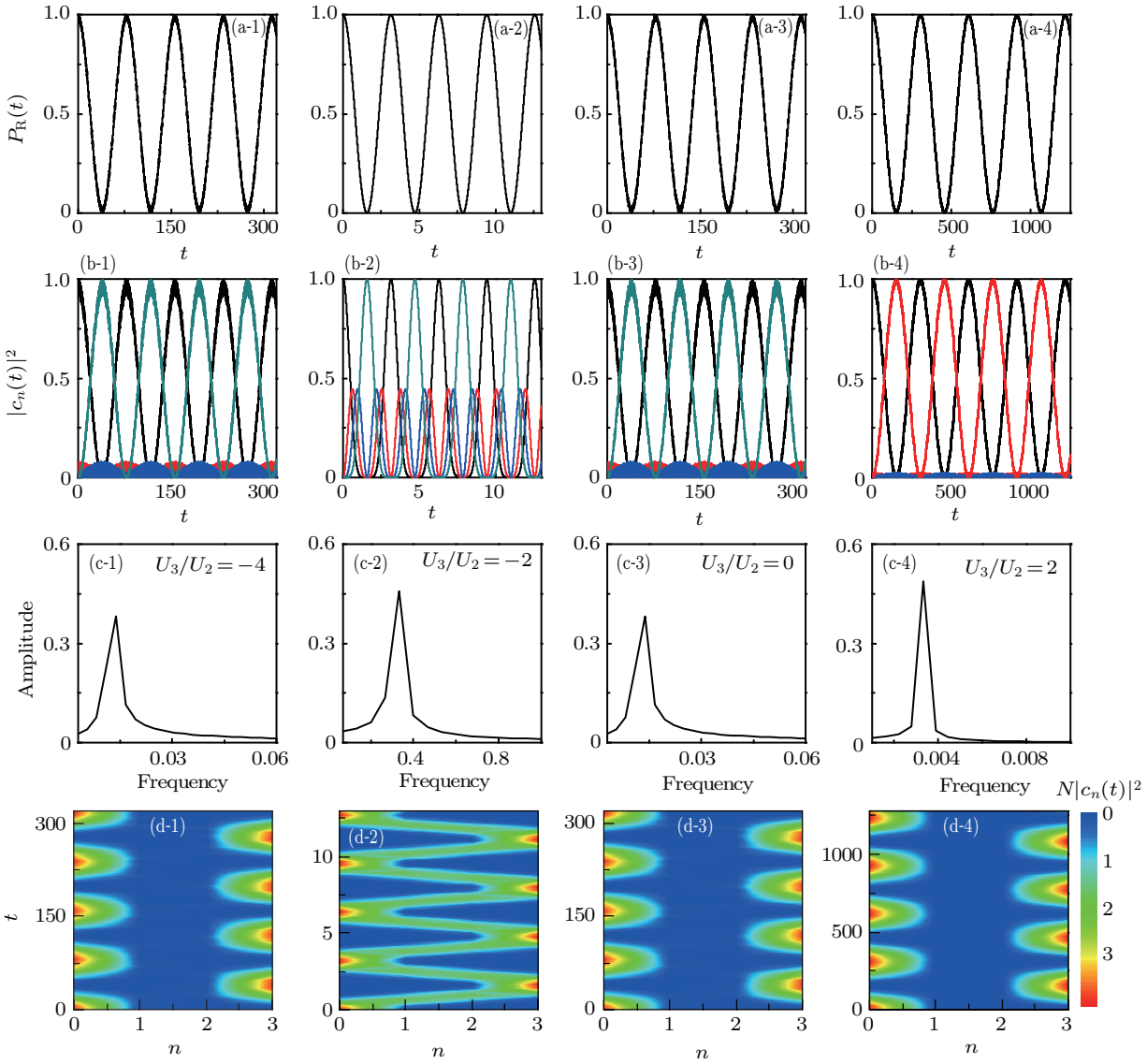


Fig. 2. (color online) The time evolution of (a) $P_R(t)$ and (b) $|c_n(t)|^2$, note that $|c_0(t)|^2$, $|c_1(t)|^2$, $|c_2(t)|^2$, and $|c_3(t)|^2$ are denoted by black, red, blue, and green lines, respectively. (c) The corresponding frequency spectra of $P_R(t)$. (d) The evolution of $N|c_n(t)|^2$ with the atomic number in the left well and time t. The plots are all for different U_3/U_2 with $U_2 = 6$ and $N = 3$.

Uncorrelated Rabi oscillation of three bosons ($N = 3$) with the minimum oscillation period happens only when U_3/U_2 satisfies the critical condition of Eq. (5) (see the second column in Fig. 2). According to the energy spectra given in Fig. 1, we find that the oscillation period T_R increases when U_3/U_2 is away from the critical condition ($U_3/U_2 = -2$ for $N = 3$), which is further illustrated in Figs. 3(a)–3(c). Figure 3 indicates that the oscillation period T_R is always an approximate constant 3.14 (the minimum period for Rabi oscillation) when U_3/U_2 satisfies the condition of Eq. (5), which is independent of the number of bosons. However, bosons will do correlated Rabi oscillation when U_3/U_2 is away from the critical condition of Eq. (5) (the first, the third, and the fourth columns in Fig. 2), but the period of Rabi oscillation becomes larger, which implies the occurrence of the trapping scenario. We find that T_R increases when N increases or when U_3/U_2 is

far away from the corresponding critical condition of Eq. (5). Hence, when the condition of Eq. (5) is not satisfied, the evolution of T_R not only depends on the number of bosons but also results from U_3/U_2.

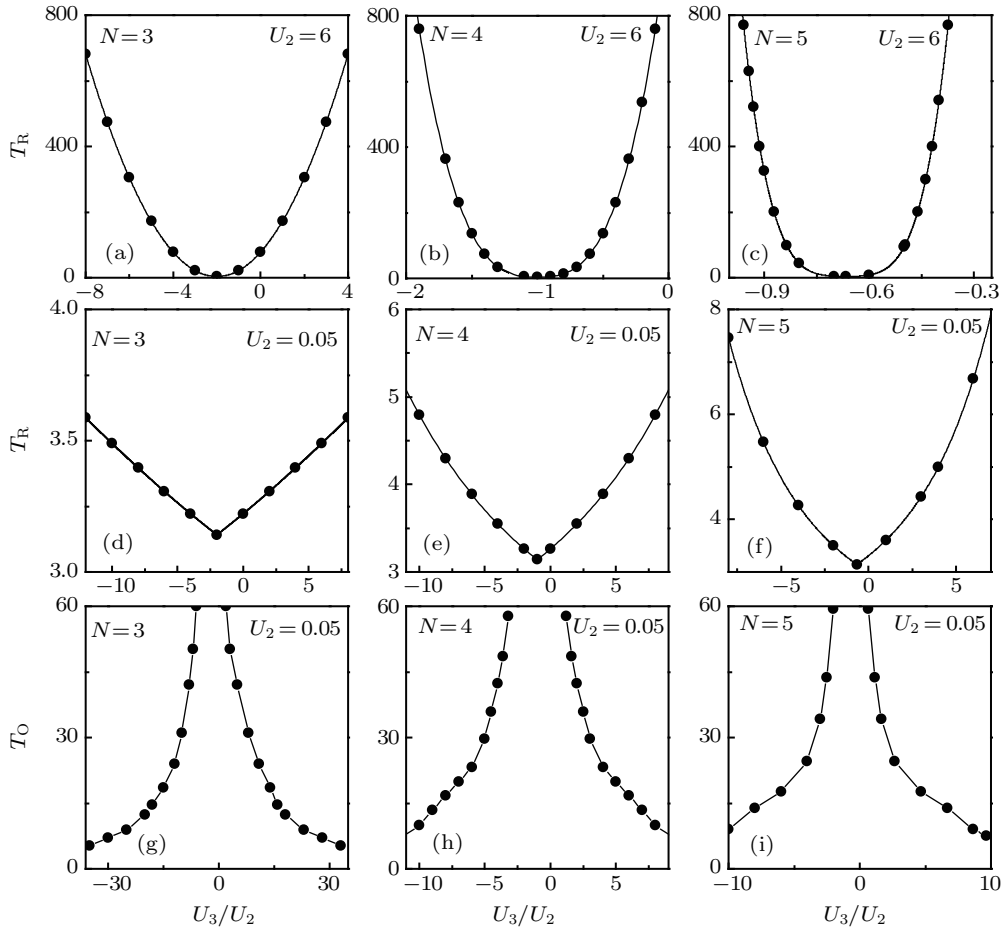


Fig. 3. (a)–(f) The oscillation period T_R and (g)–(i) the collapse-revival period T_O against U_3/U_2.

For the weak interaction regime ($U_2 = 0.05$), the evolution of $P_R(t)$, $|c_n(t)|^2$, the corresponding frequency spectra, and $N|c_n(t)|^2$ are demonstrated in Fig. 4. When U_3/U_2 satisfies the critical condition of Eq. (5), uncorrelated Rabi oscillation with the single minimum period still happens (see the second column of Fig. 4). Otherwise, when U_3/U_2 does not satisfy the critical condition of Eq. (5), collapse-revival oscillation of $P_R(t)$ occurs (the first, the third, and the fourth columns in Fig. 4). The oscillating period T_R (the period of collapse-revival T_O) increases (decreases) when U_3/U_2 is away from the corresponding critical condition or when N increases. This is clearly shown in Figs. 3(d)–3(i) for different N. For the case of the weak interaction and where the condition of Eq. (5) is not satisfied, the individual tunneling with different frequencies (three peaks occur in the frequency spectra in Figs. 4(c-1), 4(c-3), and 4(c-4)) dominates, as is the case of normal Josephson junctions, which results in the collapse-revival phenomena. This agrees with the energy band structure shown in Fig. 1 for $N = 3$ and $U_2 = 0.05$, as U_3/U_2 is away from the critical point of $U_3/U_2 = -2$, the energy levels are slightly non-equidistance (see the inset in Fig. 1), so the superposition of four very close tunneling frequencies results in the collapse-revival phenomenon. Initially, the system is prepared in a definite state and correlated tunneling dominates, i.e., the oscillation of $|c_0(t)|^2$ and $|c_3(t)|^2$ dominates. However, as time goes on, the oscillations corresponding to different initial excitations of $|c_n(t)|^2$ with different frequencies take place, thereby leading to the uncorrelated tunneling collapse. As time further increases, the correlation is partly restored and revived. This process is repeated and the collapses-revivals occur.

Figures 2 and 4 indicate that, when the atomic interaction changes from strong ($U_2 = 6$) to weak (U_2=0.05), the tunneling dynamics of a few bosons in the double well undergoes the transition from correlated tunneling (Rabi oscillation) to individual uncorrelated tunneling (collapse-revival). This is further clearly illustrated in Fig. 5. Initially, the system is equipped with a definite state (1000) for $N = 3$, which implies that all three bosons are in the right well. U_2 gradually varies from a strong regime to a weak regime ($U_2 = 6$, 4, 2, 1.5, 1, 0.5, 0.2, 0.1, 0.05 respectively) on the condition

that $U_3/U_2 = -4$. The transitive process from correlated Rabi oscillation to the uncorrelated collapse-revival is presented in Fig. 5. As U_2 decreases and at $U_2 = 2$, there are faster oscillations with smaller amplitude, however, $P_R(t)$ eventually varies from the maximum amplitude (one) to the minimum amplitude (zero) indicating the Rabi oscillation, i.e., the correlated tunneling. There is an increasing temporary decay of the oscillation amplitude (see for $U_2 = 1.5$, 1, and 0.5). Suffering from the further decrease of U_2, the tunnelings experience the transformation from Rabi oscillation to the phenomenon of collapses-revivals, i.e., the uncorrelated tunneling. The tunneling period decreases with the decrease of U_2, gradually, correlated tunneling vanishes and uncorrelated tunneling occurs simultaneously.

The two classical quantum phenomena, i.e., the Rabi tunneling and self-trapping phenomenon are further presented in Fig. 6. Note that equation (5) illustrates a critical condition for complete Rabi oscillation, which is exhibited with the black line in Fig. 6, for instance, $U_3 = -2U_2$ for $N = 3$ (1000), $U_3 = -U_2$ for $N = 4$ (10000), and $U_3 = -2U_2/3$ for $N = 5$ (100000). Figure 6 shows the transition from complete Rabi oscillation (Eq. (5) is satisfied) to self-trapping (Eq. (5) is not satisfied), which depends on the atomic interactions and particle number. Note that the red region is the self-trapping region, where the minimum $P_R(t)$ is around 1, the blue region is the Rabi tunneling region, where the minimum $P_R(t)$ is around zero. It is clear that, when the minimum $P_R(t)$ is around zero, ($P_R(t)$ varies from 1 to 0, i.e., bosons do Rabi oscillation (blue region); otherwise, tunneling halts and the trapping phenomenon (red region) occurs when $P_R(t)$ varies from 1 to 1. We find that the tunneling region becomes narrow gradually with the increase of the number of bosons, which is in good agreement with the tunneling period T_R shown in Fig. 3.

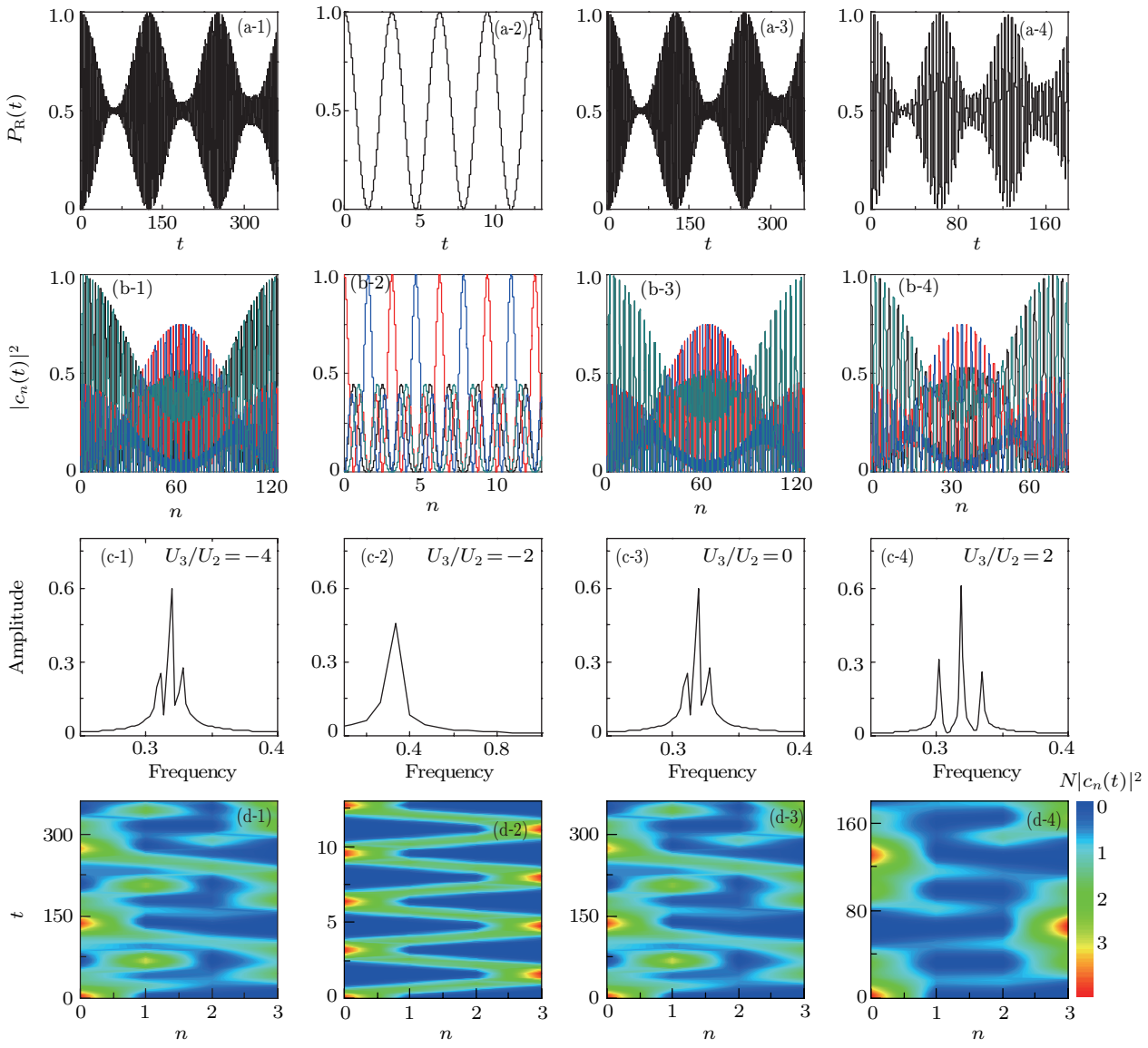


Fig. 4. (color online) Same as in Fig. 2, but for weak two-body interaction with $U_2 = 0.05$.

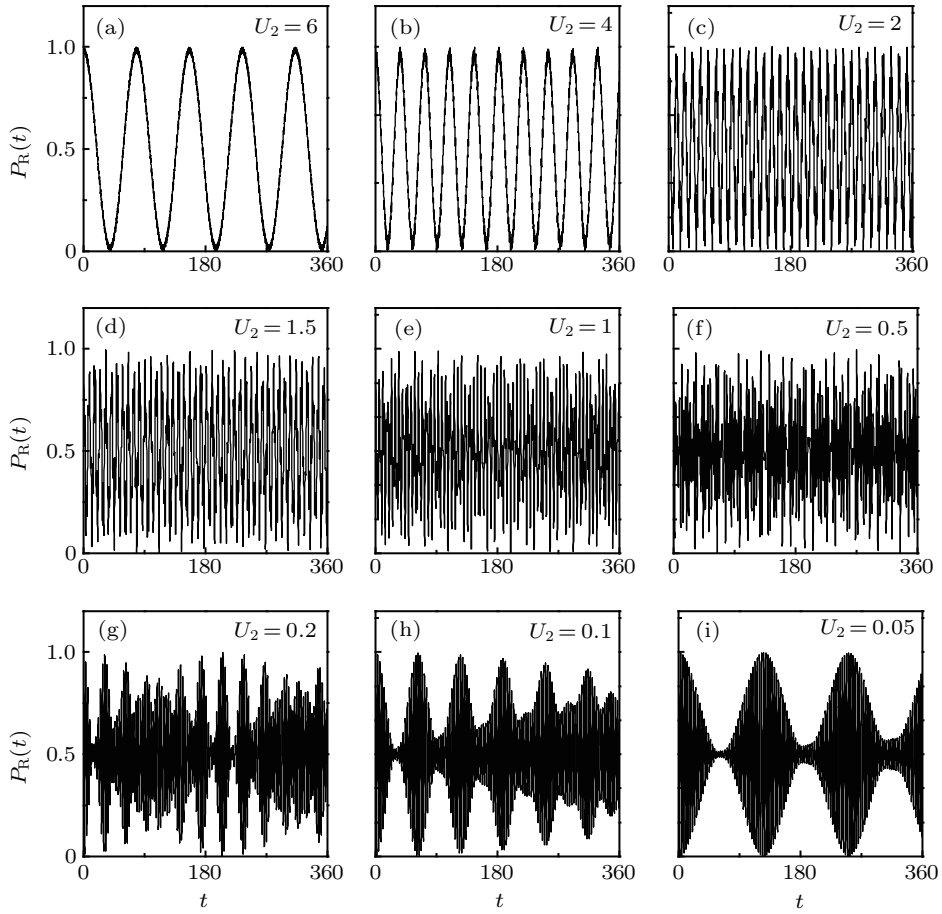


Fig. 5. The time evolution of $P_R(t)$ with $N=3$ and $U_3/U_2=-4$ for (a)–(i) different U_2.

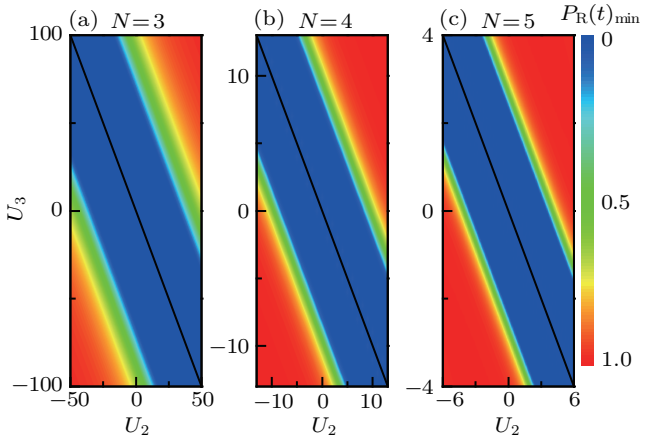


Fig. 6. (color online) The dynamical phase diagram in the U_3–U_2 plane for (a) $N=3$, (b) $N=4$, and (c) $N=5$. Note that the red region represents the self-trapping region while the blue one stands for the Rabi tunneling region.

Furthermore, to have a better understanding of the tunneling dynamics for a large number of bosons, the time averaged dynamics is obtained by using a time averaged $P_R(t)$, which is denoted as

$$\alpha = \frac{1}{T}\int_0^T P_R(t)\,dt, \tag{10}$$

where $T \equiv 3000$. Obviously, the bosons are entirely trapped in the left well when $\alpha = 0$, when $\alpha = 0.5$, the bosons tunnel back and forth between the double wells, while the bosons are trapped in the right well when $\alpha = 1$. α is a function of $\lg(1/N)$ in the weak regime and $1/N$ in the strong interaction regime, and different two- and three-body interactions are demonstrated in Fig. 7. For the weak interaction in Fig. 7(a), when $U_3 = U_{3c} = U_2/(1-0.5N)$, there is a critical boson number N_c that $\alpha = 0.5$ for $N < N_c$ and $\alpha < 0.5$ for $N > N_c$, namely, all bosons tunnel between right and left well for $N < N_c$ while the bosons are trapped in the left well for $N > N_c$. When $U_3 \neq U_{3c}$, there are two critical boson numbers N_{c1} and N_{c2} that $\alpha = 0.5$ for $N < N_{c1}$, $\alpha > 0.5$ for $N_{c1} < N < N_{c2}$, and $\alpha < 0.5$ for $N > N_{c2}$, that is, all bosons tunnel between the right and left wells for $N < N_{c1}$, the bosons are trapped in the right well for $N_{c1} < N < N_{c2}$, and the bosons are trapped in the left well for $N > N_{c2}$. Note that N_{c1} and N_{c2} depend on the atomic interaction. For the strong interaction in Fig. 7(b), the time averaged dynamics is similar to the case of the weak interaction, but the critical boson number decreases significantly. Particularly, when $U_3 = U_{3c}$, $N_c = 100$ for weak interaction (see Fig. 7(a)) while $N_c = 10$ for strong interaction (see Fig. 7(b)). Furthermore, a large enough boson number can result in self-trapping. In a word, tunneling dynamics as well as the tunneling period are not only dependent on the interac-

tion between the bosons, but they are also affected by boson number N, which is in good agreement with that in Figs. 3 and 6.

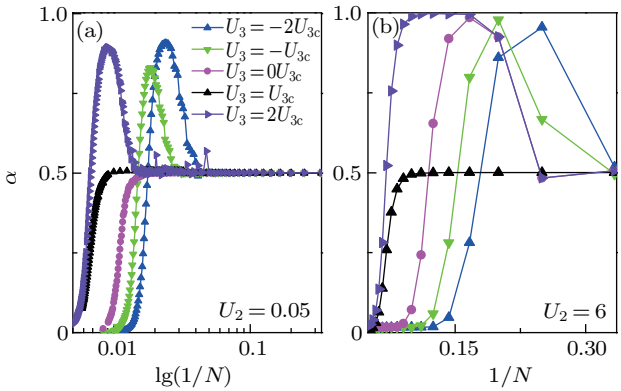


Fig. 7. (color online) The integrated value of $P_R(t)$, i.e., α, as a function of (a) $\lg(1/N)$ in the weak and (b) $1/N$ in the strong interaction regimes.

5. The effect of initial configurations

Motivated by the prospects of creating different initial configurations successfully in experiments, we study the tunneling dynamics subject to different initial states for both odd ($N=3$) and even ($N=4$) numbers of bosons. With a close scrutiny of different initial conditions, we find that, not only in the strong ($U_2=6$) but also in the weak ($U_2=0.05$) interaction regime, interesting observations (see Fig. 8) are hunted for in the symmetric double well no matter whether U_3/U_2 satisfies the critical condition of Eq. (5) or not.

For the odd number of bosons ($N=3$), the initial condition of having unequal numbers of bosons in the two wells, i.e., (0100) denoted by the blue lines in Figs. 8(a)–8(d), results in some interesting consequences. Both when $U_2=6$ and $U_2=0.05$, $P_R(t)$, corresponding to (0100), starts with 2/3 no matter whether U_3/U_2 satisfies the critical condition of Eq. (5) or not, which is revealed in Figs. 8(a)–8(d). When $U_2=6$ with any U_3/U_2 or when $U_2=0.05$ with $U_3/U_2=-2$ (the condition of Eq. (5) is satisfied), we find that it is most discernible when the population difference between the two wells is unity, which is shown in Figs. 8(a)–8(c). However, when $U_2=0.05$ with $U_3/U_2=2$ (the condition of Eq. (5) is not satisfied), as time progresses, $P_R(t)$ modulates between 0.75 and 0.25 (shown in Fig. 8(d)). Thus in the weak coupling regime ($U_2=0.05$) and the condition of Eq. (5) is not satisfied, the fraction of the total number of bosons occupying the right well becomes larger (smaller) than 2/3 (1/3), thereby indicating a tendency of accumulation of particles in one of the wells, which can be observed during $t\approx 12$–50 in Fig. 8(d).

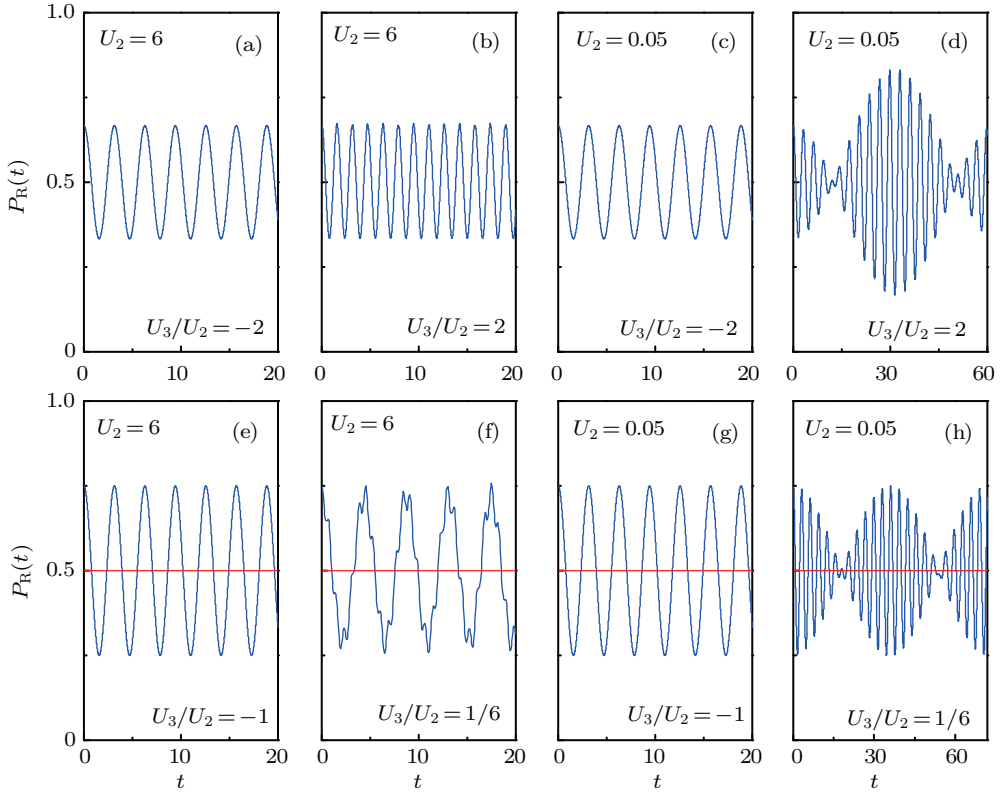


Fig. 8. (color online) The time evolution of $P_R(t)$ for some interesting initial configurations: (a)–(d) $N=3$ with (0100), while $U_3/U_2=-2$ (the condition of Eq. (5) is satisfied) in (a) and (c), and $U_3/U_2=2$ (the condition of Eq. (5) is not satisfied) in (b) and (d); (e)–(h) $N=4$ with $U_3/U_2=-1$ (the condition of Eq. (5) is satisfied) in (e) and (g), and $U_3/U_2=1/6$ (the condition of Eq. (5) is not satisfied) in (f) and (h), (01000) is denoted by the blue line while (00100) is denoted by the red line.

The tunneling dynamics in the first panel of Fig. 8 is further clearly depicted by the temporal evolution of $|c_n(t)|^2$ in Fig. 9, which is to ensure that there are 0, 1, 2, and 3 bosons in the left well, respectively. The first panel describes the case of $U_3/U_2 = -2$ (the condition of Eq. (5) is satisfied) both in the strong ($U_2 = 6$) and in the weak ($U_2 = 0.05$) interaction regimes, whose tunneling dynamics is completely identical, i.e., Rabi oscillation. Note that when equation (5) is satisfied (indicated by the dotted gray lines in Fig. 1), the levels are equidistant, and the system should do the Rabi oscillation with a single frequency $\omega = \omega_{01} = \omega_{12} = \omega_{23}$. However, the tunneling dynamics (see Fig. 8(d)) for $U_2 = 0.05$ with $U_3/U_2 = 2$ (the condition of Eq. (5) is not satisfied) is illustrated by the time evolution of $|c_n(t)|^2$ in the second panel of Fig. 9. We find that, the evolutions of $|c_1(t)|^2$ and $|c_2(t)|^2$ are dominated and their oscillation is close to unity when the bosons are distributed unbalanced (0100) initially. That is to say, for an odd number of bosons, such as three (or five), an initial configuration of having unequal numbers of bosons in the two wells has an interesting result when $U_3/U_2 = -2$ (the condition of Eq. (5) is satisfied) both in the strong (U_2=6) and weak ($U_2 = 0.05$) interaction regimes, or when U_3/U_2 does not satisfy the critical condition of Eq. (5) with $U_2 = 6$, which is most discernible when the population difference between the two wells is unity; however, when U_3/U_2 does not satisfy the critical condition of Eq. (5) in the limit of weak repulsion ($U_2 = 0.05$), an accumulation tendency seems to present, which could be resulted from the time evolution of $|c_1(t)|^2$ and $|c_2(t)|^2$ in the second panel of Fig. 9.

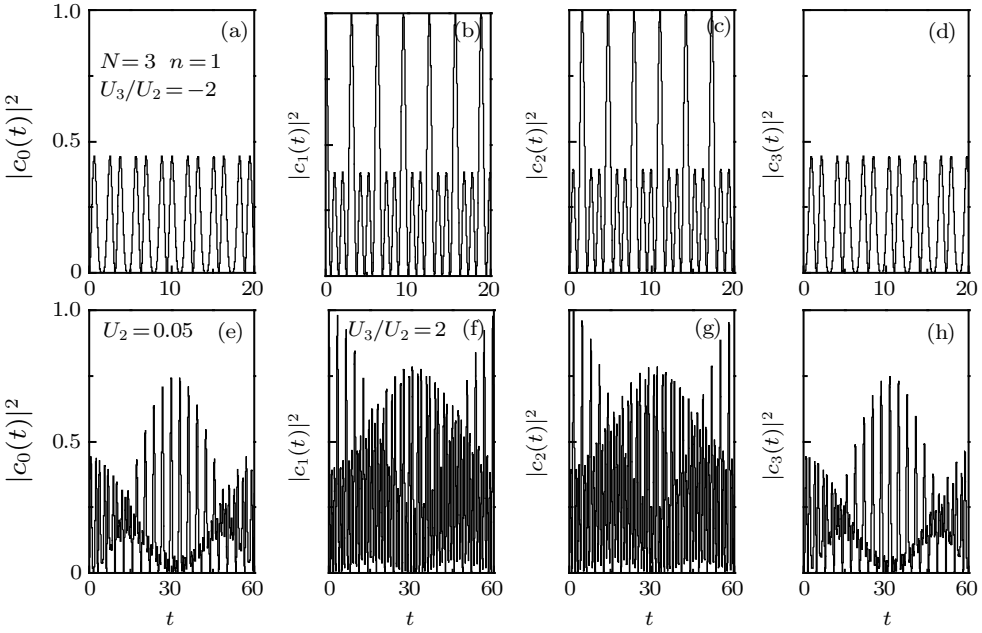


Fig. 9. The time evolution of $|c_n(t)|^2$ for $N = 3$ with the initial configuration (0100), i.e., there is only one boson in the left well while two bosons in the right well, which is corresponding to the blue lines in the first panel of Fig. 8. The first panel is for $U_3/U_2 = -2$ (the condition of Eq. (5) is satisfied) with both $U_2 = 6$ and $U_2 = 0.05$, which correspond to panels (a) and (c) in Fig. 8, respectively; the second panel is with respect to $U_2 = 0.05$ with $U_3/U_2 = 2$ (the condition of Eq. (5) is not satisfied), which correspond to Fig. 8(d).

The tunneling dynamics with different initial conditions for $N = 4$ is also discussed by the evolution of $P_R(t)$. In both strong ($U_2 = 6$) and weak ($U_2 = 0.05$) interaction regimes, no matter whether the condition of Eq. (5) is satisfied or not for $N = 4$, $P_R(t) = 0.5$ holds invariably when the bosons are initially distributed equally in the two wells ($n = N/2$), namely, (00100), denoted by the red lines in Figs. 8(e)–8(h), or, $P_R(t)$ oscillates between 0.25 and 0.75 when the bosons are initially distributed unequally (01000), denoted by the blue lines in Figs. 8(e)–8(h). Consequently, pair tunneling occurs when the bosons are not initially distributed completely in one well for an even (here $N = 4$) number of bosons. However, the tendency of an accumulation of particles in one well never happens when $N = 4$.

Similarly, these pair tunneling dynamics are also vividly exhibited by the temporal evolution of $|c_n(t)|^2$ in Figs. 10 and 11, which are to ensure that there are 0, 1, 2, 3, and 4 bosons in the left well, respectively. Figure 10 indicates that, for the case of (00100), i.e., the bosons are initially distributed equally in the two wells, the time evolution of $|c_2(t)|^2$ is always predominant (see Fig. 10(c)). Especially, when $U_2 = 6$ with $U_3/U_2 = 1/6$ (the condition of Eq. (5) is not satisfied), which is shown in the third panel of Fig. 10, the time evolutions of $|c_0(t)|^2$ (see Fig. 10(a-3)) and $|c_4(t)|^2$ (see Fig. 10(e-3)) are always around zero while $|c_2(t)|^2$ (see Fig. 10(c-3)) oscillates between 0.5 and 1, which further implies the occurrence of pair tunneling. Figure 11 indicates that, when the bosons are distributed as (01000), i.e., there is only one boson in the left well while three bosons are in

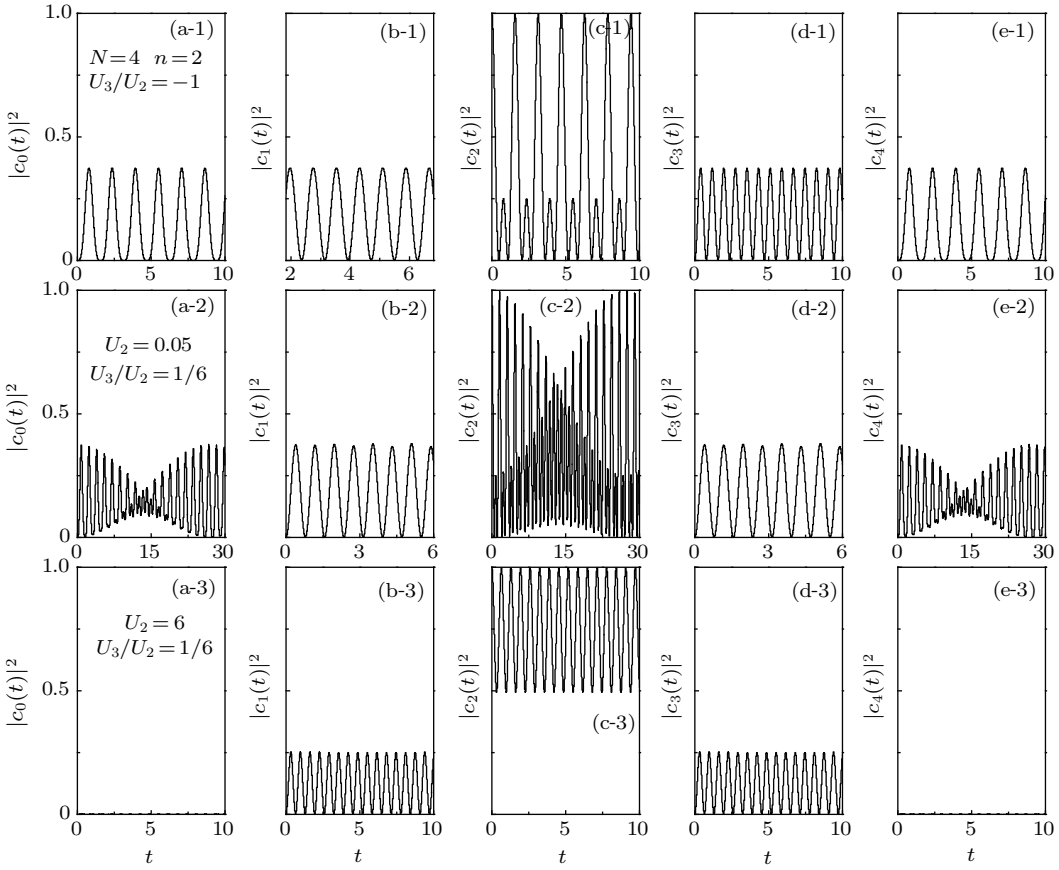


Fig. 10. The time evolution of $|c_n(t)|^2$ for $N=4$ with the initial configuration (00100), i.e., bosons are divided equally and there are two bosons in each well, which correspond to the red lines of the second panel in Fig. 8. The first panel is for $U_3/U_2=-1$ (the condition of Eq. (5) is satisfied) with both $U_2=6$ and $U_2=0.05$, which correspond to the red lines in Figs. 8(e) and 8(g); the second and the third panels are with respect to $U_3/U_2=2$ (the condition of Eq. (5) is not satisfied) with $U_2=0.05$ and $U_2=6$, respectively, which correspond to the red lines in Figs. 8(f) and 8(h).

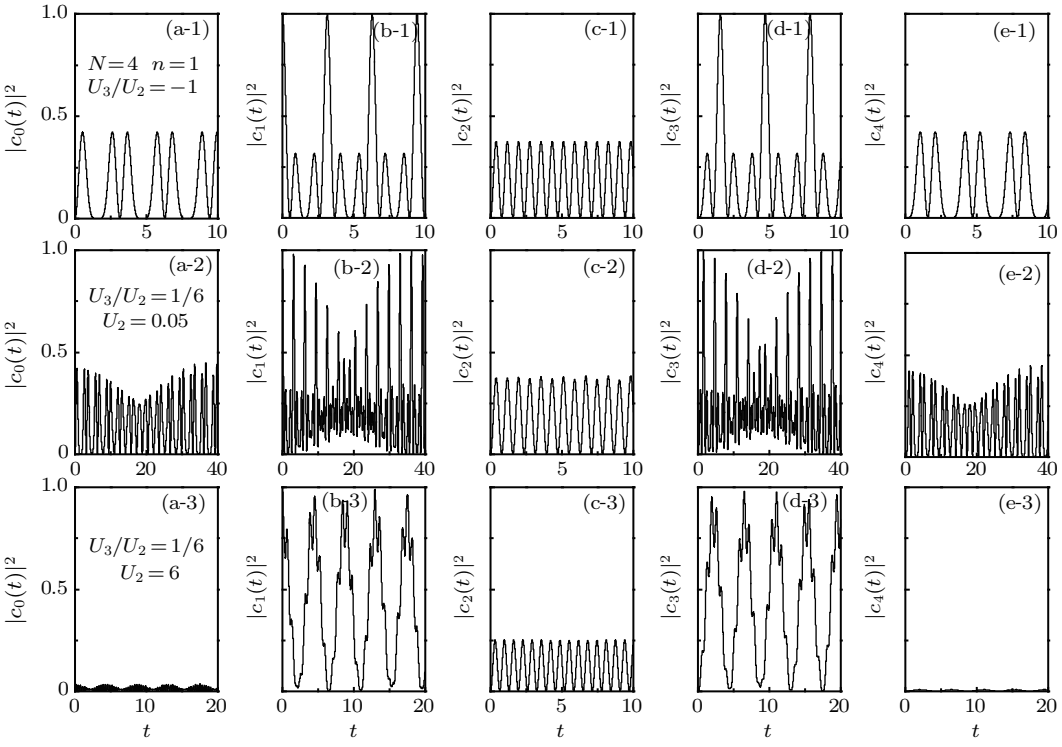


Fig. 11. The time evolution of $|c_n(t)|^2$ for $N=4$ with the initial configuration (01000), i.e., there is only one boson in the left well while three bosons are in the right well. The first panel is for $U_3/U_2=-1$ (the condition of Eq. (5) is satisfied) with both $U_2=6$ and $U_2=0.05$, which correspond to the blue lines in Figs. 8(e) and 8(g); the second and the third panels are with respect to $U_3/U_2=1/6$ (the condition of Eq. (5) is not satisfied) with $U_2=0.05$ and $U_2=6$, respectively, which correspond to the blue lines in Figs. 8(f) and 8(h).

the right well, the time evolutions of $|c_1(t)|^2$ (see Fig. 11(b)) and $|c_3(t)|^2$ (see Fig. 11(d)) are dominated no matter whether U_3/U_2 satisfies the critical condition of Eq. (5) or not both in the strong ($U_2 = 6$) and weak ($U_2 = 0.05$) interaction regimes. We find that, the time evolution of $|c_1(t)|^2$ ($|c_3(t)|^2$) in Fig. 11 can oscillate between 1 (0) to 0 (1) when $U_3/U_2 = -1$ (the condition of Eq. (5) is satisfied) or $U_3/U_2 = 1/6$ (the condition of Eq. (5) is not satisfied) with both $U_2 = 0.05$ and $U_2 = 6$, however, the other counterparts could not reach to 1 (the oscillation exists). Surprisingly, when $U_3/U_2 = 1/6$ with $U_2 = 6$, the time evolutions of $|c_1(t)|^2$ (see Fig. 11(b-3)) and $|c_3(t)|^2$ (see Fig. 11(d-3)) are completely dominated while $|c_0(t)|^2$ (see Fig. 11(a-3)) and $|c_4(t)|^2$ (see Fig. 11(e-3)) are around zero, which implies that pair tunneling occurs while there is only one still boson in each well. The signature of the tunneling pairs has been recognized in the characteristic momentum distribution and through spectroscopic measurements.[40,41]

More interestingly, T_R decreases sharply when the initial states are configured as (0100) or (0010) for $N = 3$, and (01000) or (00010) for $N = 4$ on the condition that U_3/U_2 do not satisfy the critical condition of Eq. (5), which are exhibited in Figs. 8(b) and 8(d) for $N = 3$ while Figs. 8(f) and 8(h) for $N = 4$ (here $n = 1$).

In brief, the effect of initial conditions on tunneling dynamics seems significant for both odd ($N = 3$) and even ($N = 4$) numbers of bosons in a symmetric double well potential. It indicates that for an odd number of bosons (here $N = 3$), when the bosons are unbalanced and distributed between the two wells, the population difference of bosons between the wells differs by unity. Moreover, an accumulation of bosons may be observed in the weak interacting regime when U_3/U_2 do not satisfy the critical condition of Eq. (5) (here $U_3/U_2 \neq -2$ for $N = 3$). For an even number of bosons ($N = 4$), both when $U_2 = 6$ and when $U_2 = 0.05$ with the equal distribution of bosons in the double well, i.e., (00100), $P_R(t) = 0.5$ holds invariably. Consequently, pair tunneling occurs, which is fairly insensitive to U_3/U_2. For an even number of bosons ($N = 4$), when the bosons are not distributed equationally, i.e., (01000), and U_3/U_2 does not satisfy the critical condition of Eq. (5) (here $U_3/U_2 \neq -1$ for $N = 4$) with $U_2 = 6$, we find that pair tunneling still occurs.

6. Conclusion

The tunneling dynamics of a few bosons with both two- and three-body interactions in a double well potential have been investigated. We find that, uncorrelated tunneling of Rabi oscillation with the minimum period can happen only when the two- and three-body interactions satisfy a critical condition, i.e., the effective interaction energy is minimized. When the atomic interactions are slightly away from the critical condition in the weak interaction regime, the uncorrelated tunneling exhibits a collapse-revival character. When the atomic interactions are strong and far away from the critical condition, correlated tunneling with Rabi oscillation still occurs. The tunneling period (the period of collapse-revival) increases (decreases) when the rate between the two-body and three-body interactions is away from the corresponding critical condition or when the number of bosons increases. The tunneling properties are understood with the help of the energy spectrum of the system. Eventually, the effect of the initial configuration on the tunneling dynamics of a few bosons for both odd and even numbers of bosons is studied, which shows intriguing consequences.

Understanding the few-body mechanism of tunneling with both two- and three-body interactions can be helpful to design atomic devices, i.e., atom switch, for selective transport concerning a definite number of particles between different wells. Furthermore, our study could stimulate the investigations of quantum dynamics in the presence of quantum many body interaction and even be extended to multi-well systems. As a result, this engineering may provide a quantitatively theoretical foundation for precise manipulation of particles in experiments.

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