Establishment of infinite dimensional Hamiltonian system of multilayer quasi-geostrophic flow & study on its linear stability*

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A multilayer flow is a stratified fluid composed of a finite number of layers with densities homogeneous within one layer but different from each other. It is an intermediate system between the single-layer barotropic model and the continuously stratified baroclinic model. Since this system can simulate the baroclinic effect simply, it is widely used to study the large-scale dynamic process in atmosphere and ocean. The present paper is concerned with the linear stability of the multilayer quasi-geostrophic flow, and the associated linear stability criteria are established. Firstly, the nonlinear model is turned into the form of a Hamiltonian system, and a basic flow is defined. But it cannot be an extreme point of the Hamiltonian function since the system is an infinite-dimensional one. Therefore, it is necessary to reconstruct a new Hamiltonian function so that the basic flow becomes an extreme point of it. Secondly, the linearized equations of disturbances in the multilayer quasi-geostrophic flow are derived by introducing infinitesimal disturbances superposed on the basic flows. Finally, the properties of the linearized system are discussed, and the linear stability criteria in the sense of Liapunov are derived under two different conditions with respect to certain norms.

Keywords: infinite dimensional Hamiltonian system, multilayer quasi-geostrophic flow, linear stability

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1. Introduction

Flow stability is one of the classical problems in fluid mechanics and atmosphere–ocean dynamics,[1–4] and the linear stability theory is the most mature. Two main approaches are usually adopted in the field of flow stability. The first one is the normal method. The most famous application is the linear instability criterion of incompressible flow, i.e., Rayleigh's criterion (it is the necessary conditions of instability, since there exist inflection points in the basic flow $\bar{U}(y)$, it can be called inflection point theorem). Kuo[5] applied this method to meteorology and derived atmospheric barotropic stability criteria in 1949. Pedlosky[6] studied the atmospheric baroclinic stability. The second one is the variational method along with eigenvalue estimate, which was proposed by Synge *et al.*[1–3] when studying the Orr–Sommerfeld equations' eigenvalue problem of viscous fluid. Chaney and Eady studied the baroclinic instability problems in the process of atmospheric movement.[7] Oliver and Ulrich[8] discussed the linear stability of compressible flow, and solved the eigenvalue problem of linear stability theory for three different cases. Wang and Huang[9] used the variational method to deduce the sufficient conditions of linear stability and instability when studying ionospheric variations in equatorial and low latitude areas.

With the development of linear stability, the theories of weak nonlinear and nonlinear stability were also developed accordingly. Among them, the perturbation method is widely used, which can be referred to Refs. [6] and [7].

In the development of nonlinear stability theory, Arnol'd[10,11] is the first one to deduce the nonlinear stability of planar inviscid steady incompressible flow through the variational method along with priori estimate, which opens a new phase for the research. Afterwards, based on Arnol'd's idea, Mu *et al.*[12,13] did many valuable researches when studying multilayer quasi-geostrophic flow and continuous stratified quasi-geostrophic flow. They took the generalized energy conservation and generalized energy–enstropy conservation into consideration, established a new conserved term, and derived Arnol'd's first and second theorems by making use of the prior estimate and Poincare inequality. Zhang and Xiang[14,15] investigated nonlinear saturation of baroclinic instability in the generalized Phillips model in the context of Arnol'd's second theorem and obtained the upper and lower bounds on the disturbance energy and potential enstrophy in the case of nonlinearly unstable basic flow. Based on the previous work, Swaters[16] studied linear and nonlinear stability of the barotropic vorticity equation by using a Hamiltonian system, and gave a new idea for stability research.

As we all know, the Hamiltonian principle can be counted as the greatest principle not only in mechanics, but even in the whole of physics. It integrally describes the motion of a mechanical system and reveals basic laws that energy transformation should follow. At the same time, because of its

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uniformity and conciseness, the Hamiltonian principle has excellent universality. For the above reasons, many scholars are devoted to studying it.[17–21] However, it is a difficult work to transform nonlinear partial differential equations into an infinite-dimensional Hamiltonian system, since the following two parts must be accomplished. The first one is the establishment of a conservative term H and a differential operator J, which is the key to transform nonlinear partial differential equations into the form $\boldsymbol{q}_t = J\frac{\delta H}{\delta \boldsymbol{q}}$; the second one is the definition of Poisson bracket $[F,G] = \left\langle \frac{\delta F}{\delta \boldsymbol{q}}, J\frac{\delta G}{\delta \boldsymbol{q}} \right\rangle_{L^2(\Omega)}$, such that J satisfies the five properties of Poisson bracket (self-Poisson commutation, skew-symmetry, distributive property, associative property, and Jacobi identity).

In this paper, the linear stability of multilayer quasi-geostrophic flow is discussed. First, the multilayer quasi-geostrophic flow nonlinear equations are turned into the form of an infinite-dimensional Hamiltonian system $\boldsymbol{q}_t = J\frac{\delta H_2}{\delta \boldsymbol{q}}$, the problem of finding the operator J is involved, which is related to the Poisson bracket. And then, Hamiltonian function H_2 is required to be conservative, and its gradient should be calculated. After these works, a Hamiltonian system can be established. Second, basic flow $\boldsymbol{Q}$ is introduced. Since J is a differential operator and irreversible, the equation $\frac{\delta H_2}{\partial \boldsymbol{q}}(\boldsymbol{Q}) = \boldsymbol{0}$ can not be derived from the expression $J_{[\boldsymbol{Q}]}\frac{\delta H_2}{\partial \boldsymbol{q}}(\boldsymbol{Q}) = \boldsymbol{0}$, namely, $\boldsymbol{Q}$ cannot be an extreme point of Hamiltonian function H_2. It is essential to reconstitute Hamiltonian function H, which should satisfy the following three conditions: (i) H is time independent, i.e. $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$, (ii) H concludes an extreme point $\boldsymbol{Q}$, (iii) H maintains the Hamiltonian system invariant. From the analyses above, H contains not only the kinetic energy of the system, the potential vorticity energy, the boundary circulation energy, but also the basic flow energy. Third, general disturbances are superposed to the basic flows, and the linearized equations of multilayer quasi-geostrophic flow are derived. Unlike nonlinear stability, now the linearized equations do not have a Hamiltonian structure, and we do not make use of the normal method to discuss the stability. Fourth, stability criteria are established. The deduction of the second order variation of $H(\boldsymbol{Q})$ is the key, that is $\delta^2 H(\boldsymbol{Q})$, and it is necessary to prove that it is invariable over time, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 H(\boldsymbol{Q}) = 0$. Based on these works, the linear stability criteria in the sense of Liapunov are derived for two different cases that the basic flow is a maximum point ($\delta^2 H(\boldsymbol{Q}) < 0$) and a minimum point ($\delta^2 H(\boldsymbol{Q}) > 0$) of the new Hamiltonian function H by using the methods of prior estimate and eigenvalue estimate.

For the nonlinear stability, Mu *et al.*[12,13] have made a lot of contributions by taking advantage of the Arnol'd method, but so far no one has discussed multilayer linear stability on the basis of the Hamiltonian system. It should be noted that the results of nonlinear stability are not completely suitable for the linear stability. There exist many essential difficulties, so it is necessary to study linear stability.

2. Mathematical model of multiplayer quasi-geostrophic flow

Consider a stably stratified fluid of N superimposed layers of constant densities $0 < \rho_1 < \cdots < \rho_N$, with equal density jumps $\rho' = \rho_{i+1} - \rho_i$, and mean layer depth d_i. The multiplayer quasi-geostrophic vorticity equations are given by[6]

$$\frac{\partial q_i}{\partial t} + \partial(\psi_i, q_i) = 0, \tag{1}$$

$$q_i(t,x,y) = \Delta\psi_i + F_i\sum_{j=1}^{N} T_{ij}\psi_j + f_i(x,y), \quad (i = 1,\ldots,N), \tag{2}$$

where ψ_i and q_i are the stream function and the potential vorticity of the i^{th} layer, respectively,

$$\partial(\psi, q) \triangleq \frac{\partial(\psi, q)}{\partial(x,y)} = \psi_x q_y - \psi_y q_x,$$

and F_i denotes the Froude number, i.e., $F_i = \frac{f_0^2\rho_0}{g\rho' d_i}$. It is obvious that $d_i F_i = \frac{f_0^2\rho_0}{g\rho'} = a$ (a is a constant) and $f_i(x,y)$ follows

$$f_i(x,y) = \delta_{iN}\left(\frac{f_0}{d_N}\right)h(x,y) + f_0 + \beta y, \tag{3}$$

where $h(x,y)$ denotes the lowest layer orographic function, δ_{iN} is the Dirac function, f_0 and β are the Coriolis parameters, respectively, and $\boldsymbol{T} = (T_{ij})_{N\times N}$ is given by

$$\boldsymbol{T} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}. \tag{4}$$

By calculation, we have $\langle \psi, -\boldsymbol{T}\psi \rangle = \sum_{i=1}^{N-1}(\psi_{i+1} - \psi_i)^2$, so $-\boldsymbol{T}$ is a nonnegative symmetric matrix and $\det \boldsymbol{T} = 0$.

In Eqs. (1) and (2), the region $\Omega \subset \mathrm{R}^2$ presents a multiply (or simple) connected domain in plane, $\partial\Omega$ is the boundary of Ω, which is smooth, $\partial\Omega$ consists of $m+1$ simple closed curves $\partial\Omega_j (j = 0,1,\ldots,m)$, and $\partial\Omega_j \subseteq \partial\Omega_0$, and $\boldsymbol{n}$ denotes the unit outward normal vector on the boundary curve $\partial\Omega$.

The boundary conditions of the fluid are as follows. The boundary conditions of the rigid wall are

$$\left.\frac{\partial \psi_i}{\partial s}\right|_{\partial\Omega_j} = 0, \quad (\text{or} \quad \psi_i|_{\partial\Omega_j} = \lambda_j^i, \quad \lambda_j^i \text{ is a constant}). \tag{5}$$

The conservation conditions of the boundary circulation are

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial\Omega_j}\frac{\partial\psi_i}{\partial n}\mathrm{d}s = 0, \quad i = 1,\ldots,N; \quad j = 0,\ldots,m. \tag{6}$$

If the following expressions in vector form

$$\begin{aligned}
&\boldsymbol{q}=(q_1,\ldots,q_N)^{\mathrm{T}},\quad \boldsymbol{\psi}=(\psi_1,\ldots,\psi_N)^{\mathrm{T}},\\
&\boldsymbol{f}=(f_1,\ldots,f_N)^{\mathrm{T}},\boldsymbol{F}=\mathrm{diag}(\sqrt{F_1},\ldots,\sqrt{F_N}),\\
&\boldsymbol{D}=\mathrm{diag}(\sqrt{d_1},\ldots,\sqrt{d_N}),\quad \boldsymbol{DF}=\sqrt{a}\boldsymbol{I},\hat{f}_i=-\partial(q_i,\cdot),\\
&\hat{f}_{[\boldsymbol{q}]}\triangleq\hat{f}=\mathrm{diag}(\hat{f}_1,\ldots,\hat{f}_N),\\
&\hat{f}(-\boldsymbol{\psi})=\mathrm{diag}(\hat{f}_1(-\psi_1),\ldots,\hat{f}_N(-\psi_N)),\\
&\hat{f}_{[Q_i]}=-\partial(Q_i,\cdot),\quad \hat{f}_{[\boldsymbol{Q}]}=\mathrm{diag}(\hat{f}_{[Q_1]},\ldots,\hat{f}_{[Q_N]}),\\
&\hat{f}_{[\boldsymbol{Q}]}(-\boldsymbol{\Psi})=\mathrm{diag}(\hat{f}_{[Q_1]}(-\Psi_1),\ldots,\hat{f}_{[Q_N]}(-\Psi_N)),\\
&\partial(\boldsymbol{\psi},\boldsymbol{\varphi})\triangleq(\partial(\psi_1,\varphi_1),\ldots,\partial(\psi_N,\varphi_N))^{\mathrm{T}},\\
&J=\hat{f}\boldsymbol{D}^{-2},\quad J_{[\boldsymbol{Q}]}=\hat{f}_{[\boldsymbol{Q}]}\boldsymbol{D}^{-2}
\end{aligned}\tag{7}$$

are introduced into Eqs. (1) and (2), then we have

$$\boldsymbol{q}_t=-\partial(\boldsymbol{q},-\boldsymbol{\psi})=\hat{J}(-\boldsymbol{\psi}),\tag{8a}$$

$$\boldsymbol{q}=\Delta\boldsymbol{\psi}+\boldsymbol{F}^2\boldsymbol{T}\boldsymbol{\psi}+\boldsymbol{f},\tag{8b}$$

$$\left.\frac{\partial\boldsymbol{\psi}}{\partial s}\right|_{\partial\Omega_j}=\boldsymbol{0}\quad(\text{or}\quad \boldsymbol{\psi}|_{\partial\Omega_j}=\boldsymbol{\lambda}_j,\quad j=0,\ldots,m;\ \boldsymbol{\lambda}_j\ \text{is a constant vector}),\tag{8c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial\Omega_j}\frac{\partial\boldsymbol{\psi}}{\partial n}\mathrm{d}s=0,\quad j=0,\ldots,m.\tag{8d}$$

3. Construction of Hamiltonian system

Lemma 1 For $\partial(\cdot,\cdot)$, suppose ψ, q, and φ are differentiable functions in $\bar{\Omega}\subset\mathrm{R}^2$, then we can obtain

$$\int_\Omega\psi\partial(\varphi,q)\mathrm{d}\Omega=\int_\Omega q\partial(\psi,\varphi)\mathrm{d}\Omega-\oint_{\partial\Omega}\psi q\frac{\partial\varphi}{\partial s}\mathrm{d}s.\tag{9}$$

Proof: We have

$$\begin{aligned}
&\int_\Omega\psi\partial(\varphi,q)\mathrm{d}\Omega\\
=&\int_\Omega\psi(\varphi_xq_y-\varphi_yq_x)\mathrm{d}\Omega\\
=&\int_\Omega[(\psi\varphi_xq)_y-(\psi\varphi_yq)_x]\mathrm{d}\Omega-\int_\Omega[(\psi\varphi_x)_y-(\psi\varphi_y)_x]q\mathrm{d}\Omega\\
=&\int_\Omega q\partial(\psi,\varphi)\mathrm{d}\Omega+\int_{\partial\Omega}q(-\psi\varphi_y,\psi\varphi_x)\cdot\boldsymbol{n}\mathrm{d}s\\
=&\int_\Omega q\partial(\psi,\varphi)\mathrm{d}\Omega-\int_{\partial\Omega}\psi q\frac{\partial\varphi}{\partial s}\mathrm{d}s.
\end{aligned}\tag{10}$$

Specially, under the condition of $\psi=1$, we obtain

$$\int_\Omega\partial(\varphi,q)\mathrm{d}\Omega=-\oint_{\partial\Omega}q\frac{\partial\varphi}{\partial s}\mathrm{d}s.\tag{11}$$

Then the following expression

$$\int_\Omega\partial(\varphi,q)\mathrm{d}\Omega=0\tag{12}$$

is established if the condition $\left.\frac{\partial\varphi}{\partial s}\right|_{\partial\Omega}=0$ is introduced into Eq. (11).

In the process of establishing infinite-dimensional Hamiltonian system $\boldsymbol{q}_t=J\frac{\delta H}{\delta\boldsymbol{q}}$, there are three problems to be solved. The first one is the establishment of J that J satisfies the properties of the Poisson bracket; the second one is the conservative property of H, i.e., $\mathrm{d}H/\mathrm{d}t=0$; and the last one is the gradient calculation of H, that is $\delta H/\delta\boldsymbol{q}$, which should ensure that the equation $\boldsymbol{q}_t=J\delta H/\delta\boldsymbol{q}$ is in accordance with the original nonlinear problem.

3.1. Construction of Poisson bracket formulation

Introduce $(\boldsymbol{a},\boldsymbol{b})\triangleq\sum_{i=1}^{N}a_ib_i$ and $\langle f,g\rangle_{L^2(\Omega)}\triangleq\int_\Omega fg\mathrm{d}\Omega$, and define $[*,*]$ as follows:

$$\begin{aligned}
[F,G]=&\left\langle\frac{\delta F}{\delta\boldsymbol{q}},J\frac{\delta G}{\delta\boldsymbol{q}}\right\rangle_{L^2(\Omega)}\\
=&-\sum_{i=1}^{N}\frac{1}{d_i}\int_\Omega\frac{\delta F}{\delta q_i}\partial\left(q_i,\frac{\delta G}{\delta q_i}\right)\mathrm{d}\Omega\\
=&\sum_{i=1}^{N}\frac{1}{d_i}\left\langle\frac{\delta F}{\delta q_i},\hat{f}_i\frac{\delta G}{\delta q_i}\right\rangle_{L^2(\Omega)}.
\end{aligned}\tag{13}$$

The following five conditions of Poisson bracket $[*,*]$ can be proved by using Lemma 1:

1) self-Poisson commutation

$$[F,F]=0,\tag{14}$$

2) skew-symmetry

$$[F,G]=-[G,F],\tag{15}$$

3) distributive property

$$\begin{aligned}
&[\alpha F_1+\beta F_2,G]=\alpha[F_1,G]+\beta[F_2,G],\\
&(\alpha\ \text{and}\ \beta\ \text{are constants}),
\end{aligned}\tag{16}$$

4) associative property

$$[FG,Q]=F[G,Q]+[F,Q]G,\tag{17}$$

5) Jacobi identity

$$[F,[G,Q]]+[G,[Q,F]]+[Q,[F,G]]=0.\tag{18}$$

We will prove the second one, the others can be proved similarly.

Proof: It follows from Eq. (13) that

$$\begin{aligned}
[F,G]=&-\sum_{i=1}^{N}\frac{1}{d_i}\int_\Omega\frac{\delta F}{\delta q_i}\partial\left(q_i,\frac{\delta G}{\delta q_i}\right)\mathrm{d}\Omega\\
=&-\sum_{i=1}^{N}\frac{1}{d_i}\int_\Omega\frac{\delta G}{\delta q_i}\partial\left(\frac{\delta F}{\delta q_i},q_i\right)\mathrm{d}\Omega
\end{aligned}$$

$$+\sum_{i=1}^{N}\frac{1}{d_i}\oint_{\partial\Omega}\frac{\delta F}{\delta q_i}\frac{\delta G}{\delta q_i}\frac{\partial q_i}{\partial s}\mathrm{d}s$$

$$=-[G,F]+\sum_{i=1}^{N}\frac{1}{d_i}\oint_{\partial\Omega}\frac{\partial}{\partial s}\left(\int^{q_i}\chi_i(\xi)\mathrm{d}\xi\right)\mathrm{d}s, \quad (19)$$

where $\chi_i=\frac{\delta F}{\delta q_i}\frac{\delta G}{\delta q_i}$. Based on the closure of curves and the periodicity of χ_i, we have the result $\oint_{\partial\Omega}\frac{\partial}{\partial s}(\int^{q_i}\chi_i(\xi)\mathrm{d}\xi)\mathrm{d}s=0$, then the property $[F,G]=-[G,F]$ is established.

Next, we will find a new Hamiltonian function H to satisfy the equation $\boldsymbol{q}_t=J\frac{\partial H}{\partial \boldsymbol{q}}$. After that, the multiplayer quasi-geostrophic flow can be turned into the form of an infinite-dimensional Hamiltonian system.

3.2. Construction of the Hamiltonian function

Introduce $|\nabla\psi|^2=(\nabla\psi,\nabla\psi)=\sum\limits_{i=1}^{N}|\nabla\psi_i|^2$, the Hamiltonian function will be constructed by the following two steps.

Step 1 Introduce kinetic energy H_1, check it is conservative or not?

$$H_1=\frac{1}{2}\int_{\Omega}|\nabla\boldsymbol{D}\psi|^2\,\mathrm{d}\Omega=\frac{1}{2}\int_{\Omega}(\nabla\boldsymbol{D}\psi,\nabla\boldsymbol{D}\psi)\mathrm{d}\Omega$$

$$=\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}d_i|\nabla\psi_i|^2\,\mathrm{d}\Omega. \quad (20)$$

Based on the Green formula, a direct calculation of $\mathrm{d}H_1/\mathrm{d}t$ gives

$$\frac{\mathrm{d}H_1}{\mathrm{d}t}=\int_{\Omega}(\nabla\boldsymbol{D}\psi,\nabla\boldsymbol{D}\psi_t)\mathrm{d}\Omega$$

$$=-\int_{\Omega}(\boldsymbol{D}\psi,\Delta\boldsymbol{D}\psi_t)\mathrm{d}\Omega+\int_{\partial\Omega}\left(\boldsymbol{D}\psi,\boldsymbol{D}\frac{\partial\psi_t}{\partial n}\right)\mathrm{d}s. \quad (21)$$

From the expressions

$$\boldsymbol{q}=\Delta\psi+\boldsymbol{F}^2\boldsymbol{T}\psi+\boldsymbol{f},\quad \boldsymbol{q}_t=\Delta\psi_t+\boldsymbol{F}^2\boldsymbol{T}\psi_t,$$
$$\boldsymbol{q}_t=\hat{J}(-\psi),\quad \Delta\boldsymbol{D}\psi_t=\boldsymbol{D}\boldsymbol{q}_t-\sqrt{a}\boldsymbol{F}\boldsymbol{T}\psi_t, \quad (22)$$

we have

$$\frac{\mathrm{d}H_1}{\mathrm{d}t}=-\int_{\Omega}(\boldsymbol{D}\psi,\boldsymbol{D}\boldsymbol{q}_t)\mathrm{d}\Omega+\sqrt{a}\int_{\Omega}(\boldsymbol{D}\psi,\boldsymbol{F}\boldsymbol{T}\psi_t)\mathrm{d}\Omega$$
$$+\sum_{j=0}^{m}\left(\boldsymbol{\lambda}_j,\boldsymbol{D}^2\int_{\partial\Omega_j}\frac{\partial\psi_t}{\partial n}\mathrm{d}s\right)$$
$$=-\int_{\Omega}(\boldsymbol{D}\psi,\boldsymbol{D}\hat{J}(-\psi))\mathrm{d}\Omega+a\int_{\Omega}(\psi,\boldsymbol{T}\psi_t)\mathrm{d}\Omega$$
$$+\sum_{j=0}^{m}\left(\boldsymbol{\lambda}_j,\boldsymbol{D}^2\int_{\partial\Omega_j}\frac{\partial\psi_t}{\partial n}\mathrm{d}s\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}a\int_{\Omega}(\psi,\boldsymbol{T}\psi)\mathrm{d}\Omega\right.$$
$$\left.+\sum_{j=0}^{m}\left(\boldsymbol{\lambda}_j,\boldsymbol{D}^2\int_{\partial\Omega_j}\frac{\partial\psi}{\partial n}\mathrm{d}s\right)\right]. \quad (23)$$

Denote

$$H_2=\frac{1}{2}\int_{\Omega}[|\nabla\boldsymbol{D}\psi|^2-a(\psi,\boldsymbol{T}\psi)]\mathrm{d}\Omega$$
$$-\sum_{j=0}^{m}\left(\boldsymbol{\lambda}_j,\boldsymbol{D}^2\int_{\partial\Omega_j}\frac{\partial\psi}{\partial n}\mathrm{d}s\right), \quad (24)$$

we obtain the result $\mathrm{d}H_2/\mathrm{d}t=0$, that is, H_2 is conservative.

Step 2 Calculate $\delta H_2/\delta\boldsymbol{q}$. We have

$$\delta H_2=\int_{\Omega}(\nabla\boldsymbol{D}\psi,\nabla\boldsymbol{D}\delta\psi)\mathrm{d}\Omega-a\int_{\Omega}(\psi,\boldsymbol{T}\delta\psi)\mathrm{d}\Omega$$
$$-\sum_{j=0}^{m}\left(\boldsymbol{\lambda}_j,\boldsymbol{D}^2\int_{\partial\Omega_j}\frac{\partial\delta\psi}{\partial n}\mathrm{d}s\right)$$
$$=-\int_{\Omega}(\boldsymbol{D}\psi,\Delta\boldsymbol{D}\delta\psi)\mathrm{d}\Omega-a\int_{\Omega}(\psi,\boldsymbol{T}\delta\psi)\mathrm{d}\Omega$$
$$=-\int_{\Omega}(\boldsymbol{D}\psi,\boldsymbol{D}\delta\boldsymbol{q}-\boldsymbol{D}\boldsymbol{F}^2\boldsymbol{T}\delta\psi)\mathrm{d}\Omega-a\int_{\Omega}(\psi,\boldsymbol{T}\delta\psi)\mathrm{d}\Omega$$
$$=\int_{\Omega}(\boldsymbol{D}^2(-\psi),\delta\boldsymbol{q})\mathrm{d}\Omega, \quad (25)$$

where $\delta\boldsymbol{q}=\Delta\delta\psi+\boldsymbol{F}^2\boldsymbol{T}\delta\psi$ has been substituted into. Based on the definition of variation

$$\delta H_2=\int_{\Omega}\left(\frac{\delta H_2}{\delta\boldsymbol{q}},\delta\boldsymbol{q}\right)\mathrm{d}\Omega,$$

we can obtain

$$\frac{\delta H_2}{\delta\boldsymbol{q}}=\boldsymbol{D}^2(-\psi)\quad\left(\text{or }\ \boldsymbol{D}^{-2}\frac{\delta H_2}{\delta\boldsymbol{q}}=-\psi\right). \quad (26)$$

It follows from Eq. (8a) that

$$\boldsymbol{q}_t=\hat{J}(-\psi)=\hat{J}\boldsymbol{D}^{-2}\frac{\delta H_2}{\delta\boldsymbol{q}}=J\frac{\delta H_2}{\delta\boldsymbol{q}}. \quad (27)$$

Therefore, equation (1) can be transformed into the following forms:

$$\boldsymbol{q}_t=J\frac{\delta H_2}{\delta\boldsymbol{q}}, \quad (28a)$$

$$\boldsymbol{q}=\Delta\psi+\boldsymbol{F}^2\boldsymbol{T}\psi+\boldsymbol{f}, \quad (28b)$$

$$\left.\frac{\partial\psi}{\partial s}\right|_{\partial\Omega_j}=\boldsymbol{0},\ \ (\text{or}\ \ \psi|_{\partial\Omega_j}=\boldsymbol{\lambda}_j, j=0,\ldots,m,$$
$$\boldsymbol{\lambda}_j\ \ \text{is a constant vector}), \quad (28c)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial\Omega_j}\frac{\partial\psi}{\partial n}\mathrm{d}s=\boldsymbol{0},\quad j=0,\ldots,m, \quad (28d)$$

where

$$H_2=\frac{1}{2}\int_{\Omega}[|\nabla\boldsymbol{D}\psi|^2-a(\psi,\boldsymbol{T}\psi)]\mathrm{d}\Omega$$
$$-\sum_{j=0}^{m}\left(\boldsymbol{\lambda}_j,\boldsymbol{D}^2\int_{\partial\Omega_j}\frac{\partial\psi}{\partial n}\mathrm{d}s\right). \quad (29)$$

4. Derivation of the linearized equations for infinitesimal disturbances

4.1. Introduction of the basic flows $\boldsymbol{\Psi}(x,y)$ and $\boldsymbol{Q}(x,y)$

It is necessary to introduce the basic flows $\boldsymbol{\Psi}(x,y) = (\Psi_1,\ldots,\Psi_N)^{\mathrm{T}}$ and $\boldsymbol{Q}(x,y) = (Q_1,\ldots,Q_N)^{\mathrm{T}}$ when studying the stability, they should satisfy Eq. (28a), that is,

$$J_{[\boldsymbol{Q}]}\frac{\delta H_2}{\delta \boldsymbol{q}}(\boldsymbol{Q}) = \hat{J}_{[\boldsymbol{Q}]}(-\boldsymbol{\Psi}) = \boldsymbol{0}. \tag{30}$$

Clearly, $\boldsymbol{\Psi}(x,y)$ and $\boldsymbol{Q}(x,y)$ are functionally dependent based on the definition of $J_{[\boldsymbol{Q}]}$. Hence, there exists a differential function Φ_i, such that $\Psi_i = \Phi_i(Q_i)$, or it can be written as follows:

$$\boldsymbol{\Psi} = \boldsymbol{\Phi}(\boldsymbol{Q}_i) \triangleq (\Phi_1(Q_1),\ldots,\Phi_N(Q_N))^{\mathrm{T}}. \tag{31}$$

Since $J_{[\boldsymbol{Q}]}$ is irreversible, we cannot obtain $\frac{\delta H_2}{\delta \boldsymbol{q}}(\boldsymbol{Q}) = \boldsymbol{0}$ from the equation $J_{[\boldsymbol{Q}]}\frac{\delta H_2}{\delta \boldsymbol{q}}(\boldsymbol{Q}) = \boldsymbol{0}$, namely, the basic flow $\boldsymbol{Q}$ cannot be an extreme point of the Hamiltonian function H_2, which makes it insignificant to study the stability. So H_2 must be modified into a new one H, denoting

$$H(\boldsymbol{q}) = H_2(\boldsymbol{q}) + C(\boldsymbol{q}), \tag{32}$$

where $C(\boldsymbol{q})$ is an undetermined functional. The function $H(\boldsymbol{q})$ must satisfy the following two conditions.

1) $H(\boldsymbol{q})$ is conservative, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}H(\boldsymbol{q}) = 0$, thus, the functional $C(\boldsymbol{q})$ should be also conservative, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}C(\boldsymbol{q}) = 0. \tag{33}$$

2) The basic flow $\boldsymbol{Q}$ should be an extreme point of $H(\boldsymbol{q})$, that is,

$$\begin{aligned}\frac{\delta H}{\delta \boldsymbol{q}}(\boldsymbol{Q}) &= \frac{\delta H_2}{\delta \boldsymbol{q}}(\boldsymbol{Q}) + \frac{\delta C}{\delta \boldsymbol{q}}(\boldsymbol{Q}) \\ &= \boldsymbol{D}^2(-\boldsymbol{\Psi}) + \frac{\delta C}{\delta \boldsymbol{q}}(\boldsymbol{Q}) = \boldsymbol{0}.\end{aligned} \tag{34}$$

Therefore, the functional $C(\boldsymbol{q})$ is introduced, such that

$$\frac{\delta C}{\delta \boldsymbol{q}}(\boldsymbol{Q}) = \boldsymbol{D}^2(\boldsymbol{\Psi}). \tag{35}$$

The functional $C(\boldsymbol{q})$ is usually named Casimir functional. The new Hamilton function H contains the system energy, the potential vorticity energy, the boundary circulation energy, and the system basic flow energy.

4.2. Construction of functional $C(\boldsymbol{q})$

Construct $C(\boldsymbol{q})$ as follows:

$$C(\boldsymbol{q}) = \sum_{i=1}^{N}\int_{\Omega}\left(d_i\int^{q_i}\Phi_i(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\Omega. \tag{36}$$

It follows from Eq. (8a) and Lemma 1 that

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}t}(\boldsymbol{q}) &= \int_{\Omega}(\boldsymbol{D}^2\Phi(\boldsymbol{q}),\boldsymbol{q}_t)\,\mathrm{d}\Omega = -\int_{\Omega}(\boldsymbol{D}^2\Phi(\boldsymbol{q}),\hat{J}(\psi))\,\mathrm{d}\Omega \\ &= \sum_{i=1}^{N}\int_{\Omega} d_i\Phi_i(q_i)\partial(q_i,\psi_i)\,\mathrm{d}\Omega \\ &= \sum_{i=1}^{N}\int_{\Omega} d_i\partial\left(\int^{q_i}\Phi_i(\xi)\,\mathrm{d}\xi,\psi_i\right)\mathrm{d}\Omega = 0.\end{aligned} \tag{37}$$

The expression implies that $C(\boldsymbol{q})$ is conservative, therefore, $C(\boldsymbol{q})$ satisfies expression (33). On the other hand,

$$\delta C(\boldsymbol{q}) = \sum_{i=1}^{N}\int_{\Omega} d_i\Phi_i(q_i)\delta q_i\,\mathrm{d}\Omega = \int_{\Omega}(\boldsymbol{D}^2\boldsymbol{\Phi}(\boldsymbol{q}),\delta\boldsymbol{q})\,\mathrm{d}\Omega, \tag{38}$$

so based on the definition of variation

$$\delta C(\boldsymbol{q}) = \int_{\Omega}\left(\frac{\delta C(\boldsymbol{q})}{\delta \boldsymbol{q}},\delta\boldsymbol{q}\right)\mathrm{d}\Omega,$$

we have

$$\frac{\delta C(\boldsymbol{q})}{\delta \boldsymbol{q}} = \boldsymbol{D}^2\boldsymbol{\Phi}(\boldsymbol{q}), \frac{\delta C(\boldsymbol{Q})}{\delta \boldsymbol{q}} = \boldsymbol{D}^2\boldsymbol{\Phi}(\boldsymbol{Q}) = \boldsymbol{D}^2\boldsymbol{\Psi}, \tag{39}$$

namely, $C(\boldsymbol{q})$ satisfies Eq. (35).

We can conclude that, after introducing the new Hamiltonian function $H(\boldsymbol{q}) = H_2(\boldsymbol{q}) + C(\boldsymbol{q})$, the Hamiltonian system is transformed into the following form:

$$\boldsymbol{q}_t = J\frac{\delta H}{\delta \boldsymbol{q}}. \tag{40}$$

The formula $\frac{\delta H}{\delta \boldsymbol{q}}(\boldsymbol{Q}) = \boldsymbol{0}$ is established, that is, $\boldsymbol{Q}$ is an extreme point of $H(\boldsymbol{q})$.

4.3. Construction of linearized function for infinitesimal disturbances

Denote

$$\begin{aligned}\psi(t,x,y) &= \boldsymbol{\Psi}(x,y) + \delta\psi, \\ \boldsymbol{q}(t,x,y) &= \boldsymbol{Q}(x,y) + \delta\boldsymbol{q},\end{aligned} \tag{41}$$

where $\delta\psi$ and $\delta\boldsymbol{q}$ present infinitesimal disturbances superposing on the basic flows $\boldsymbol{\Psi}(x,y)$ and $\boldsymbol{Q}(x,y)$, respectively, and they satisfy the following relation expressions:

$$\delta\boldsymbol{q} = \Delta\delta\psi + \boldsymbol{F}^2\boldsymbol{T}\delta\psi, \quad \boldsymbol{\Psi} = \boldsymbol{\Phi}(\boldsymbol{Q}). \tag{42}$$

Introduce Eq. (41) into Eq. (40), and ignore h.o.t., we obtain

$$\frac{\partial\delta\boldsymbol{q}}{\partial t} + \partial(\boldsymbol{\Psi},\delta\boldsymbol{q}) + \partial(\delta\psi,\boldsymbol{Q}) = \boldsymbol{0}. \tag{43}$$

Since

$$\partial(\boldsymbol{\Psi},\delta\boldsymbol{q}) = \partial(\boldsymbol{\Phi}(\boldsymbol{Q}),\delta\boldsymbol{q}) = \boldsymbol{\Phi}'(\boldsymbol{Q})\partial(\boldsymbol{Q},\delta\boldsymbol{q}), \tag{44}$$

where $\boldsymbol{\Phi}^{(i)}(\boldsymbol{Q}) \triangleq \mathrm{diag}(\Phi_1^{(i)}(Q_1),\ldots,\Phi_N^{(i)}(Q_N))$, $(i=1,2)$, we can obtain the following formula:

$$\frac{\partial \delta \boldsymbol{q}}{\partial t} + \boldsymbol{\Phi}'(\boldsymbol{Q})\partial(\boldsymbol{Q},\delta\boldsymbol{q}) - \partial(\boldsymbol{Q},\delta\boldsymbol{\psi}) = \boldsymbol{0}. \tag{45}$$

Thus, the linearized equations for infinitesimal disturbances are established as

$$\frac{\partial \delta \boldsymbol{q}}{\partial t} + \partial(\boldsymbol{Q},\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q} - \delta\boldsymbol{\psi}) = \boldsymbol{0}, \tag{46}$$

where the expression

$$\begin{aligned}\boldsymbol{\Phi}'(\boldsymbol{Q})\partial(\boldsymbol{Q},\delta\boldsymbol{q}) &= \partial(\boldsymbol{Q},\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q}) - \boldsymbol{\Phi}''(\boldsymbol{Q})\delta\boldsymbol{q}\partial(\boldsymbol{Q},\boldsymbol{Q})\\ &= \partial(\boldsymbol{Q},\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q})\end{aligned}$$

has been used.

5. Stability criterion (I) of the linearized equations for infinitesimal disturbances

For the given basic flows $\boldsymbol{\Psi}$ and $\boldsymbol{Q}$, they satisfy the second order variation of the Hamiltonian function $\delta^2 H(\boldsymbol{Q}) > 0$, that is, $\boldsymbol{Q}$ is a minimal value of $H(\boldsymbol{q})$, and then the linear stability criterion can be deduced, i.e., Arnol'd's first linear stability criterion.

Firstly, we derive the expression $\delta^2 H(\boldsymbol{q})$, and then prove that it is time independent, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 H(\boldsymbol{Q}) = 0$, the linear stability criterion under the condition that $\boldsymbol{Q}$ is the minimal value of $H(\boldsymbol{q})$ with respect to the norm $\|\cdot\|_+$ can be obtained.

Lemma 2 Suppose $H(\boldsymbol{q})$ is a Hamiltonian function, then

$$\begin{aligned}\delta^2 H(\boldsymbol{q}) &= \int_\Omega [(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{q})\delta\boldsymbol{q},\delta\boldsymbol{q}) + |\nabla \boldsymbol{D}\delta\boldsymbol{\psi}|^2\\ &\quad - a(\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi})]\mathrm{d}\Omega, \qquad (47a)\\ \delta^2 H(\boldsymbol{Q}) &= \int_\Omega [(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q}) + |\nabla \boldsymbol{D}\delta\boldsymbol{\psi}|^2\\ &\quad - a(\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi})]\mathrm{d}\Omega. \qquad (47b)\end{aligned}$$

Proof: Based on the expressions (32) and (29), we have

$$\begin{aligned}H(\boldsymbol{q}) &= H_2(\boldsymbol{q}) + C(\boldsymbol{q})\\ &= \frac{1}{2}\int_\Omega [|\nabla \boldsymbol{D}\boldsymbol{\psi}|^2 - a(\boldsymbol{\psi},\boldsymbol{T}\boldsymbol{\psi})]\mathrm{d}\Omega\\ &\quad - \sum_{j=0}^{m}\left(\boldsymbol{\lambda}_j, \boldsymbol{D}^2\int_{\partial\Omega_j}\frac{\partial\boldsymbol{\psi}}{\partial n}\mathrm{d}s\right) + C(\boldsymbol{q}). \qquad (48)\end{aligned}$$

It follows from Eqs. (26), (39), (42) and $\delta\boldsymbol{\psi}|_{\partial\Omega} = 0$ that

$$\begin{aligned}\delta H(\boldsymbol{q}) &= \left\langle \frac{\delta H_2(\boldsymbol{q})}{\delta \boldsymbol{q}}, \delta\boldsymbol{q}\right\rangle + \left\langle \frac{\delta C(\boldsymbol{q})}{\delta \boldsymbol{q}}, \delta\boldsymbol{q}\right\rangle\\ &= \int_\Omega (\boldsymbol{D}^2(\boldsymbol{\Phi}(\boldsymbol{q}) - \boldsymbol{\psi}), \delta\boldsymbol{q})\mathrm{d}\Omega, \qquad (49)\end{aligned}$$

$$\begin{aligned}\delta^2 H(\boldsymbol{q}) &= \int_\Omega (\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{q})\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega - \int_\Omega (\boldsymbol{D}^2\delta\boldsymbol{\psi},\delta\boldsymbol{q})\mathrm{d}\Omega\\ &= \int_\Omega (\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{q})\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega\\ &\quad - \int_\Omega (\boldsymbol{D}^2\delta\boldsymbol{\psi}, \Delta\delta\boldsymbol{\psi} + \boldsymbol{F}^2\boldsymbol{T}\delta\boldsymbol{\psi})\mathrm{d}\Omega\\ &= \int_\Omega (\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{q})\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega + \int_\Omega |\boldsymbol{D}\nabla\delta\boldsymbol{\psi}|^2\mathrm{d}\Omega\\ &\quad - \int_{\partial\Omega}(\boldsymbol{D}^2\delta\boldsymbol{\psi}, \frac{\partial\delta\boldsymbol{\psi}}{\partial n})\mathrm{d}\Omega\\ &\quad - \int_\Omega (\boldsymbol{D}^2\delta\boldsymbol{\psi}, \boldsymbol{F}^2\boldsymbol{T}\delta\boldsymbol{\psi})\mathrm{d}\Omega\\ &= \int_\Omega [(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{q})\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega + \int_\Omega |\boldsymbol{D}\nabla\delta\boldsymbol{\psi}|^2\mathrm{d}\Omega\\ &\quad - a\int_\Omega (\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi})\mathrm{d}\Omega. \qquad (50)\end{aligned}$$

Therefore, it can be concluded that

$$\begin{aligned}\delta^2 H(\boldsymbol{Q}) &= \int_\Omega [(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q}) + |\nabla\boldsymbol{D}\delta\boldsymbol{\psi}|^2\\ &\quad - a(\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi})]\mathrm{d}\Omega. \qquad (51)\end{aligned}$$

Lemma 3 $\delta^2 H(\boldsymbol{Q})$ is time independent, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 H(\boldsymbol{Q}) = 0$.

Proof: Based on the above result of Eq. (51), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 H(\boldsymbol{Q}) &= \int_\Omega [(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q}_t) + (\nabla\boldsymbol{D}\delta\boldsymbol{\psi},\nabla\boldsymbol{D}\delta\boldsymbol{\psi}_t)\\ &\quad - a(\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi}_t)]\mathrm{d}\Omega = J_1 + J_2 - J_3. \qquad (52)\end{aligned}$$

It follows from Lemma 1 and Eq. (46) that

$$\begin{aligned}J_1 &= \int_\Omega (\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q}_t)\mathrm{d}\Omega\\ &= -\int_\Omega (\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q}, \partial(\boldsymbol{Q},\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q} - \delta\boldsymbol{\psi}))\mathrm{d}\Omega\\ &= -\sum_{i=1}^{N}\int_\Omega d_i\Phi_i'(Q_i)\delta q_i\cdot\partial(Q_i,\Phi_i'(Q_i)\delta q_i)\mathrm{d}\Omega\\ &\quad + \int_\Omega (\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\partial(\boldsymbol{Q},\delta\boldsymbol{\psi}))\mathrm{d}\Omega\\ &= -\sum_{i=1}^{N}\int_\Omega d_i\partial\left(Q_i,\frac{1}{2}(\Phi_i'(Q_i)\delta q_i)^2\right)\mathrm{d}\Omega\\ &\quad + \int_\Omega (\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\partial(\boldsymbol{Q},\delta\boldsymbol{\psi}))\mathrm{d}\Omega\\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=0}^{m}\int_{\partial\Omega_j} d_i(\Phi_i'(Q_i)\delta q_i)^2\frac{\partial Q_i}{\partial s}\mathrm{d}s\\ &\quad + \int_\Omega (\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\partial(\boldsymbol{Q},\delta\boldsymbol{\psi}))\mathrm{d}\Omega\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=0}^{m}\int_{\partial\Omega_j} d_i\Phi_i'(Q_i)(\delta q_i)^2\frac{\partial\Psi_i}{\partial s}\mathrm{d}s \\
&\quad + \int_{\Omega}(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\partial(\boldsymbol{Q},\delta\boldsymbol{\psi}))\mathrm{d}\Omega \\
&= \int_{\Omega}(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\partial(\boldsymbol{Q},\delta\boldsymbol{\psi}))\mathrm{d}\Omega, \qquad (53)
\end{aligned}
$$

where the formulas $\frac{\partial\Psi_i}{\partial s}\Big|_{\partial\Omega_j}=0$ and $\Psi_i=\Phi_i(Q_i)$ have been used.

The following result can be deduced from Lemma 1, $\delta\Psi|_{\partial\Omega}=0$, and $\delta\boldsymbol{q}_t=\Delta\delta\boldsymbol{\psi}_t+\boldsymbol{F}^2\boldsymbol{T}\delta\boldsymbol{\psi}_t=-\partial(\boldsymbol{Q},\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q}-\delta\boldsymbol{\psi})$:

$$
\begin{aligned}
&J_2-J_3 \\
&= \int_{\Omega}(\nabla\boldsymbol{D}\delta\boldsymbol{\psi},\nabla\boldsymbol{D}\delta\boldsymbol{\psi}_t)\mathrm{d}\Omega - J_3 \\
&= -\int_{\Omega}(\boldsymbol{D}^2\delta\boldsymbol{\psi},\Delta\delta\boldsymbol{\psi}_t)\mathrm{d}\Omega + \int_{\partial\Omega}(\boldsymbol{D}^2\delta\boldsymbol{\psi},\frac{\partial\delta\boldsymbol{\psi}_t}{\partial n})\mathrm{d}\Omega - J_3 \\
&= \int_{\Omega}(\boldsymbol{D}^2\delta\boldsymbol{\psi},\partial(\boldsymbol{Q},\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q})+\boldsymbol{F}^2\boldsymbol{T}\delta\boldsymbol{\psi}_t)\mathrm{d}\Omega - J_3 \\
&= \int_{\Omega}(\boldsymbol{D}^2\delta\boldsymbol{\psi},\partial(\boldsymbol{Q},\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q}))\mathrm{d}\Omega. \qquad (54)
\end{aligned}
$$

From the formula $\frac{\partial\Psi_i}{\partial s}\Big|_{\partial\Omega_j}=0$, we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2H(\boldsymbol{Q}) &= \int_{\Omega}(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\partial(\boldsymbol{Q},\delta\boldsymbol{\psi}))\mathrm{d}\Omega \\
&\quad + \int_{\Omega}(\boldsymbol{D}^2\delta\boldsymbol{\psi},\partial(\boldsymbol{Q},\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q}))\mathrm{d}\Omega \\
&= \sum_{i=1}^{N}\int_{\Omega} d_i\partial(Q_i,\Phi_i'(Q_i)\delta q_i\delta\psi_i)\mathrm{d}\Omega \\
&\quad (\text{using } a\partial(b,c)+c\partial(b,a)=\partial(b,ac)) \\
&= -\sum_{i=1}^{N}\sum_{j=0}^{m}\int_{\partial\Omega_j} d_i\frac{\partial Q_i}{\partial s}(\Phi_i'(Q_i)\delta q_i\delta\psi_i)\mathrm{d}\Omega \\
&= -\sum_{i=1}^{N}\sum_{j=0}^{m}\int_{\partial\Omega_j} d_i\frac{\partial\Psi_i}{\partial s}\delta q_i\delta\psi_i\mathrm{d}\Omega = 0. \qquad (55)
\end{aligned}
$$

Theorem 1 Suppose $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ satisfy

$$
\boldsymbol{\Psi}=\boldsymbol{\Phi}(\boldsymbol{Q}), \qquad (56)
$$

$$
\boldsymbol{Q}=\Delta\boldsymbol{\Psi}+\boldsymbol{F}^2\boldsymbol{T}\boldsymbol{\Psi},\ \frac{\partial\boldsymbol{\Psi}}{\partial s}\Big|_{\partial\Omega_j}=\boldsymbol{0},
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial\Omega_j}\frac{\partial\boldsymbol{\Psi}}{\partial n}\mathrm{d}s=\boldsymbol{0},\ \forall j=0,\dots m, \qquad (57)
$$

$$
\boldsymbol{\Phi}'(\boldsymbol{Q})>0,\ \text{i.e. } \mu_1\|\boldsymbol{y}\|^2\le(\boldsymbol{\Phi}'(\boldsymbol{Q})\boldsymbol{y},\boldsymbol{y})\le\mu_2\|\boldsymbol{y}\|^2,
$$

$$
\forall\boldsymbol{y}\in\mathrm{R}^N,\ \mu_i>0,\ i=1,2. \qquad (58)
$$

Denote

$$
\begin{aligned}
\|\delta\boldsymbol{\psi}\|_+^2 &= \int_{\Omega}[(\boldsymbol{D}\delta\boldsymbol{q},\boldsymbol{D}\delta\boldsymbol{q})+|\nabla\boldsymbol{D}\delta\boldsymbol{\psi}|^2 \\
&\quad -a(\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi})]\mathrm{d}\Omega. \qquad (59)
\end{aligned}
$$

Then, it can be concluded that the basic flow $\boldsymbol{\Psi}$ is stable in the sense of Liapunov with respect to the norm $\|\cdot\|_+$.

Expression (59) presents the sum of perturbation energy and energy–enstropy.

Proof: Direct calculation of $\delta^2H(\boldsymbol{Q})$ gives

$$
\begin{aligned}
\delta^2H(\boldsymbol{Q}) &= \int_{\Omega}[(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q})+|\nabla\boldsymbol{D}\delta\boldsymbol{\psi}|^2 \\
&\quad -a(\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi})]\mathrm{d}\Omega. \qquad (60)
\end{aligned}
$$

Based on the assumption that $\boldsymbol{\Phi}'(\boldsymbol{Q})$ is a positive definite symmetric matrix, and $-\boldsymbol{T}$ is a nonnegative matrix, we obtain

$$
\delta^2H(\boldsymbol{Q})>0,\quad\forall\delta\boldsymbol{q},\forall\delta\boldsymbol{\psi}\neq\boldsymbol{0}. \qquad (61)
$$

On the other hand, it can be deduced that $\delta^2H(\boldsymbol{Q})\big|_t=\delta^2H(\boldsymbol{Q})\big|_{t=0}$ from Lemma 3. Denote

$$
\Gamma_1=\min(1,\mu_1),\quad\Gamma_2=\max(1,\mu_2), \qquad (62)
$$

It follows that

$$
\begin{aligned}
\|\delta\boldsymbol{\psi}\|_+^2\Big|_t &= \int_{\Omega}[(\boldsymbol{D}\delta\boldsymbol{q},\boldsymbol{D}\delta\boldsymbol{q})+|\nabla\boldsymbol{D}\delta\boldsymbol{\psi}|^2-a(\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi})]\mathrm{d}\Omega\Big|_t \\
&\le \frac{1}{\Gamma_1}\int_{\Omega}[(\boldsymbol{D}\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\boldsymbol{D}\delta\boldsymbol{q})+|\nabla\boldsymbol{D}\delta\boldsymbol{\psi}|^2 \\
&\quad -a(\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi})]\mathrm{d}\Omega\Big|_t \\
&= \frac{1}{\Gamma_1}\delta^2H(\boldsymbol{Q})\Big|_t=\frac{1}{\Gamma_1}\delta^2H(\boldsymbol{Q})\big|_{t=0} \\
&\le \frac{\Gamma_2}{\Gamma_1}\int_{\Omega}[(\boldsymbol{D}\delta\boldsymbol{q},\boldsymbol{D}\delta\boldsymbol{q})+|\nabla\boldsymbol{D}\delta\boldsymbol{\psi}|^2 \\
&\quad -a(\delta\boldsymbol{\psi},\boldsymbol{T}\delta\boldsymbol{\psi})]\mathrm{d}\Omega\Big|_{t=0} \\
&= \frac{\Gamma_2}{\Gamma_1}\|\delta\boldsymbol{\psi}\|_+^2\Big|_{t=0}. \qquad (63)
\end{aligned}
$$

Thus, for every $\varepsilon>0$ and for all $t\ge0$, when $\|\delta\boldsymbol{\psi}\|_+^2\Big|_{t=0}<\varepsilon\Gamma_1/\Gamma_2$, we have $\|\delta\boldsymbol{\psi}\|_+^2\Big|_t<\varepsilon$, i.e., the linear stability of $\boldsymbol{\Psi}(x,y)$ holds in the sense of Liapunov with respect to the energy–enstropy norm (59).

6. Stability criterion (II) of the linearized equations for infinitesimal disturbances

For the given basic flows $\boldsymbol{\Psi}$ and $\boldsymbol{Q}$, they satisfy the second order variation of the Hamiltonian function $\delta^2H(\boldsymbol{Q})<0$, that is, $\boldsymbol{Q}$ is a maximal value of $H(\boldsymbol{q})$, and then the linear stability criterion can be deduced, i.e., Arnol'd's second linear stability criterion. Actually, the linear stability criterion is more meaningful than the first one in physics.

Lemma 4 Suppose Ω is as indicated earlier, φ satisfies

$$-\Delta\varphi=\lambda\varphi,\quad \left.\frac{\partial\varphi}{\partial s}\right|_{\partial\Omega_j}=0,\ \oint\limits_{\partial\Omega_j}\frac{\partial\varphi}{\partial n}\mathrm{d}s=0,\quad \forall j=0,\ldots,m. \tag{64}$$

then there exist denumerable eigenvalues $0<\lambda_1\le\ldots\le\lambda_n\to\infty(n\to\infty)$, φ_n $(n=1,2,\ldots)$ are the corresponding eigenfunctions, $\{\varphi_n,c_0\}$ consist of the complete orthogonal system of $L^2(\Omega)$, where c_0 is a constant and φ_n satisfies $\int\limits_\Omega\varphi_n\mathrm{d}\Omega=0$ $(n=1,2,\ldots)$. So for an arbitrary $\psi\in L^2(\Omega)$, it can be expanded as the following form:

$$\psi=\sum_{n=1}^{\infty}c_n\varphi_n+c_0,\quad (c_n\ \text{is a constant}). \tag{65}$$

If $\int\limits_\Omega\psi\mathrm{d}\Omega=0$, then we have $c_0=0$, the proof can be referenced in the theory of partial differential equations.

Since $-\boldsymbol{FTF}$ is a nonnegative symmetrical matrix, denote the eigenvalues of $-\boldsymbol{FTF}$ as α_i, and they satisfy $0=\alpha_1\le\alpha_2\le\ldots\le\alpha_N$, thus, there is an orthogonal matrix $\boldsymbol{U}$, such that $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{FTFU}\triangleq-\boldsymbol{A}=-\mathrm{diag}(\alpha_1,\ldots,\alpha_N)$. From the theory of algebra, $-\boldsymbol{F}^2\boldsymbol{T}$ and $-\boldsymbol{FTF}$ have the same eigenvalues. Denote

$$\boldsymbol{\Lambda}=\mathrm{diag}\left(\frac{1}{\lambda_1+\alpha_1},\ldots,\frac{1}{\lambda_1+\alpha_N}\right) \tag{66}$$

Lemma 5 Suppose $\delta\psi\in\left[H^2(\Omega)\right]^N$, and $\int\limits_\Omega\delta\psi\mathrm{d}\Omega=0$, $\delta\psi$ satisfies

$$\Delta\delta\psi+\boldsymbol{F}^2\boldsymbol{T}\delta\psi=\delta\boldsymbol{q},\quad \left.\frac{\partial\delta\psi}{\partial s}\right|_{\partial\Omega_j}=\boldsymbol{0},\ \oint\limits_{\partial\Omega_j}\frac{\partial\delta\psi}{\partial n}\mathrm{d}s=\boldsymbol{0},\quad \forall j=0,\ldots,m, \tag{67}$$

then the following inequality

$$-\int\limits_\Omega(\boldsymbol{D}^2\delta\psi,\delta\boldsymbol{q})\mathrm{d}\Omega\le\int\limits_\Omega(\boldsymbol{D}^2\delta\boldsymbol{q},\boldsymbol{F}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{F}^{-1}\delta\boldsymbol{q})\mathrm{d}\Omega \tag{68}$$

can be derived.

Proof: Step 1 When $N=1$, take the following boundary problem into account:

$$\Delta\delta\psi-\gamma\delta\psi=\delta q,(\gamma\ge0),\quad \left.\frac{\partial\delta\psi}{\partial s}\right|_{\partial\Omega_j}=0,\ \oint\limits_{\partial\Omega_j}\frac{\partial\delta\psi}{\partial n}\mathrm{d}s=0,\quad \forall j=0,\ldots,m. \tag{69}$$

Based on $\int\limits_\Omega\delta\psi\mathrm{d}\Omega=0$ and Lemma 4, $\delta\psi$ can be expanded as follows:

$$\delta\psi=\sum_{i=1}^{\infty}c_i\varphi_i,\quad (-\Delta\varphi_i=\lambda_i\varphi_i,\ \left.\frac{\partial\delta\psi}{\partial s}\right|_{\partial\Omega_j}=0,\ \oint\limits_{\partial\Omega_j}\frac{\partial\delta\psi}{\partial n}\mathrm{d}s=0,\quad \forall j=0,\ldots,m), \tag{70}$$

then we obtain

$$\delta q=\Delta\delta\psi-\gamma\delta\psi=-\sum_{i=1}^{\infty}c_i(\lambda_i+\gamma)\varphi_i. \tag{71}$$

So it follows that

$$\begin{aligned}-\int\limits_\Omega\delta\psi\delta q\mathrm{d}\Omega&=\int\limits_\Omega\left[\sum_{i=1}^{\infty}c_i\varphi_i\right]\left[\sum_{j=1}^{\infty}c_j(\lambda_j+\gamma)\varphi_j\right]\mathrm{d}\Omega\\&=\sum_{i=1}^{\infty}c_i^2(\lambda_i+\gamma)\\&\le\frac{1}{\lambda_1+\gamma}\left[\sum_{i=1}^{\infty}c_i^2(\lambda_i+\gamma)^2\right]\\&=\frac{1}{\lambda_1+\gamma}\int\limits_\Omega[\delta q]^2\mathrm{d}\Omega.\end{aligned} \tag{72}$$

Step 2 Substituting the expressions $\delta\psi=\boldsymbol{FUw}$ and $\boldsymbol{b}=\boldsymbol{U}^{-1}\boldsymbol{F}^{-1}\delta\boldsymbol{q}$ into the first equation of Eq. (67) yields the following formula:

$$\Delta\boldsymbol{w}-\mathrm{A}\boldsymbol{w}=\boldsymbol{b}. \tag{73}$$

From $\int\limits_\Omega\delta\psi\mathrm{d}\Omega=\boldsymbol{0}$, we can deduce $\int\limits_\Omega\boldsymbol{w}\mathrm{d}\Omega=\boldsymbol{0}$, and based on step 1, it can be concluded that

$$\begin{aligned}&-\int\limits_\Omega(\boldsymbol{D}^2\delta\psi,\delta\boldsymbol{q})\mathrm{d}\Omega\\&=-\int\limits_\Omega(\boldsymbol{D}^2\boldsymbol{FUw},\boldsymbol{FUb})\mathrm{d}\Omega\\&=-a\int\limits_\Omega(\boldsymbol{w},\boldsymbol{b})\mathrm{d}\Omega\\&\le a\int\limits_\Omega(\boldsymbol{b},\boldsymbol{\Lambda b})\mathrm{d}\Omega\\&=a\int\limits_\Omega(\boldsymbol{U}^{-1}\boldsymbol{F}^{-1}\delta\boldsymbol{q},\boldsymbol{\Lambda U}^{-1}\boldsymbol{F}^{-1}\delta\boldsymbol{q})\mathrm{d}\Omega\\&=\int\limits_\Omega(\boldsymbol{D}^2\delta\boldsymbol{q},\boldsymbol{F}^{\mathrm{T}}\boldsymbol{U\Lambda U}^{\mathrm{T}}\boldsymbol{F}^{-1}\delta\boldsymbol{q})\mathrm{d}\Omega.\end{aligned} \tag{74}$$

Theorem 2 Suppose the basic flows $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ satisfy

$$\boldsymbol{\Psi}=\boldsymbol{\Phi}(\boldsymbol{Q}), \tag{75}$$

$$\boldsymbol{Q}=\Delta\boldsymbol{\Psi}+\boldsymbol{F}^2\boldsymbol{T\Psi},\ \left.\frac{\partial\boldsymbol{\Psi}}{\partial s}\right|_{\partial\Omega_j}=\boldsymbol{0},\ \oint\limits_{\partial\Omega_j}\frac{\partial\boldsymbol{\Psi}}{\partial n}\mathrm{d}s=\boldsymbol{0},\quad \forall j=0,\ldots,m, \tag{76}$$

$$\boldsymbol{\Phi}'(\boldsymbol{Q})<0,\quad \text{i.e.,}$$

$$(\boldsymbol{C}_1\boldsymbol{y},\boldsymbol{y})\le(-\boldsymbol{\Phi}'(\boldsymbol{Q})\boldsymbol{y},\boldsymbol{y})\le(\boldsymbol{C}_2\boldsymbol{y},\boldsymbol{y}),\ \forall\boldsymbol{y}\in\mathrm{R}^N, \tag{77}$$

where $\boldsymbol{C}_i=\mathrm{diag}(c_{i1},\ldots,c_{iN})>0$, $(i=1,2)$;

$$\boldsymbol{M}=\boldsymbol{C}_1-\boldsymbol{F}^{\mathrm{T}}\boldsymbol{U\Lambda U}^{\mathrm{T}}\boldsymbol{F}^{-1}>0; \tag{78}$$

$$\oint_{\partial\Omega_j}\frac{\partial\delta\psi}{\partial n}\mathrm{d}s=\mathbf{0},\quad \forall j=0,\ldots,m. \tag{79}$$

Denote

$$\|\delta\psi\|_-^2=\int_\Omega(\boldsymbol{D}\delta\boldsymbol{q},D\delta\boldsymbol{q})\mathrm{d}\Omega. \tag{80}$$

Then the basic flow $\boldsymbol{\Psi}$ is stable in the sense of Liapunov with respect to the norm $\|\cdot\|_-$.

The formula (80) presents the perturbed enstropy.

Proof: Step 1 If we can prove $\delta^2H(\boldsymbol{Q})<0$, then $\boldsymbol{Q}$ is a maximal value of $H(\boldsymbol{q})$. Calculate

$$\begin{aligned}
&\int_\Omega[|\nabla\boldsymbol{D}\delta\psi|^2-a(\delta\psi,\boldsymbol{T}\delta\psi)]\mathrm{d}\Omega\\
&=-\int_\Omega[(\boldsymbol{D}^2\delta\psi,\Delta\delta\psi)+a(\delta\psi,\boldsymbol{T}\delta\psi)]\mathrm{d}\Omega\\
&\quad+\int_{\partial\Omega}\left(\boldsymbol{D}^2\delta\psi,\frac{\partial\delta\psi}{\partial n}\right)\mathrm{d}\Omega\\
&=-\int_\Omega[(\boldsymbol{D}^2\delta\psi,\delta\boldsymbol{q}-\boldsymbol{F}^2\boldsymbol{T}\delta\psi)+a(\delta\psi,\boldsymbol{T}\delta\psi)]\mathrm{d}\Omega\\
&=-\int_\Omega[(\boldsymbol{D}^2\delta\psi,\delta\boldsymbol{q})]\mathrm{d}\Omega.
\end{aligned}\tag{81}$$

Based on Lemma 5 and the expression of $\delta^2H(\boldsymbol{Q})$

$$\begin{aligned}
\delta^2H(\boldsymbol{Q})=&\int_\Omega[(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q})+|\nabla\boldsymbol{D}\delta\psi|^2\\
&-a(\delta\psi,\boldsymbol{T}\delta\psi)]\mathrm{d}\Omega,
\end{aligned}\tag{82}$$

we can obtain

$$\begin{aligned}
\delta^2H(\boldsymbol{Q})&=\int_\Omega(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega-\int_\Omega(\boldsymbol{D}^2\delta\psi,\delta\boldsymbol{q})\mathrm{d}\Omega\\
&\le\int_\Omega[\boldsymbol{D}^2(\boldsymbol{\Phi}'(\boldsymbol{Q})+\boldsymbol{F}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{F}^{-1})\delta\boldsymbol{q},\delta\boldsymbol{q}]\mathrm{d}\Omega\\
&\le\int_\Omega[\boldsymbol{D}^2(-\boldsymbol{C}_1+\boldsymbol{F}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{F}^{-1})\delta\boldsymbol{q},\delta\boldsymbol{q}]\mathrm{d}\Omega\\
&=-\int_\Omega(\boldsymbol{D}^2\boldsymbol{M}\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega.
\end{aligned}\tag{83}$$

From Eq. (78), it can be obtained that $\boldsymbol{M}>0$, therefore $\delta^2H(\boldsymbol{Q})<0(\forall\delta\boldsymbol{q}\neq\mathbf{0})$, namely, $\boldsymbol{Q}$ is a maximal point of $\delta^2H(\boldsymbol{q})$.

Step 2 Estimate $\|\delta\psi\|_-$. Denote $\mu>0$ as the minimal eigenvalue of $\boldsymbol{M}$, and $\sigma>0$ as the maximal eigenvalue of $\boldsymbol{C}_2$, then

$$\begin{aligned}
\delta^2H(\boldsymbol{Q})|_t&\le-\int_\Omega(\boldsymbol{D}^2\boldsymbol{M}\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega\Big|_t\\
&\le-\mu\int_\Omega(\boldsymbol{D}^2\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega|_t=-\mu\|\delta\psi\|_-^2\Big|_t,
\end{aligned}\tag{84}$$

$$\begin{aligned}
&-\int_\Omega(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega\Big|_{t=0}\\
&\le\sigma\int_\Omega(\boldsymbol{D}^2\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega\Big|_{t=0}.
\end{aligned}\tag{85}$$

So

$$\begin{aligned}
\|\delta\psi\|_-^2|_t&\le-\frac{1}{\mu}\delta^2H(\boldsymbol{Q})\Big|_t=-\frac{1}{\mu}\delta^2H(\boldsymbol{Q})\Big|_{t=0}\\
&=-\frac{1}{\mu}\int_\Omega[(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q})+|\nabla\boldsymbol{D}\delta\psi|^2\\
&\quad-a(\delta\psi,\boldsymbol{T}\delta\psi)]\mathrm{d}\Omega\Big|_{t=0}\\
&\le-\frac{1}{\mu}\int_\Omega(\boldsymbol{D}^2\boldsymbol{\Phi}'(\boldsymbol{Q})\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega\Big|_{t=0}\\
&\le\frac{\sigma}{\mu}\int_\Omega(\boldsymbol{D}^2\delta\boldsymbol{q},\delta\boldsymbol{q})\mathrm{d}\Omega\Big|_{t=0}=\frac{\sigma}{\mu}\|\delta\psi\|_-^2\Big|_{t=0}.
\end{aligned}\tag{86}$$

Thus, for every $\varepsilon>0$ and for all $t\ge0$, when $\|\delta\psi\|_-^2\Big|_{t=0}<\varepsilon\frac{\mu}{\sigma}$, we have $\|\delta\psi\|_-^2\Big|_t<\varepsilon$, i.e., the linear stability of $\boldsymbol{\Psi}(x,y)$ holds in the sense of Liapunov with respect to the enstropy norm (80).

Remark 1 In Theorem 2, we assume that the disturbance vorticity $\delta\psi$ satisfies the condition (79) $\oint_{\partial\Omega_j}\frac{\partial\delta\psi}{\partial n}\mathrm{d}s=\mathbf{0}$, while $\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial\Omega_j}\frac{\partial\delta\psi}{\partial n}\mathrm{d}s=\mathbf{0}$ in the original problem. Actually, the condition (79) is not essential for the reason that we can divide the disturbances into $(\delta\psi_1+\psi_2,\delta\boldsymbol{q}_1+\boldsymbol{q}_2)$, such that $\delta\psi_1$ satisfies Eq. (79) and ψ_2 satisfies

$$\oint_{\partial\Omega_j}\frac{\partial\psi_2}{\partial n}\mathrm{d}s=\oint_{\partial\Omega_j}\frac{\partial\psi_0}{\partial n}\mathrm{d}s,$$

where ψ_0 denotes the original stream function value, $\boldsymbol{q}_2=\frac{\int_{\Omega_j}\boldsymbol{q}_0\mathrm{d}\Omega}{|\Omega_j|}$, and $|\Omega_j|$ presents the measure of Ω_j. Now, the problem can be divided into two parts as follows:

$$\begin{aligned}
&\delta\boldsymbol{q}_1=\Delta\delta\psi_1+\boldsymbol{F}^2\boldsymbol{T}\delta\psi_1,\\
&\frac{\partial\delta\psi_1}{\partial s}\Big|_{\partial\Omega_j}=\mathbf{0},(j=0,\ldots,m),\\
&\oint_{\partial\Omega_j}\frac{\partial\delta\psi_1}{\partial n}\mathrm{d}s=\mathbf{0},\quad(j=0,\ldots,m),
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{q}_2=\Delta\psi_2+\boldsymbol{F}^2\boldsymbol{T}\psi_2,\\
&\frac{\partial\psi_2}{\partial s}\Big|_{\partial\Omega_j}=\mathbf{0},(j=0,\ldots,m),\\
&\oint_{\partial\Omega_j}\frac{\partial\psi_2}{\partial n}\mathrm{d}s=\oint_{\partial\Omega_j}\frac{\partial\psi_0}{\partial n}\mathrm{d}s,\quad(j=0,\ldots,m).
\end{aligned}$$

Estimate $\delta\psi_1$ and ψ_2 respectively, and the stability criterion can be deduced similarly.

Remark 2 When discussing the nonlinear stability, the functional

$$N(\boldsymbol{q}) = H(\boldsymbol{Q}+\delta\boldsymbol{q}) - H(\boldsymbol{Q}) - \delta H(\boldsymbol{Q})$$

is introduced to substitute $\delta^2 H(\boldsymbol{Q})$. First of all, we should calculate $\delta H(\boldsymbol{Q}) = 0$ and the conserved quantity of $N(\boldsymbol{q})$, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}N(\boldsymbol{q}) = 0$, then on the basis of the estimation of the operator $\boldsymbol{\Phi}'(\boldsymbol{q})$, the stability criteria can be obtained, too.

7. Conclusions

The linear stability of multilayer quasi-geostrophic flow is studied in this paper. Firstly, the equations are turned into the form of an infinite-dimensional Hamiltonian system $\boldsymbol{q}_t = J\frac{\delta H_2}{\delta \boldsymbol{q}}$. To make the basic flow $\boldsymbol{Q}$ an extreme point of the Hamiltonian function, a new Hamiltonian function H is reconstructed. It is time independent and maintains the structure of the system invariant, namely, $\boldsymbol{q}_t = J\frac{\delta H}{\delta \boldsymbol{q}}$, $J \triangleq \hat{J}\boldsymbol{D}^{-2}$. Secondly, the linearized equations are derived by superposing infinitesimal disturbances on the basic flows. Finally, the linear stability criteria are deduced in the sense of Liapunov under two different conditions with respect to certain norms. That is, the basic flow is a minimum point or a maximal point of H, the energy–enstropy and the enstropy are introduced to be norms, respectively. The nonlinear stability can be researched by using the same method, which we will discuss in other articles.

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