Free-matrix-based time-dependent discontinuous Lyapunov functional for synchronization of delayed neural networks with sampled-data control*

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This paper is concerned with the synchronization of delayed neural networks via sampled-data control. A new technique, namely, the free-matrix-based time-dependent discontinuous Lyapunov functional approach, is adopted in constructing the Lyapunov functional, which takes advantage of the sampling characteristic of sawtooth input delay. Based on this discontinuous Lyapunov functional, some less conservative synchronization criteria are established to ensure that the slave system is synchronous with the master system. The desired sampled-data controller can be obtained through the use of the linear matrix inequality (LMI) technique. Finally, two numerical examples are provided to demonstrate the effectiveness and the improvements of the proposed methods.

Keywords: neural networks, synchronization, sampled-data control, free-matrix-based inequality

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1. Introduction

Since the master-slave concept was proposed for the synchronization of coupled chaotic systems in Ref. [1], a variety of alternative schemes have been proposed for ensuring the synchronization of such systems.[2–6] Neural networks have received increasing attention as they have been successfully applied in many areas such as signal processing, pattern recognition, associative memories, and fixed-point computations. Various issues about neural networks have been investigated and many important results have been obtained.[7–19] As a special class of complex networks, delayed neural networks (DNNs) have also been found to exhibit complicated dynamics and chaotic behaviors, such as being extremely sensitive to variations of the initial conditions, having bounded trajectories in phase space, and so on. Therefore, many approaches have been developed for the synchronization of chaotic neural networks, such as time-delay feedback control, impulsive control, and sampled-data control (see Refs. [20]–[24] and the references therein).

In Ref. [25], the problem of synchronization was investigated for stochastic neural networks with time delay. Based on the Lyapunov functional approach, some sufficient conditions were derived to ensure the synchronization of the slaver system with the master system. Since the conditions proposed in Ref. [25] are delay-independent, they are rather conservative. The problem of delay-dependent $\mathcal{H}_\infty$ synchronization of delay neural networks was studied in Ref. [26], where several delay-dependent criteria were proposed. In Ref. [27], the synchronization problem was investigated for a class of master-slave delayed neural networks with heterogeneous dimensions, and improved synchronization conditions were derived by using the free-matrix-based inequality approach.[28,29] In Ref. [30], the problem of adaptive exponential synchronization was investigated for a general class of memristive neural networks with mixed time-varying delays, where an adaptive controller with feedback control law was designed to achieve exponential synchronization. In Ref. [31], the synchronization problem of fractional-order complex-valued neural networks with time delays was investigated by means of linear delay feedback control. As for discrete-time chaotic neural networks, the problem of exponential $\mathcal{H}_\infty$ synchronization was studied in Ref. [32], where a state feedback controller was obtained to not only ensure the exponential synchronization between two general neural networks, but also to reduce the effect of external disturbance on the synchronization error such that it satisfies a minimal $\mathcal{H}_\infty$ norm constraint. In Ref. [33], the local and global synchronization conditions have been derived for delayed discrete-time neural networks subject to saturated time-delay feedback.

Generally, networked control systems (NCSs) can be modeled as sampled-data systems under variable sampling with an additive network-induced delay.[34,35] Thus, sampled-data control in the presence of a constant input delay has been an important research field. However, the input delay approach

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neglects information concerning the actual sampling pattern, which leads to some conservative results. In Ref. [36], a time-dependent Lyapunov functional approach was proposed to improve the input delay approach. The advantage of the time-dependent Lyapunov functional approach lies in the fact that it considers the sawtooth characteristics of the time-varying delay induced by the sample holder. In Ref. [37], a new two-sided looped-function approach was proposed for stability analysis of sampled-data systems. In Ref. [38], the sampled-data feedback control was investigated for the synchronization of neural networks with discrete and distributed delays under the framework of an input delay approach. However, the signal transmission delay has not been taken into account. Recently, the sampled-data control in the presence of a constant input delay was proposed for the synchronization of neural networks with time-varying delay in Ref. [39], and some sufficient conditions were derived based on a Wirtinger inequality-based discontinuous Lyapunov functional.

In this paper, we revisit the problem of master-slave synchronization of neural networks with time-varying delay. By adopting a free-matrix-based time-dependent discontinuous Lyapunov functional approach,[42–44] less conservative synchronization conditions are derived, which take advantage of the sampling characteristic of sawtooth input delay. Two numerical examples are provided to demonstrate the effectiveness of the proposed results and their improvement over existing ones.

Notation Through this paper, $\boldsymbol{N}^{\mathrm{T}}$ and $\boldsymbol{N}^{-1}$ stand for the transpose and the inverse of the matrix $\boldsymbol{N}$, respectively; $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the n-dimensional Euclidean space and the set of all $m\times n$ real matrices, respectively. $\boldsymbol{P}>0$ ($\boldsymbol{P}\geq 0$) means that the matrix, $\boldsymbol{P}$, is symmetric and positive definite (positive semidefinite). $\boldsymbol{I}$ and 0 represent the identity matrix and a zero matrix, respectively. diag$\{\cdots\}$ stands for a block-diagonal matrix. $||\cdot||$ denotes the Euclidean norm of a vector and its induced norm of a matrix. Sym$\{\boldsymbol{X}\}=\boldsymbol{X}+\boldsymbol{X}^{\mathrm{T}}$ and the symmetric terms in a symmetric matrix are denoted by $*$, e.g., $\begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}\\ * & \boldsymbol{Z}\end{bmatrix}=\begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}\\ \boldsymbol{Y}^{\mathrm{T}} & \boldsymbol{Z}\end{bmatrix}$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Preliminaries

Consider a master system, which is a neural network defined as follows:

$$\dot{\boldsymbol{x}}(t)=-\boldsymbol{C}x(t)+\boldsymbol{A}\boldsymbol{g}(\boldsymbol{x}(t))+\boldsymbol{B}\boldsymbol{g}(\boldsymbol{x}(t-d(t)))+\boldsymbol{J}(t), \quad (1)$$

where $\boldsymbol{x}(t)=\left[x_1(t)\ x_2(t)\ \cdots\ x_n(t)\right]^{\mathrm{T}}$, $\boldsymbol{g}(\boldsymbol{x}(t))=\left[g_1(x_1(t))\ g_2(x_2(t))\ \cdots\ g_n(x_n(t))\right]^{\mathrm{T}}$, $x_i(t)$ is the state of the i-th neuron, and $g_i(x_i(t))$ denotes the neuron activation function; $\boldsymbol{C}=\mathrm{diag}\{c_1,c_2,\ldots,c_n\}$ is a diagonal matrix with positive entries; $\boldsymbol{A}=(a_{ij})_{n\times n}$ and $\boldsymbol{B}=(b_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively; $\boldsymbol{J}(t)=\left[j_1(t)\ j_2(t)\ \cdots\ j_n(t)\right]^{\mathrm{T}}$ is the external input; $d(t)$ is the time-varying delay and satisfies

$$0\leq d(t)\leq \bar{d},\ \ \dot{d}(t)\leq \mu, \quad (2)$$

where $\bar{d}$ and μ are known constants. The activation functions $g_i(\cdot), i=1,2,\ldots,n$, are continuous and satisfy

$$F_i^-\leq\frac{g_i(\alpha_1)-g_i(\alpha_2)}{\alpha_1-\alpha_2}\leq F_i^+,\ \ g_i(0)=0,\ \ i=1,2,\ldots,n, \quad (3)$$

where $\alpha_1,\ \alpha_2\in\mathbb{R}$, and $\alpha_1\neq\alpha_2$; F_i^- and F_i^+ are known real scalars. In this paper, the slave system for Eq. (1) is described as follows:

$$\begin{aligned}\dot{\boldsymbol{y}}(t)\ =\ &-\boldsymbol{C}\boldsymbol{y}(t)+\boldsymbol{A}\boldsymbol{g}(\boldsymbol{y}(t))+\boldsymbol{B}\boldsymbol{g}(\boldsymbol{y}(t-d(t)))\\ &+\boldsymbol{J}(t)+\boldsymbol{u}(t),\end{aligned} \quad (4)$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are matrices which follow the same definitions as those in Eq. (1), and $\boldsymbol{u}(t)\in\mathbb{R}^n$ is the control input.

By defining an error signal as $\boldsymbol{r}(t)=\boldsymbol{y}(t)-\boldsymbol{x}(t)$, the corresponding error system can be obtained as follows:

$$\dot{\boldsymbol{r}}(t)=-\boldsymbol{C}\boldsymbol{r}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{r}(t))+\boldsymbol{B}\boldsymbol{f}(\boldsymbol{r}(t-d(t)))+\boldsymbol{u}(t), \quad (5)$$

where $\boldsymbol{f}(\boldsymbol{r}(t))=\boldsymbol{g}(\boldsymbol{r}(t)+\boldsymbol{x}(t))-\boldsymbol{g}(\boldsymbol{x}(t))$. It can be derived from Eq. (3) that the functions $f_i(\cdot)$ satisfy

$$F_i^-\leq\frac{f_i(\alpha)}{\alpha}\leq F_i^+,\ \ f_i(0)=0,\ \ \ i=1,2,\ldots,n, \quad (6)$$

where $\alpha\in\mathbb{R}$ and $\alpha\neq 0$.

Suppose that there is a constant transmission delay, η, when the updating signal transmitted from the sampler to the controller, and the updating instant time of the zero-order-hold (ZOH) is denoted by t_k. The sampling intervals are assumed to satisfy

$$t_{k+1}-t_k=h_k\leq\bar{h}, \quad (7)$$

for any integer $k\geq 0$, where $\bar{h}>0$ represents the largest sampling interval. In this paper, we take the state-feedback controller as follows:

$$\boldsymbol{u}(t)=\boldsymbol{K}\boldsymbol{r}(t_k-\eta),\ \ \ t_k\leq t<t_{k+1}, \quad (8)$$

where K is the sampled-data feedback controller gain matrix to be determined.

By substituting Eq. (8) into Eq. (5), the error system is reduced to

$$\begin{aligned}\dot{\boldsymbol{r}}(t)\ =\ &-\boldsymbol{C}\boldsymbol{r}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{r}(t))+\boldsymbol{B}\boldsymbol{f}(\boldsymbol{r}(t-d(t)))+\boldsymbol{K}\boldsymbol{r}(t_k-\eta),\\ &t_k\leq t<t_{k+1}.\end{aligned} \quad (9)$$

Our goal is to design a controller in the form of Eq. (8) to ensure that the slave system (4) synchronizes with the master

system (1). In other words, our purpose is to find a feedback gain matrix $\boldsymbol{K}$ such that the error system (9) is stable.

Before presenting our main results, we introduce the following lemmas.

Lemma 1[40] For any matrix $\bar{U} = \begin{bmatrix} \boldsymbol{Z} & \boldsymbol{S} \\ * & \boldsymbol{Z} \end{bmatrix} \geqslant 0$, scalars $d_2 > d_1 > 0$, $d(t) > 0$ satisfying $d_1 < d(t) < d_2$, vector function $\dot{\boldsymbol{x}} : [-d_2, 0] \to \mathbb{R}^n$ such that the concerned integrations are well defined, then

$$-(d_2 - d_1)\int_{t-d_2}^{t-d_1} \dot{\boldsymbol{x}}^{\mathrm{T}}(\alpha) Z \dot{\boldsymbol{x}}(\alpha)\mathrm{d}\alpha \leqslant -\boldsymbol{\varpi}^{\mathrm{T}}(t)\bar{U}\boldsymbol{\varpi}(t),$$

where

$$\boldsymbol{\varpi}(t) = \begin{bmatrix} \boldsymbol{x}(t-d_1) - \boldsymbol{x}(t-d(t)) \\ \boldsymbol{x}(t-d(t)) - \boldsymbol{x}(t-d_2) \end{bmatrix}.$$

Lemma 2[41] For any positive matrix $\boldsymbol{R}$ and for differentiable signal $\boldsymbol{x}$ in $[\alpha, \beta] \to \mathbb{R}^n$, the following inequality holds:

$$\int_{\alpha}^{\beta} \dot{\boldsymbol{x}}^{\mathrm{T}}(u)\boldsymbol{R}\dot{x}(u)\,\mathrm{d}u \geq \frac{1}{\beta-\alpha}\hat{\Omega}^{\mathrm{T}}\hat{R}\hat{\Omega}, \tag{10}$$

where $\hat{R} = \mathrm{diag}\{\boldsymbol{R}, 3\boldsymbol{R}\}$ and

$$\hat{\Omega} = \begin{bmatrix} \boldsymbol{x}(\beta) - \boldsymbol{x}(\alpha) \\ \boldsymbol{x}(\beta) + \boldsymbol{x}(\alpha) - \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} \boldsymbol{x}(u)\,\mathrm{d}u \end{bmatrix}.$$

Lemma 3 Let $\boldsymbol{x}$ be a differentiable function: $[\alpha, \beta] \to \mathbb{R}^n$. For any vector $\boldsymbol{\xi} \in \mathbb{R}^m$, symmetric matrices $\boldsymbol{R} \in \mathbb{R}^{n\times n}$, and $\boldsymbol{Z}_1, \boldsymbol{Z}_3 \in \mathbb{R}^{m\times m}$, and any matrices $\boldsymbol{Z}_2 \in \mathbb{R}^{m\times m}$, and $\boldsymbol{N}_1, \boldsymbol{N}_2 \in \mathbb{R}^{m\times n}$ satisfying

$$\begin{bmatrix} \boldsymbol{Z}_1 & \boldsymbol{Z}_2 & \boldsymbol{N}_1 \\ * & \boldsymbol{Z}_3 & \boldsymbol{N}_2 \\ * & * & \boldsymbol{R} \end{bmatrix} \geq 0,$$

the following inequality holds:

$$\begin{aligned} &-\int_{\alpha}^{\beta} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)R\dot{\boldsymbol{x}}(s)\mathrm{d}s \\ &\leq (\beta-\alpha)\boldsymbol{\xi}^{\mathrm{T}}[\boldsymbol{Z}_1 + \frac{(\beta-\alpha)^2}{3}\boldsymbol{Z}_3]\boldsymbol{\xi} \\ &\quad + 2\boldsymbol{\xi}^{\mathrm{T}}\left[\boldsymbol{N}_1(\boldsymbol{x}(\beta) - \boldsymbol{x}(\alpha)) - 2\boldsymbol{N}_2\int_{\alpha}^{\beta}\boldsymbol{x}(s)\mathrm{d}s\right] \\ &\quad + 2(\beta-\alpha)\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{N}_2[\boldsymbol{x}(\beta) + \boldsymbol{x}(\alpha)]. \end{aligned} \tag{11}$$

Proof The proof is omitted here since it is similar to Lemma 2 in Ref. [42].

Remark 1 It should be pointed out that Lemma 2 in Ref. [42] is a special case of Lemma 3 with $\boldsymbol{\xi} = \left[\boldsymbol{x}^{\mathrm{T}}(\beta) \;\; \boldsymbol{x}^{\mathrm{T}}(\alpha) \;\; \int_{\alpha}^{\beta}\boldsymbol{x}^{\mathrm{T}}(s)\mathrm{d}s\right]^{\mathrm{T}}$. By introducing more information about state vectors in $\boldsymbol{\xi}$, less conservative results may be obtained.

3. Main results

In this section, the problem of synchronization of the master system (1) and slave system (4) will be investigated via a discontinuous Lyapunov functional approach. To present the main results, we first denote

$$\begin{aligned}
&\boldsymbol{\xi}_1(t) = \left[\boldsymbol{r}^{\mathrm{T}}(t) \;\; \boldsymbol{r}^{\mathrm{T}}(t-\eta) \;\; \int_{t-\eta}^{t}\boldsymbol{r}^{\mathrm{T}}(s)\mathrm{d}s\right]^{\mathrm{T}}, \\
&\boldsymbol{\xi}_2(t) = \left[\boldsymbol{r}^{\mathrm{T}}(t) \;\; \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{r}(t))\right]^{\mathrm{T}}, \\
&\boldsymbol{\xi}_3(t) = \left[\boldsymbol{r}^{\mathrm{T}}(t) \;\; \dot{\boldsymbol{r}}^{\mathrm{T}}(t)\right]^{\mathrm{T}}, \\
&\boldsymbol{\xi}_4(t) = \left[\boldsymbol{r}^{\mathrm{T}}(t-\eta) - \boldsymbol{r}^{\mathrm{T}}(t_k-\eta) \;\; \int_{t_k-\eta}^{t-\eta}\boldsymbol{r}^{\mathrm{T}}(s)\mathrm{d}s\right]^{\mathrm{T}}, \\
&\boldsymbol{\xi}_5(t) = \left[\boldsymbol{\xi}_1^{\mathrm{T}}(t) \;\; \boldsymbol{r}^{\mathrm{T}}(t_k-\eta) \;\; \int_{t_k-\eta}^{t-\eta}\boldsymbol{r}^{\mathrm{T}}(s)\mathrm{d}s\right]^{\mathrm{T}}, \\
&\boldsymbol{\chi}_1(t) = \left[\boldsymbol{r}^{\mathrm{T}}(t) \;\; \boldsymbol{r}^{\mathrm{T}}(t-d(t)) \;\; \boldsymbol{r}^{\mathrm{T}}(t-\bar{d}) \;\; \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{r}(t))\right]^{\mathrm{T}}, \\
&\boldsymbol{\chi}_2(t) = \left[\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{r}(t-d(t))) \;\; \boldsymbol{r}^{\mathrm{T}}(t-\eta) \;\; \frac{1}{\eta}\int_{t-\eta}^{t}\boldsymbol{r}^{\mathrm{T}}(s)\mathrm{d}s\right]^{\mathrm{T}}, \\
&\boldsymbol{\chi}_3(t) = \left[\dot{\boldsymbol{r}}^{\mathrm{T}}(t) \;\; \dot{\boldsymbol{r}}^{\mathrm{T}}(t-\eta) \;\; \boldsymbol{r}^{\mathrm{T}}(t_k-\eta) \;\; \int_{t_k-\eta}^{t-\eta}\boldsymbol{r}^{\mathrm{T}}(s)\mathrm{d}s\right]^{\mathrm{T}}, \\
&\boldsymbol{\chi}(t) = \left[\boldsymbol{\chi}_1^{\mathrm{T}}(t) \;\; \boldsymbol{\chi}_2^{\mathrm{T}}(t) \;\; \boldsymbol{\chi}_3^{\mathrm{T}}(t)\right]^{\mathrm{T}}, \\
&\boldsymbol{e}_i = \left[0_{n\times(i-1)n} \;\; \boldsymbol{I}_n \;\; 0_{n\times(11-i)n}\right], \;\; i = 1,2,\ldots,11, \\
&\boldsymbol{K}_1 = \mathrm{diag}\left\{F_1^+, F_2^+, \ldots, F_n^+\right\}, \\
&\boldsymbol{K}_2 = \mathrm{diag}\left\{F_1^-, F_2^-, \ldots, F_n^-\right\}.
\end{aligned}$$

Now, we present the following main results.

Theorem 1 Given scalars γ_1, γ_2, if there exist $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R}_1 > 0$, $\boldsymbol{R}_2 > 0$, $\boldsymbol{Z}_1 > 0$, $\boldsymbol{Z}_2 > 0$, $\boldsymbol{Z}_3 > 0$, $\boldsymbol{Z}_4 > 0$, $\boldsymbol{S}$, $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$, $\boldsymbol{Y}_3$, $\boldsymbol{N}_1$, $\boldsymbol{N}_2$, $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, $\boldsymbol{H}_3$, $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{G}$, $\boldsymbol{L}$, and diagonal matrices $\boldsymbol{\Lambda}_1 > 0$, $\boldsymbol{\Lambda}_2 > 0$ such that

$$\boldsymbol{\Xi}_1(h_k) = \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2 + h_k\boldsymbol{\Phi}_3 + \boldsymbol{\Phi}_5 + \boldsymbol{\Phi}_6 < 0, \tag{12}$$

$$\boldsymbol{\Xi}_2(h_k) = \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2 + h_k(\boldsymbol{\Phi}_4 + \boldsymbol{\Phi}_7) + \boldsymbol{\Phi}_5 + \boldsymbol{\Phi}_6 < 0, \tag{13}$$

$$\boldsymbol{U}_1 = \begin{bmatrix} \boldsymbol{Z}_1 & \boldsymbol{S} \\ * & \boldsymbol{Z}_1 \end{bmatrix} \geq 0, \tag{14}$$

$$\boldsymbol{U}_2 = \begin{bmatrix} \boldsymbol{Y}_1 & \boldsymbol{Y}_2 & \boldsymbol{N}_1 \\ * & \boldsymbol{Y}_3 & \boldsymbol{N}_2 \\ * & * & \boldsymbol{Z}_3 \end{bmatrix} \geq 0, \tag{15}$$

$$\boldsymbol{U}_3 = \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2 & \boldsymbol{M}_1 \\ * & \boldsymbol{H}_3 & \boldsymbol{M}_2 \\ * & * & \boldsymbol{Z}_4 \end{bmatrix} \geq 0 \tag{16}$$

are feasible for $h_k \in [0, \bar{h}]$, where

$$\begin{aligned}
\boldsymbol{\Phi}_1 &= \mathrm{Sym}\{\boldsymbol{\Pi}_1^{\mathrm{T}}P\boldsymbol{\Pi}_2\} + \boldsymbol{\Pi}_3^{\mathrm{T}}Q\boldsymbol{\Pi}_3 - (1-\mu)\boldsymbol{\Pi}_4^{\mathrm{T}}Q\boldsymbol{\Pi}_4 \\
&\quad + \boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{e}_1 - \boldsymbol{e}_3^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{e}_3 + \bar{d}^2\boldsymbol{e}_8^{\mathrm{T}}\boldsymbol{Z}_1\boldsymbol{e}_8 - \boldsymbol{\Pi}_5^{\mathrm{T}}\boldsymbol{U}_1\boldsymbol{\Pi}_5 \\
&\quad + \boldsymbol{\Pi}_6^{\mathrm{T}}\boldsymbol{R}_2\boldsymbol{\Pi}_6 - \boldsymbol{\Pi}_7^{\mathrm{T}}\boldsymbol{R}_2\boldsymbol{\Pi}_7 + \eta^2\boldsymbol{e}_8^{\mathrm{T}}\boldsymbol{Z}_2\boldsymbol{e}_8 - \boldsymbol{\Pi}_8^{\mathrm{T}}\bar{Z}\boldsymbol{\Pi}_8, \\
\boldsymbol{\Phi}_2 &= \mathrm{Sym}\{\boldsymbol{N}_1\boldsymbol{\Pi}_{10} - 2\boldsymbol{N}_2\boldsymbol{e}_{11} - \boldsymbol{\Pi}_9^{\mathrm{T}}\boldsymbol{X}_2\boldsymbol{e}_{10}\} - \boldsymbol{\Pi}_9^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{\Pi}_9, \\
\boldsymbol{\Phi}_3 &= \mathrm{Sym}\{\boldsymbol{\Pi}_9^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{\Pi}_{12} + \boldsymbol{\Pi}_{12}^{\mathrm{T}}\boldsymbol{X}_2\boldsymbol{e}_{10}\} + \boldsymbol{e}_9^{\mathrm{T}}\boldsymbol{Z}_3\boldsymbol{e}_9, \\
\boldsymbol{\Phi}_4 &= \mathrm{Sym}\{\boldsymbol{N}_2\boldsymbol{\Pi}_{11}\} + \boldsymbol{Y}_1 + \frac{h^2}{3}\boldsymbol{Y}_3, \\
\boldsymbol{\Phi}_5 &= \mathrm{Sym}\{\boldsymbol{\Pi}_{13}^{\mathrm{T}}\boldsymbol{\Pi}_{14} + \boldsymbol{\Pi}_{15}^{\mathrm{T}}\boldsymbol{\Lambda}_1\boldsymbol{\Pi}_{16} + \boldsymbol{\Pi}_{17}^{\mathrm{T}}\boldsymbol{\Lambda}_2\boldsymbol{\Pi}_{18}\}, \\
\boldsymbol{\Phi}_6 &= \boldsymbol{\Pi}_{19}(\boldsymbol{H}_1 + \frac{\bar{h}^2}{3}\boldsymbol{H}_3 + \bar{M}_1)\boldsymbol{\Pi}_{19} + \boldsymbol{e}_9^{\mathrm{T}}\boldsymbol{Z}_4\boldsymbol{e}_9
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{Sym}\{\boldsymbol{\Pi}_{19}^{\mathrm{T}}\bar{M}_2\boldsymbol{\Pi}_{20}\},\\
\boldsymbol{\Phi}_7 &= \mathrm{Sym}\{\boldsymbol{\Pi}_{19}(\boldsymbol{H}_1+\frac{\bar{h}^2}{3}\boldsymbol{H}_3+\bar{M}_1)\boldsymbol{\Pi}_{20}\},\\
\boldsymbol{\Pi}_1 &= \left[\boldsymbol{e}_1^{\mathrm{T}}\ \ \boldsymbol{e}_6^{\mathrm{T}}\ \ \eta\boldsymbol{e}_7^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_2 &= \left[\boldsymbol{e}_8^{\mathrm{T}}\ \ \boldsymbol{e}_9^{\mathrm{T}}\ \ \boldsymbol{e}_1^{\mathrm{T}}-\boldsymbol{e}_6^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_3 &= \left[\boldsymbol{e}_1^{\mathrm{T}}\ \ \boldsymbol{e}_4^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_4 &= \left[\boldsymbol{e}_2^{\mathrm{T}}\ \ \boldsymbol{e}_5^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_5 &= \left[\boldsymbol{e}_1^{\mathrm{T}}-\boldsymbol{e}_2^{\mathrm{T}}\ \ \boldsymbol{e}_2^{\mathrm{T}}-\boldsymbol{e}_3^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_6 &= \left[\boldsymbol{e}_1^{\mathrm{T}}\ \ \boldsymbol{e}_8^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_7 &= \left[\boldsymbol{e}_6^{\mathrm{T}}\ \ \boldsymbol{e}_9^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_8 &= \left[\boldsymbol{e}_1^{\mathrm{T}}-\boldsymbol{e}_6^{\mathrm{T}}\ \ \boldsymbol{e}_1^{\mathrm{T}}+\boldsymbol{e}_6^{\mathrm{T}}-2\boldsymbol{e}_7^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_9 &= \left[\boldsymbol{e}_6^{\mathrm{T}}-\boldsymbol{e}_{10}^{\mathrm{T}}\ \ \boldsymbol{e}_{11}^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{10} &= \left[\boldsymbol{e}_6^{\mathrm{T}}-\boldsymbol{e}_{10}^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{11} &= \left[\boldsymbol{e}_6^{\mathrm{T}}+\boldsymbol{e}_{10}^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{12} &= \left[\boldsymbol{e}_9^{\mathrm{T}}\ \ \boldsymbol{e}_6^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{13} &= \left[\ \boldsymbol{e}_1^{\mathrm{T}}+\gamma_1\boldsymbol{e}_8^{\mathrm{T}}+\gamma_2\boldsymbol{e}_{10}^{\mathrm{T}}\ \right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{14} &= \left[\ -\boldsymbol{G}\boldsymbol{e}_8-\boldsymbol{G}\boldsymbol{C}\boldsymbol{e}_1+\boldsymbol{G}\boldsymbol{A}\boldsymbol{e}_4+\boldsymbol{G}\boldsymbol{B}\boldsymbol{e}_5+\boldsymbol{L}\boldsymbol{e}_{10}\ \right],\\
\boldsymbol{\Pi}_{15} &= \left[\boldsymbol{e}_4^{\mathrm{T}}-\boldsymbol{K}_2\boldsymbol{e}_1^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{16} &= \left[\boldsymbol{K}_1\boldsymbol{e}_1^{\mathrm{T}}-\boldsymbol{e}_4^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{17} &= \left[\boldsymbol{e}_5^{\mathrm{T}}-\boldsymbol{K}_2\boldsymbol{e}_2^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{18} &= \left[\boldsymbol{K}_1\boldsymbol{e}_2^{\mathrm{T}}-\boldsymbol{e}_5^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{19} &= \left[\boldsymbol{e}_1^{\mathrm{T}}\ \ \boldsymbol{e}_6^{\mathrm{T}}\ \ \eta\boldsymbol{e}_7^{\mathrm{T}}\ \ \boldsymbol{e}_{10}^{\mathrm{T}}\ \ \boldsymbol{e}_{11}^{\mathrm{T}}\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_{20} &= \left[\boldsymbol{e}_8^{\mathrm{T}}\ \ \boldsymbol{e}_9^{\mathrm{T}}\ \ \boldsymbol{e}_1^{\mathrm{T}}-\boldsymbol{e}_6^{\mathrm{T}}\ \ 0\ \ \boldsymbol{e}_6^{\mathrm{T}}\right]^{\mathrm{T}},\\
\bar{Z} &= \mathrm{diag}\{\boldsymbol{Z}_2,3\boldsymbol{Z}_2\}\\
\bar{M}_1 &= \mathrm{Sym}\left\{\left[0\ \ \boldsymbol{M}_2\ \ 0\ \ \boldsymbol{M}_2\ \ 0\right]\right\},\\
\bar{M}_2 &= \mathrm{Sym}\left\{\left[0\ \ \boldsymbol{M}_1\ \ 0\ \ -\boldsymbol{M}_1\ \ -2\boldsymbol{M}_2\right]\right\}.
\end{aligned}
$$

Then, the master system (1) and slave system (4) are synchronous. Moreover, the desired controller gain matrix in Eq. (8) is given by $\boldsymbol{K}=\boldsymbol{G}^{-1}\boldsymbol{L}$.

Proof Consider the following Lyapunov functional for the error system (1):

$$\boldsymbol{V}(t)=\sum_{i=1}^{5}\boldsymbol{V}_i(t),\ t\in[t_k,t_{k+1}), \tag{17}$$

where

$$
\begin{aligned}
\boldsymbol{V}_1(t) &= \boldsymbol{\xi}_1^{\mathrm{T}}(t)P\boldsymbol{\xi}_1(t),\\
\boldsymbol{V}_2(t) &= \int_{t-d(t)}^{t}\boldsymbol{\xi}_2^{\mathrm{T}}(s)\boldsymbol{Q}\boldsymbol{\xi}_2(s)\,\mathrm{d}s+\int_{t-\bar{d}}^{t}\boldsymbol{r}^{\mathrm{T}}(s)\boldsymbol{R}_1\boldsymbol{r}(s)\,\mathrm{d}s\\
&\quad+\bar{d}\int_{-\bar{d}}^{0}\int_{t+\theta}^{t}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_1\dot{\boldsymbol{r}}(s)\mathrm{d}s\,\mathrm{d}\theta,\\
\boldsymbol{V}_3(t) &= \int_{t-\eta}^{t}\boldsymbol{\xi}_3^{\mathrm{T}}(s)\boldsymbol{R}_2\boldsymbol{\xi}_3(s)\,\mathrm{d}s+\eta\int_{-\eta}^{0}\int_{t+\theta}^{t}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_2\dot{\boldsymbol{r}}(s)\,\mathrm{d}s\,\mathrm{d}\theta,\\
\boldsymbol{V}_4(t) &= (t_{k+1}-t)\boldsymbol{\xi}_4^{\mathrm{T}}(t)\left[\boldsymbol{X}_1\boldsymbol{\xi}_4(t)+2\boldsymbol{X}_2\boldsymbol{r}(t_k-\eta)\right]\\
&\quad+(t_{k+1}-t)\int_{t_k-\eta}^{t-\eta}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_3\dot{\boldsymbol{r}}(s)\mathrm{d}s,\\
\boldsymbol{V}_5(t) &= (t-t_k)[\boldsymbol{\xi}_5^{\mathrm{T}}(t)(\boldsymbol{H}_1+\frac{\bar{h}^2}{3}\boldsymbol{H}_3)\boldsymbol{\xi}_5(t)\\
&\quad+2\boldsymbol{\xi}_5^{\mathrm{T}}(t)\boldsymbol{M}_2(\boldsymbol{r}(t-\eta)+\boldsymbol{r}(t_k-\eta))]\\
&\quad+2\boldsymbol{\xi}_5^{\mathrm{T}}(t)\Big[\boldsymbol{M}_1(\boldsymbol{r}(t-\eta)-\boldsymbol{r}(t_k-\eta))\\
&\quad-2\boldsymbol{M}_2\int_{t_k-\eta}^{t-\eta}\boldsymbol{r}(s)\mathrm{d}s\Big]+\int_{t_k-\eta}^{t-\eta}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_4\dot{\boldsymbol{r}}(s)\mathrm{d}s.
\end{aligned}
$$

According to Lemma 3, we get from Eq. (16) that $\boldsymbol{V}_5(t)\geq 0$. Moreover, it can be obtained that $\lim_{t\to t_k^-}\boldsymbol{V}_5(t)\geq\boldsymbol{V}_5(t_k)=0$. Taking the derivative of Eq. (17 along the solution of system (9), we get

$$
\begin{aligned}
\dot{\boldsymbol{V}}_1(t) &= 2\boldsymbol{\xi}_1^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{\xi}}_1(t)=2\boldsymbol{\chi}^{\mathrm{T}}(t)\boldsymbol{\Pi}_1^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Pi}_2\boldsymbol{\chi}(t), \qquad (18)\\
\dot{\boldsymbol{V}}_2(t) &= \boldsymbol{\xi}_2^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\xi}_2^{\mathrm{T}}(t)-(1-\dot{d}(t))\boldsymbol{\xi}_2^{\mathrm{T}}(t-d(t))\boldsymbol{Q}\boldsymbol{\xi}_2^{\mathrm{T}}(t-d(t))\\
&\quad+\boldsymbol{r}^{\mathrm{T}}(t)\boldsymbol{R}_1\boldsymbol{r}(t)-\boldsymbol{r}^{\mathrm{T}}(t-\bar{d})\boldsymbol{R}_1\boldsymbol{r}(t-\bar{d})\\
&\quad+\bar{d}^2\dot{\boldsymbol{r}}^{\mathrm{T}}(t)\boldsymbol{Z}_1\dot{\boldsymbol{r}}(t)-\bar{d}\int_{t-\bar{d}}^{t}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_1\dot{\boldsymbol{r}}(s)\mathrm{d}s\\
&\leq \boldsymbol{\chi}^{\mathrm{T}}(t)[\boldsymbol{\Pi}_3^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Pi}_3-(1-\mu)\boldsymbol{\Pi}_4^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Pi}_4+\boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{e}_1\\
&\quad-\boldsymbol{e}_3^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{e}_3+\bar{d}^2\boldsymbol{e}_8^{\mathrm{T}}\boldsymbol{Z}_1\boldsymbol{e}_8]\boldsymbol{\chi}(t)\\
&\quad-\bar{d}\int_{t-\bar{d}}^{t}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_1\dot{\boldsymbol{r}}(s)\mathrm{d}s, \qquad (19)\\
\dot{\boldsymbol{V}}_3(t) &= \boldsymbol{\xi}_3^{\mathrm{T}}(t)R_2\xi_3(\mathrm{t})-\boldsymbol{\xi}_3^{\mathrm{T}}(\mathrm{t}-\eta)\mathrm{R}_2\boldsymbol{\xi}_3(\mathrm{t}-\eta)\\
&\quad+\eta^2\dot{\boldsymbol{r}}^{T}(t)\boldsymbol{Z}_2\dot{\boldsymbol{r}}(t)-\eta\int_{t-\eta}^{t}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_2\dot{\boldsymbol{r}}(s)\mathrm{d}s\\
&= \boldsymbol{\chi}^{\mathrm{T}}(t)[\boldsymbol{\Pi}_6^{\mathrm{T}}\boldsymbol{R}_2\boldsymbol{\Pi}_6-\boldsymbol{\Pi}_7^{\mathrm{T}}\boldsymbol{R}_2\boldsymbol{\Pi}_7+\eta^2\boldsymbol{e}_8^{\mathrm{T}}\boldsymbol{Z}_2\boldsymbol{e}_8]\boldsymbol{\chi}(t)\\
&\quad-\eta\int_{t-\eta}^{t}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_2\dot{\boldsymbol{r}}(s)\mathrm{d}s, \qquad (20)\\
\dot{\boldsymbol{V}}_4(t) &= -\left[\boldsymbol{\xi}_4^{\mathrm{T}}(t)(\boldsymbol{X}_1\boldsymbol{\xi}_4(t)+2\boldsymbol{X}_2\boldsymbol{r}(t_k-\eta))\right]\\
&\quad+(t_{k+1}-t)[2\boldsymbol{\xi}_4^{\mathrm{T}}(t)\boldsymbol{X}_1\dot{\boldsymbol{\xi}}_4(t)+2\dot{\boldsymbol{\xi}}_4^{\mathrm{T}}(t)\boldsymbol{X}_2\boldsymbol{r}(t_k-\eta))\\
&\quad+\dot{\boldsymbol{r}}^{\mathrm{T}}(t-\eta)Z_3\dot{r}(t-\eta)]-\int_{t_k-\eta}^{t-\eta}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_3\dot{\boldsymbol{r}}(s)\mathrm{d}s\\
&\leq \boldsymbol{\chi}^{\mathrm{T}}(t)\left[-\boldsymbol{\Pi}_9^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{\Pi}_9-2\boldsymbol{\Pi}_9^{\mathrm{T}}\boldsymbol{X}_2\boldsymbol{e}_{10}+(t_{k+1}-t)\right.\\
&\quad\left.\times\ (2\boldsymbol{\Pi}_9^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{\Pi}_{12}+2\boldsymbol{\Pi}_{12}^{\mathrm{T}}\boldsymbol{X}_2\boldsymbol{e}_{10}+\boldsymbol{e}_9^{\mathrm{T}}\boldsymbol{Z}_3\boldsymbol{e}_9)\right]\boldsymbol{\chi}(t)\\
&\quad-\int_{t_k-\eta}^{t-\eta}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_3\dot{\boldsymbol{r}}(s)\mathrm{d}s, \qquad (21)\\
\dot{\boldsymbol{V}}_5(t) &= \boldsymbol{\xi}_5^{\mathrm{T}}(t)\Big[\boldsymbol{H}_1+\frac{\bar{h}^2}{3}\boldsymbol{H}_3+\bar{M}_1]\xi_5(t)\\
&\quad+2(t-t_k)\boldsymbol{\xi}_5^{\mathrm{T}}(t)\left[\left(\boldsymbol{H}_1+\frac{\bar{h}^2}{3}\boldsymbol{H}_3+\bar{\boldsymbol{M}}_1\right]\dot{\boldsymbol{\xi}}_5(t)\right.\\
&\quad+2\boldsymbol{\xi}_5^{\mathrm{T}}(t)\bar{\boldsymbol{M}}_2\dot{\boldsymbol{\xi}}_5(t)+\dot{\boldsymbol{r}}^{\mathrm{T}}(t-\eta)\boldsymbol{Z}_4\dot{\boldsymbol{r}}(t-\eta)\\
&= \boldsymbol{\chi}^{\mathrm{T}}(t)\left[\boldsymbol{\Pi}_{19}^{\mathrm{T}}\left(\boldsymbol{H}_1+\frac{\bar{h}^2}{3}\boldsymbol{H}_3+\bar{M}_1\right)\boldsymbol{\Pi}_{19}\right.\\
&\quad+2\boldsymbol{\Pi}_{19}^{\mathrm{T}}\bar{M}_2\boldsymbol{\Pi}_{20}+\boldsymbol{e}_9^{\mathrm{T}}\boldsymbol{Z}_4\boldsymbol{e}_9\boldsymbol{\chi}(t)\\
&\quad\left.+2(t-t_k)\left(\boldsymbol{\Pi}_{19}^{\mathrm{T}}\left(\boldsymbol{H}_1+\frac{\bar{h}^2}{3}\boldsymbol{H}_3+\bar{M}_1\right)\boldsymbol{\Pi}_{20}\right)\right]\boldsymbol{\chi}(t). \qquad (22)
\end{aligned}
$$

For $\boldsymbol{U}_1=\begin{bmatrix}\boldsymbol{Z}_1 & \boldsymbol{S}\\ * & \boldsymbol{Z}_1\end{bmatrix}\geq 0$, it is followed from Lemmas 1

that

$$-\bar{d}\int_{t-\bar{d}}^{t}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_1\dot{\boldsymbol{r}}(s)\mathrm{d}s \le -\boldsymbol{\chi}^{\mathrm{T}}(t)\boldsymbol{\Pi}_5^{\mathrm{T}}\boldsymbol{U}_1\boldsymbol{\Pi}_5\boldsymbol{\chi}(t). \quad (23)$$

Replacing Eq. (23) into Eq. (19) yields

$$\begin{aligned}\dot{\boldsymbol{V}}_2(t) \le\ & \boldsymbol{\chi}^{\mathrm{T}}(t)[\boldsymbol{\Pi}_3^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Pi}_3-(1-\mu)\boldsymbol{\Pi}_4^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Pi}_4+\boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{e}_1\\ & -\boldsymbol{e}_3^{\mathrm{T}}\boldsymbol{R}_1\boldsymbol{e}_3+\bar{d}^2\boldsymbol{e}_8^{\mathrm{T}}\boldsymbol{Z}_1\boldsymbol{e}_8-\boldsymbol{\Pi}_5^{\mathrm{T}}\boldsymbol{U}_1\boldsymbol{\Pi}_5]\boldsymbol{\chi}(t). \quad (24)\end{aligned}$$

Similarly, it follows from Lemmas 2 and 3, respectively, for $\bar{\boldsymbol{Z}}=\mathrm{diag}\{\boldsymbol{Z}_2,3\boldsymbol{Z}_2\}$ and

$$\boldsymbol{U}_2=\begin{bmatrix}\boldsymbol{Y}_1 & \boldsymbol{Y}_2 & \boldsymbol{N}_1\\ * & \boldsymbol{Y}_3 & \boldsymbol{N}_2\\ * & * & \boldsymbol{Z}_3\end{bmatrix}\ge 0$$

that

$$-\eta\int_{t-\eta}^{t}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_2\dot{\boldsymbol{r}}(s)\mathrm{d}s \le -\boldsymbol{\chi}^{\mathrm{T}}(t)\boldsymbol{\Pi}_8^{\mathrm{T}}\bar{\boldsymbol{Z}}\boldsymbol{\Pi}_8\boldsymbol{\chi}(t) \quad (25)$$

and

$$\begin{aligned}&-\int_{t_k-\eta}^{t-\eta}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_3\dot{\boldsymbol{r}}(s)\mathrm{d}s\\ &\le -\boldsymbol{\chi}^{\mathrm{T}}(t)[2\boldsymbol{N}_1\boldsymbol{\Pi}_{10}-4\boldsymbol{N}_2\boldsymbol{e}_{11}\\ &\quad +(t-t_k)(\boldsymbol{Y}_1+\frac{\bar{h}^2}{3}\boldsymbol{Y}_3+2\boldsymbol{N}_2\boldsymbol{\Pi}_{11})]\boldsymbol{\chi}(t). \quad (26)\end{aligned}$$

Replacing Eq. (25) into Eq. (20), and Eq. (26) into Eq. (22), we get

$$\begin{aligned}\dot{\boldsymbol{V}}_3(t) \le\ & \boldsymbol{\chi}^{\mathrm{T}}(t)[\boldsymbol{\Pi}_6^{\mathrm{T}}\boldsymbol{R}_2\boldsymbol{\Pi}_6-\boldsymbol{\Pi}_7^{\mathrm{T}}\boldsymbol{R}_2\boldsymbol{\Pi}_7+\eta^2\boldsymbol{e}_8^{\mathrm{T}}\boldsymbol{Z}_2\boldsymbol{e}_8\\ & -\boldsymbol{\Pi}_8^{\mathrm{T}}\bar{\boldsymbol{Z}}\boldsymbol{\Pi}_8]\boldsymbol{\chi}(t), \quad (27)\end{aligned}$$

and

$$\begin{aligned}\dot{\boldsymbol{V}}_4(t) \le\ & \boldsymbol{\chi}^{\mathrm{T}}(t)\big[-\boldsymbol{\Pi}_9^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{\Pi}_9-2\boldsymbol{\Pi}_9^{\mathrm{T}}\boldsymbol{X}_2\boldsymbol{e}_{10}+2\boldsymbol{N}_1\boldsymbol{\Pi}_{10}\\ & +(t-t_k)(\boldsymbol{Y}_1+\frac{\bar{h}^2}{3}\boldsymbol{Y}_3+2\boldsymbol{N}_2\boldsymbol{\Pi}_{11})-4\boldsymbol{N}_2\boldsymbol{e}_{11}(t_{k+1}-t)\\ & \times(2\boldsymbol{\Pi}_9^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{\Pi}_{12}+2\boldsymbol{\Pi}_{12}^{\mathrm{T}}\boldsymbol{X}_2\boldsymbol{e}_{10}+\boldsymbol{e}_9^{\mathrm{T}}\boldsymbol{Z}_3\boldsymbol{e}_9)\big]\boldsymbol{\chi}(t). \quad (28)\end{aligned}$$

From the error system (9), for scalars γ_1,γ_2, and any matrix $\boldsymbol{G}$ with appropriate dimension, the following equation holds:

$$\begin{aligned}0 =\ & 2\left[\boldsymbol{r}^{\mathrm{T}}(t)\boldsymbol{G}+\gamma_1\dot{\boldsymbol{r}}^{\mathrm{T}}(t)\boldsymbol{G}+\gamma_2\boldsymbol{r}^{\mathrm{T}}(t_k-\eta)\boldsymbol{G}\right]\\ & \times[-\dot{\boldsymbol{r}}(t)-\boldsymbol{C}\boldsymbol{r}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{r}(t))\\ & +\boldsymbol{B}\boldsymbol{f}(\boldsymbol{r}(t-d(t)))+\boldsymbol{K}\boldsymbol{r}(t_k-\eta)]\\ =\ & 2\boldsymbol{\chi}^{\mathrm{T}}(t)\boldsymbol{\Pi}_{13}^{\mathrm{T}}\boldsymbol{\Pi}_{14}\boldsymbol{\chi}(t), \quad (29)\end{aligned}$$

where $L=GK$.

On the other hand, we get from Eq. (6) that for any $\lambda_{1i}>0, i=1,2,\ldots,n$,

$$\begin{aligned}0 \le\ & \sum_{i=1}^{n}\lambda_{1i}[f_i(r_i(t))-F_i^{-}r_i(t)][F_i^{+}r_i(t)-f_i(r_i(t))]\\ =\ & 2\boldsymbol{\chi}^{\mathrm{T}}(t)\boldsymbol{\Pi}_{15}^{\mathrm{T}}\boldsymbol{\Lambda}_1\boldsymbol{\Pi}_{16}\boldsymbol{\chi}(t), \quad (30)\end{aligned}$$

where $\boldsymbol{\Lambda}_1=\mathrm{diag}\{\lambda_{11},\lambda_{12},\ldots,\lambda_{1n}\}$.

Similarly, for any diagonal matrix $\boldsymbol{\Lambda}_2=\mathrm{diag}\{\lambda_{21},\lambda_{22},\ldots,\lambda_{2n}\}>0$, the following inequality also holds:

$$\begin{aligned}0 \le\ & \sum_{i=1}^{n}\{\lambda_{2i}[f_i(r_i(t-d(t)))-F_i^{-}r_i(t-d(t))]\\ & \times[\boldsymbol{F}_i^{+}\boldsymbol{r}_i(t-d(t))-f_i(r_i(t-d(t)))]\}\\ =\ & 2\boldsymbol{\chi}^{\mathrm{T}}(t)\boldsymbol{\Pi}_{17}^{\mathrm{T}}\boldsymbol{\Lambda}_2\boldsymbol{\Pi}_{18}\boldsymbol{\chi}(t). \quad (31)\end{aligned}$$

Then, adding the right sides of Eqs. (29)–(31) to $\dot{\boldsymbol{V}}(t)$, we obtain

$$\begin{aligned}\dot{\boldsymbol{V}}(t) \le\ & \boldsymbol{\chi}^{\mathrm{T}}(t)[\boldsymbol{\Phi}_1+\boldsymbol{\Phi}_2+(t_{k+1}-t)\boldsymbol{\Phi}_3\\ & +(t-t_k)(\boldsymbol{\Phi}_4+\boldsymbol{\Phi}_7)+\boldsymbol{\Phi}_5+\boldsymbol{\Phi}_6]\boldsymbol{\chi}(t). \quad (32)\end{aligned}$$

Thus, it follows from Eqs. (12) and (13) that

$$\dot{V}(t)<-\varepsilon||\boldsymbol{r}(t)||^2 \quad (33)$$

for a sufficient small scalar $\varepsilon>0$, which implies that system (9) is stable. Therefore, the master system (1) and slave system (4) are synchronous. This completes the proof.

Remark 2 Inspired by the work reported in Ref. [36], a time-dependent term is introduced in Lyapunov functional (17). Note that

$$\lim_{t\to t_k^{+}}\boldsymbol{V}_4(t)=\lim_{t\to t_k^{-}}\boldsymbol{V}_4(t)=\boldsymbol{V}_4(t_k)=0. \quad (34)$$

Thus, $\boldsymbol{V}_4(t)$ is continuous in time. Since this information on the actual sampling pattern is used in the Lyapunov functional (17), the results obtained are less conservative.

Remark 3 Recently, a free-matrix-based time-dependent discontinuous Lyapunov functional was proposed in Ref. [42]. Inspired by Ref. [42], a new term, $\boldsymbol{V}_5(t)$, is introduced in the Lyapunov functional (17). It follows from Lemma 3 that

$$\begin{aligned}&\boldsymbol{V}_5(t)\\ &\ge (t-t_k)\boldsymbol{\xi}_5^{\mathrm{T}}(t)\Big[(\boldsymbol{H}_1+\frac{(t-t_k)^2}{3}\boldsymbol{H}_3)\boldsymbol{\xi}_5(t)\\ &\quad +2\boldsymbol{M}_2(\boldsymbol{r}(t-\eta)+\boldsymbol{r}(t_k-\eta))]+\int_{t_k-\eta}^{t-\eta}\dot{\boldsymbol{r}}^{\mathrm{T}}(s)\boldsymbol{Z}_4\dot{\boldsymbol{r}}(s)\mathrm{d}s\\ &\quad +2\boldsymbol{\xi}_5^{\mathrm{T}}(t)\left[\boldsymbol{M}_1(\boldsymbol{r}(t-\eta)-\boldsymbol{r}(t_k-\eta))-2\boldsymbol{M}_2\int_{t_k-\eta}^{t-\eta}\boldsymbol{r}(s)\mathrm{d}s\right]\\ &=\int_{t_k-\eta}^{t-\eta}\boldsymbol{\xi}_6^{\mathrm{T}}(s)\boldsymbol{U}_3\boldsymbol{\xi}_6(s)\mathrm{d}s\ge 0,\end{aligned}$$

where $\boldsymbol{\xi}_6(s)=\left[\boldsymbol{\xi}_5^{\mathrm{T}}(t)\ f(s)\boldsymbol{\xi}_5^{\mathrm{T}}(t)\ \dot{\boldsymbol{r}}^{\mathrm{T}}(s)\right]^{\mathrm{T}}$ and $f(s)=2s-\beta-\alpha$. Thus, we have

$$\lim_{t\to t_k^{-}}\boldsymbol{V}_5(t)\ge\boldsymbol{V}_5(t_k)=0, \quad (35)$$

which takes advantage of the sampling characteristic of sawtooth input delay and is helpful to reduce the conservativeness of the derived results.

To show the advantage of discontinuous Lyapunov functional (17), we present a synchronization condition based on continuous functional $\bar{V}(t) = \sum_{i=1}^{4} \boldsymbol{V}_i(t)$. By setting $\boldsymbol{H}_1 = 0$, $\boldsymbol{H}_2 = 0$, $\boldsymbol{H}_3 = 0$, $\boldsymbol{M}_1 = 0$, $\boldsymbol{M}_2 = 0$, $\boldsymbol{Z}_4 = 0$ in Theorem 1, the following corollary is obtained.

Corollary 1 Given scalars γ_1, γ_2, if there exist $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R}_1 > 0$, $\boldsymbol{R}_2 > 0$, $\boldsymbol{Z}_1 > 0$, $\boldsymbol{Z}_2 > 0$, $\boldsymbol{Z}_3 > 0$, $\boldsymbol{S}$, $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$, $\boldsymbol{Y}_3$, $\boldsymbol{N}_1$, $\boldsymbol{N}_2$, $\boldsymbol{G}$, $\boldsymbol{L}$, and diagonal matrices $\boldsymbol{\Lambda}_1 > 0$, $\boldsymbol{\Lambda}_2 > 0$ such that Eqs. (14), (15), and

$$\bar{\boldsymbol{\Xi}}_1(h_k) = \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2 + h_k\boldsymbol{\Phi}_3 + \boldsymbol{\Phi}_5 < 0, \tag{36}$$

$$\bar{\boldsymbol{\Xi}}_2(h_k) = \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2 + h_k\boldsymbol{\Phi}_4 + \boldsymbol{\Phi}_5 < 0 \tag{37}$$

are feasible for $h_k \in [0, \bar{h}]$, where $\boldsymbol{\Phi}_i$, $i = 1, 2, \ldots, 5$, are defined in Theorem 1. Then, the master system (1) and slave system (4) are synchronous. Moreover, the desired controller gain matrix in Eq. (8) is given by $\boldsymbol{K} = \boldsymbol{G}^{-1}\boldsymbol{L}$.

4. Numerical examples

In this section, we provide two numerical examples to verify the effectiveness of the proposed results.

Example 1 Consider the master system (1) and slave system (4) with

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 1.8 & -0.15 \\ -5.2 & 1.5 \end{bmatrix},$$
$$\boldsymbol{B} = \begin{bmatrix} -1.7 & -0.12 \\ -0.26 & -2.5 \end{bmatrix}.$$

The time-varying delay and the activation functions are taken as

$$d(t) = \frac{e^t}{e^t + 1},$$
$$g_i(\alpha) = \tanh(\alpha), \quad i = 1, 2.$$

It can be verified that the time-varying delay satisfies Eq. (2) with $\bar{d} = 1$ and $\mu = 0.25$ and the activation functions satisfy Eq. (4) with $F_1^- = F_2^- = 0$ and $F_1^+ = F_2^+ = 1$. When the initial states are $\boldsymbol{x}(0) = \begin{bmatrix} -0.2 & -0.4 \end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{y}(0) = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix}^{\mathrm{T}}$, the chaotic behavior of the master system (1) and slave system (4) with $\boldsymbol{u}(t) = 0$ is depicted in Figs. 1 and 2. For the values of η given in Table 1, by applying Theorem 1 and Corollary 1 with $\gamma_1 = \gamma_2 = 0.24$ and the condition provided in Ref. [39], the maximum values of sampling interval $\bar{h}$ that guarantee the synchronization of the master system (1) and slave system (4), are obtained. It can be seen from Table 1 that the approach given in this paper achieves a noticeable improvement over Ref. [36]. It is also shown in the table that Theorem 1 provides less conservative results than Corollary 1. This is due to the fact that a new term, $\boldsymbol{V}_5(t)$, is introduced in the Lyapunov functional.

In case of the constant delay $\eta = 0.01$ and the largest sampling interval $\bar{h} = 0.21$, by applying Theorem 1, the corresponding gain matrix in Eq. (8) is obtained as

$$\boldsymbol{K} = \begin{bmatrix} -4.5990 & 0.0717 \\ 0.9616 & -4.7076 \end{bmatrix}.$$

Under the above controller, the response curves of control input (8) and error system (9) are given in Figs. 1 and 2, respectively. It is obvious from Fig. 2 that the error system tends to be zero, which means that the master system (1) and the slave system (4) are synchronous.

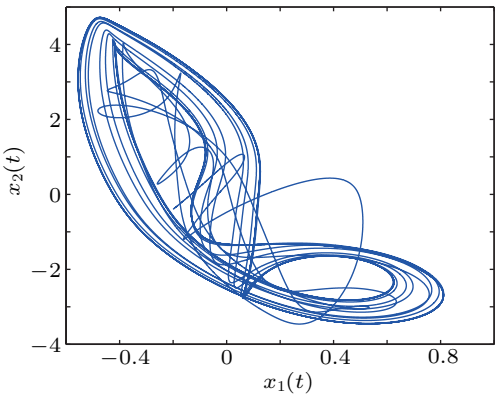


Fig. 1. (color online) Chaotic behavior of the master system (1).

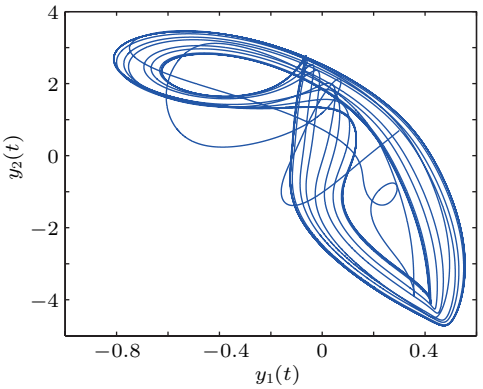


Fig. 2. (color online) Chaotic behavior of the slave system (4) with $\boldsymbol{u}(t) = 0$.

Table 1. Maximum sampling interval $\bar{h}$ for different η.

η	0.01	0.03	0.05	0.07	0.09
Ref. [39]	0.19	0.16	0.13	0.10	0.06
Corollary 1	0.19	0.16	0.14	0.11	0.09
Theorem 1	0.21	0.17	0.15	0.12	0.09

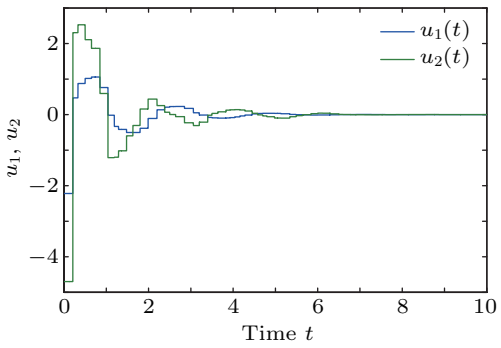


Fig. 3. (color online) Control input $\boldsymbol{u}(t)$.

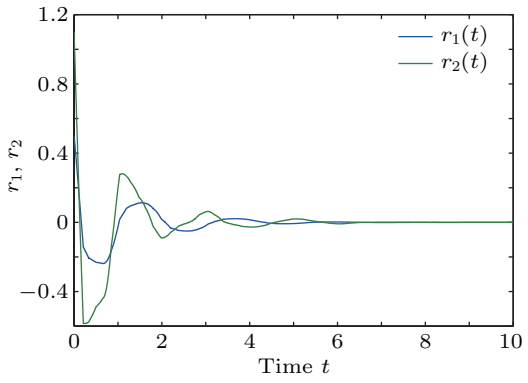


Fig. 4. (color online) State responses of error system (9).

Example 2 Consider the master system (1) and slave system (4) with the following parameters:

$$\boldsymbol{C}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\boldsymbol{A}=\begin{bmatrix}1+\pi/4 & 20\\ 0.1 & 1+\pi/4\end{bmatrix},$$
$$\boldsymbol{B}=\begin{bmatrix}\dfrac{-1.3\sqrt{2}\pi}{4} & 0.1\\ 0.1 & \dfrac{-1.3\sqrt{2}\pi}{4}\end{bmatrix}.$$

Choose the activation functions as

$$g_i(\alpha)=(|\alpha+1|-|\alpha-1|)/2,\quad i=1,2,$$

and the time-varying delay to be the same as in Example 1. It is easy to verify that the activation functions satisfy (4) with $F_1^-=F_2^-=0$ and $F_1^+=F_2^+=1$. The chaotic behavior of the master system (1) and slave system (4) with $\boldsymbol{u}(t)=0$ is depicted in Figs. 5 and 6 under the initial states $\boldsymbol{x}(0)=\begin{bmatrix}0.3 & 0.4\end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{y}(0)=\begin{bmatrix}0.6 & 0.3\end{bmatrix}^{\mathrm{T}}$.

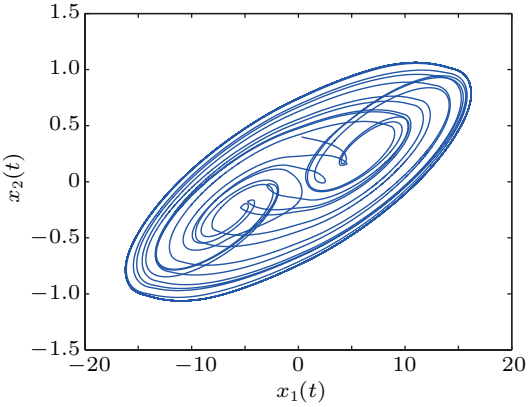


Fig. 5. (color online) Chaotic behavior of the master system (1).

When $\eta=0.04$ and $\bar{h}=0.23$, the approach proposed in Ref. [39] failed to find a feasible solution. However, applying Theorem 1 with $\gamma_1=\gamma_2=0.3$, a controller gain matrix in Eq. (8) can be obtained as

$$\boldsymbol{K}=\begin{bmatrix}-3.5108 & -2.1786\\ -0.0176 & -3.5586\end{bmatrix}.$$

Under the given controller, the response curves of control input (8) and error system (9) are depicted in Figs. 7 and 8, respectively. It is clearly observed from Fig. 8 that the error system is converging to zero. In other words, the master system (1) and the slave system (4) are synchronous.

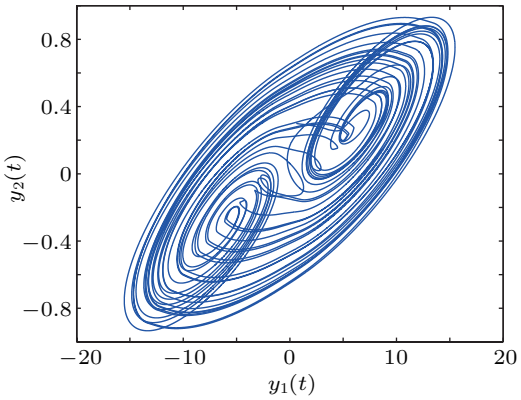


Fig. 6. (color online) Chaotic behavior of the slave system (4) with $\boldsymbol{u}(t)=0$.

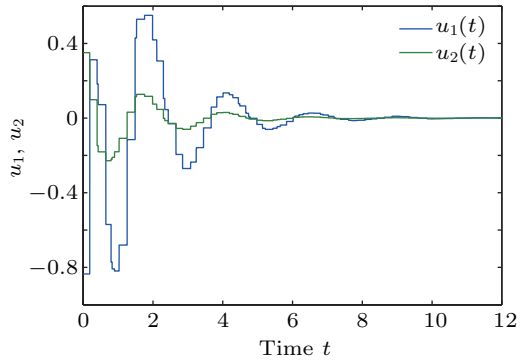


Fig. 7. (color online) Control input $\boldsymbol{u}(t)$.

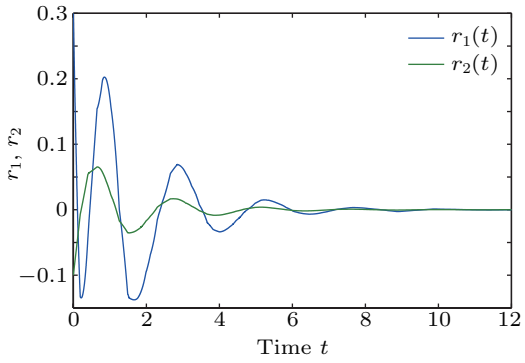


Fig. 8. (color online) State responses of error system (9).

5. Conclusions

In this paper, the synchronization problem has been investigated for delayed neural networks via sampled-data control. By employing a free-matrix-based time-dependent discontinuous Lyapunov functional, improved synchronization criteria

have been derived. Two numerical examples have been provided to show that the proposed results are less conservative than existing ones.

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