Stochastic bounded consensus of second-order multi-agent systems in noisy environment*

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This paper investigates the stochastic bounded consensus of leader-following second-order multi-agent systems in a noisy environment. It is assumed that each agent received the information of its neighbors corrupted by noises and time delays. Based on the graph theory, stochastic tools, and the Lyapunov function method, we derive the sufficient conditions under which the systems would reach stochastic bounded consensus in mean square with the protocol we designed. Finally, a numerical simulation is illustrated to check the effectiveness of the proposed algorithms.

Keywords: stochastic bounded consensus, multi-agent systems, noises, time-delays

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1. Introduction

In recent years, the consensus of multi-agent systems has attracted much attention, due to its wide applications in many areas such as cooperative control of unmanned aerial vehicles (UAVs), formation control, distributed sensor network, and so on.[1–3] As is well known, consensus is the fundamental problem of cooperative control for multi-agent systems, which means that each agent tends to the common value in a team, and the value can be position, velocity, temperature or other physical quantities. Moreover, leader-following consensus problems is very interesting, which is also called tracking control for multi-agent systems.[4–8] In tracking consensus control, it is essential to design a distributed network algorithm based on the exchange of state information with neighbor agents, whose common goal is to track a reference state that is available to only a part of agents. Most previous literatures have assumed that the information exchange between agents is accurate. However, this is only the ideal approximation of the true communication process. Recently, consensus of multi-agent systems in a noisy environment has attracted more and more attention of researchers, due to the fact that noises in communication are inevitable in the real physical world. The existence of noises has a negative impact on the performances of the entire systems, and even cause system instability. The sources of the communication noises mainly came from the sensors that the agents are equipped with and the transmission channels. In order to attenuate the noises and reduce the negative impact on consensus in multi-agent systems, a great effort has been done.[9,10] In Ref. [9], a stochastic approximation-type algorithm with the key feature of the decreasing consensus gain $\alpha(k)$ in the protocols is proposed to attenuate the measurement noises in discrete-time systems. The algorithm has a gradient descent interpretation. The approach applied in Ref. [9] has been extended to continuous-time systems in Ref. [10]. Li *et al.*[10] has investigated mean square average-consensus control with fixed topology and Gaussian communication noises.

Most literatures concerning with consensus focused on algorithms taking the form of first-order dynamics in the earlier research. However, second-order systems have more practical significance. The convergence of second-order multi-agent systems depends not only on the information exchange topologies but also on the coupling strength between the derivatives of velocities. The case of second-order multi-agent system has been studied in Refs. [11]–[13]. A distributed linear consensus protocol for second-order multi-agent systems under limited agent interaction ranges has been investigated in Ref. [11], in which two agents can interact with each other only if their distance is within a certain range. Under the linear consensus protocol with the relative state feedback, sufficient conditions for the second-order multi-agent systems have derived.

In addition, most traditional controller design approaches rely on the assumption that the measurement signals are timely transmitted. However, such an assumption is fairly conservative in many engineering practices. For example, due to network congestions, the measurements may contain time delays as well as noises. Time-delays include internal time-delays and transmission time-delays. The consensus problem of multi-agent system with time delays and noises have received considerable research attention and many important results have been reported in recent years.[14–18] In Ref. [14], the

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bounded synchronization for a class of complex network with delay feedback controller has been investigated. Based on inequality theorems, bounded synchronization criteria of complex dynamical networks have been derived. Further, finite-time synchronization for a class of the complex dynamical network with non-delayed and delayed coupling were also considered in Ref. [15]. In Ref. [16], the containment consensus control problem for multi-agent systems with measurement noises and time-varying communication delays under directed networks has been considered. By using stochastic analysis tools and algebraic graph theory, it was proved that the followers can converge to the convex hull spanned by the leaders in mean square sense if the allowed upper bound of the time-varying delays satisfied a certain sufficient condition. Mean square average consensus for multi-agent systems with measurement noises and time delays under fixed digraph was studied in Ref. [17]. By combining the tools of algebraic graph theory, matrix theory, and stochastic analysis, consensus protocols for multi-agent systems were elaborately analyzed. Song *et al.*[18] investigated the mean square consensus problem of multi-agent systems impacted by the combined uncertainty of multiplicative noises and time delays. Using tools from stochastic differential delay equation, martingale theory and stochastic inequality, sufficient conditions on mean square consensus have been obtained.

However, to the best of the authors' knowledge, the consensus problem has not been properly investigated for multi-agent systems with noises and time delays. Motivated by the above discussions, we aim to investigate the stochastic bounded consensus problem of continuous-time multi-agent systems with time-delays in a noisy environment. Due to the coexistence of noises and time delays, it is difficult in finding the optimal delay bound for stochastic consensus problems. In our model, the information received from other agents is corrupted by additive noises and time delays. We introduce the time-varying decreasing consensus gains in the follower control protocol to attenuate the noises. It is critical to maintain a trade-off in attenuating the noises to prevent long term fluctuations and meanwhile ensuring a suitable stabilizing capability so as to drive the follower agent states towards the leader agent. In this paper, the methods of the stochastic analysis and the Lyapunov function are applied to overcome the difficulties induced by the time-delays and noises. The closed-loop system is transformed into a stochastic differential delay equation driven by the additive noises. The contribution of this work can be summarized as the following three aspects. (i) The time-delays and noises are taking into account in the analysis of the consensus problems. (ii) The protocol is based on the measurement of its neighbors' states and some estimated data of the leader which are both corrupted by white noises. (iii) The sufficient conditions of stochastic bounded consensus of multi-agent system are derived in a noisy environment.

The rest of this paper is organized as follows. In Section 2, some basic definitions in graph theory and mathematical preliminary results needed for subsequent uses are presented. The consensus protocol is proposed for second-order multi-agent systems with measurement noises and time delays in Section 3. The detailed proofs for the main results are given by using Lyapunov function method. Then, numerical simulations are provided in Section 4. The paper is concluded by Section 5.

Notations: Let I_n denote an n-dimensional identity matrix; $|\cdot|$ stands for the norm for both vectors and matrices, at the same time, we denote $|\varphi|_\tau = \sup_{\theta\in[-\tau,0]}|\varphi(\theta)|$ for a function $\varphi \in C([-\tau,0],R^n)$; 1_n and 0_n denote a column vector with all ones and zeros, respectively. A^{T} denotes the transpose of the matrix A; and $\mathrm{Tr}(A)$ is its trace. Let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalues of the matrix A, respectively. For a given random variable X, $\mathbb{E}X$ denotes its mathematical expectation. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathcal{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfies that the filtration contains all $\mathcal{P}$-null sets and is right continuous. Let $\mathcal{L}^2_{\mathcal{F}_0}([-\tau,0],R^n)$, the family of all $\mathcal{F}_0$-measurable $PC([-\tau,0],R^n)$-valued random variables $\phi = \{\phi(s) : -\tau \leq s \leq 0\}$ such that $\sup_{-\tau\leq s\leq 0}\mathbb{E}\|\phi(s)\|^2 < \infty$, where $\mathbb{E}(\cdot)$ represents the mathematic expectation of the corresponding variable. Define the norm of the piecewise right continuous function as $\|\phi(t)\|_\tau = \sup_{-\tau\leq s\leq 0}\|\phi(t+s)\|$.

2. Preliminaries

Graph theory is the basic theory to analyze the interconnection topologies. A graph is denoted as $G = (v,\varepsilon,A)$, which consists of a set of nodes $v = \{v_1, v_2, \ldots, v_n\}$, an edge set $\varepsilon \subseteq v \times v$, and an adjacency matrix $A = [a_{ij}] \in R^{n\times n}$. Here, the node indexes belong to the finite index set $I = \{1,2,\ldots,n\}$. If an edge of G is denoted as $e_{ij} = (v_i, v_j) \in \varepsilon$, a directed path of G is a sequence of ordered edges $(v_{i1}, v_{i2}), (v_{i2}, v_{i3}), \ldots$ in a directed graph. A digraph is strongly connected if there is a path between any two distinct nodes. Let $N_i = \{v_j : (v_i, v_j) \in \varepsilon\}$ denote the set of neighbor nodes. A nonnegative weighted matrix A is called an adjacent matrix, which specifies the interconnection topology of network. If the edge of G satisfies $(v_i, v_j) \in \varepsilon$, the element of adjacency matrix is defined as $a_{ii} = 0$ and $a_{ij} = 1$ for all $i \neq j$. For an undirected graph, A is symmetric and every undirected graph is balanced. Moreover, a diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is called input degree matrix whose diagonal element is defined as $d_i = \sum_{j\neq i} a_{ij}$ for $i \in I$. The matrix $L = D - A$ is called the Laplacian matrix of the weighted digraph. According to the definition, the row sum of the Laplacian matrix is zero and the column vector $\mathbf{1}_n$ is the right eigenvector corresponding to a zero eigenvalue. If

the graph is balanced, there is also a left eigenvector $\mathbf{1}_n$ corresponding to a zero eigenvalue except the right one. Next we assume node 0 is the leader of the digraph $\bar{G}$, which is used to describe the multi-agent system topology with one active leader agent and follower agents. Take the diagonal matrix $B = \mathrm{diag}\{a_{10}, \ldots, a_{n0}\} \in R^{n\times n}$ as a leader adjacency matrix, where $a_{i0} = 1$ if the follower agent v_i can receive the information from the leader v_0, and $a_{i0} = 0$ otherwise. Let a new matrix $H = L + B$ denote the connectivity of the leader graph. Note that $L\mathbf{1}_n = 0$, so it follows that $H\mathbf{1}_n = B\mathbf{1}_n$.

The following lemmas play important roles in the subsequence.

Lemma 1[19] For any two real vectors x and y with the same dimension and a positive definition matrix Φ with an appropriate dimension, for any $\varepsilon > 0$ we have

$$2x^{\mathrm{T}}y \le \varepsilon x^{\mathrm{T}}\Phi x + \varepsilon^{-1}y^{\mathrm{T}}\Phi^{-1}y.$$

Lemma 2[4] The following three statements are equivalent:

1) Node 0 is a globally reachable node of directed graph $\bar{G}$, that is, there is at least one directed path from the leader node 0 to all other followers;

2) H is a positive stable matrix with positive real parts eigenvalues;

3) Moreover, $H + H^{\mathrm{T}}$ is a symmetric positive-definite matrix, if G is a balanced graph.

3. Consensus analysis

3.1. Problem formulation

In this paper, we consider a leader–following multi-agent system with a leader and n followers. The dynamics for each follower agent is described as the first-order differential system:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \ldots, n, \quad t \in R^+, \tag{1}$$

where $x_i(t) \in R^m$ and $u_i(t) \in R^m$ denote the positions and the inputs of the follower agents, respectively. The leader dynamics is described as the following second-order differential equation

$$\begin{cases} \dot{x}_0(t) = v_0(t), \\ \dot{v}_0(t) = a_0(t), \\ y_0(t) = x_0(t). \end{cases} \tag{2}$$

where $x_0(t) \in R^m$, $v_0(t) \in R^m$, and $a_0(t) \in R^m$ denote the position, the velocity, and the acceleration of the leader, respectively. Here for notation simplicity, let $m = 1$. $a_0(t)$ is assumed to be known beforehand. $y_0(t)$ denotes the output of the system.

Since the leader velocity of $v_0(t)$ cannot be measured by followers, thus we assume that $v_i(t)$ is the estimation of $v_0(t)$ for the i-th followers.

This paper aims to develop a leader–following consensus algorithm for multi-agent systems (1) under measurement noises and time-delays to ensure the tracking of the leader's position by the followers in mean square. We introduce the following definition of mean square bounded consensus for leader–following multi-agent systems.

The leader-following multi-agent systems with measurement noises (1) and (2) under the designed protocol are said to reach stochastic bounded tracking consensus in mean square if the systems satisfy the following bounded estimations:

$$\begin{cases} \limsup\limits_{t\to+\infty} \mathbb{E}[(x_i(t) - x_0(t))^2] \le C_x^*, \\ \limsup\limits_{t\to+\infty} \mathbb{E}[(v_i(t) - v_0(t))^2] \le C_v^*, \end{cases} \quad \forall i \in I, \tag{3}$$

where C_x^*, and C_v^* are bounded constants that are independent of t.

3.2. Main result

Due to that bandwidth constraints and packet loss may cause noises and time-delays, we design the tracking protocol for the leader–follower multi-agent system with time-delays in a noisy environment such that the system would reach stochastic bounded consensus in mean square sense. For system (1), we consider the neighbor-based consensus protocol as follows:

$$\begin{cases} u_i(t) = -k\alpha(t) \sum\limits_{j=0, j\neq i}^{n} a_{ij} z_{ij}(t-\tau) + \alpha(t) v_i(t), \\ \dot{v}_i(t) = a_0(t) - \gamma k \alpha(t) \sum\limits_{j=0, j\neq i}^{n} a_{ij} z_{ij}(t-\tau), \end{cases} \tag{4}$$

where $k > 0$ is a feedback gain and $0 < \gamma < 1$. We consider that the measurement of relative states by agent i have the following form

$$z_{ij}(t-\tau) = x_i(t-\tau) - x_j(t-\tau) + \rho_{ij}\omega_{ij}(t), \tag{5}$$

where a_{ij} denotes the connected weight on information link agent (i, j), τ is the time delay, $\{\omega_{ij}(t) : i, j = 1, \ldots, n\}$ are independent standard white noises and $\rho_{ij} \ge 0$ represent the noise intensity. The velocity $v_0(t)$ can be decomposed as follows:

$$\begin{cases} v_0(t) = \alpha(t) v_0(t), \\ \dot{v}_0(t) = a_0(t), \end{cases} \tag{6}$$

where $v_0(t)$ and $a_0(t)$ are called nominal velocity and nominal acceleration respectively. The continuous function $\alpha(t)$ is called consensus gain satisfying the following assumption:

Assumption 1

$$\begin{cases} \int_{t_0}^{\infty} \alpha(s)\,\mathrm{d}s = \infty, & \text{(A1)} \\ \int_{t_0}^{\infty} \alpha^2(s)\,\mathrm{d}s < \infty. & \text{(A2)} \end{cases} \tag{7}$$

In addition, we assume that $\alpha(t)$ has an upper bound μ in $[0,+\infty)$. The relationship between the acceleration $a_0(t)$ and the nominal one $\boldsymbol{a}_0(t)$ can be expressed as

$$a_0(t)=\dot{\alpha}(t)\boldsymbol{v}_0(t)+\alpha(t)\boldsymbol{a}_0(t).$$

Remark 1 Let $v_i(t)=\alpha(t)\boldsymbol{v}_i(t)$ be the estimate of $v_0(t)$, which uses a velocity decomposition scheme. Because $\alpha(t)$ has an upper bound μ in time-interval $[t_0,\infty)$, we can easily get $v_i(t)-v_0(t)\to 0$, if $\boldsymbol{v}_i(t)-\boldsymbol{v}_0(t)\to 0$.

Remark 2 The assumption $\int_{t_0}^{\infty}\alpha(s)\,\mathrm{d}s=\infty$ can guarantee the followers track the leader in a proper velocity, and $\int_{t_0}^{\infty}\alpha^2(s)\,\mathrm{d}s<\infty$ can make the tracking errors of the systems finite regardless of measurement noises.

Remark 3 There are many works on the consensus gain $\alpha(t)$ for systems with additive noises such as in Refs. [9] and [10], to name a few. A simple example for $\alpha(t)$ is $\alpha(t)=1/(t+1), t\geq 0$. Without loss of generalities, the function $\alpha(t)$ is decreasing and has an upper bound $\mu=1$.

Under the consensus protocol (4), the dynamics of multi-agent system (1) can be written as the following:

$$\begin{cases}\dot{x}_i(t)=-k\alpha(t)\sum\limits_{j=0,\ j\neq i}^{n}a_{ij}[(x_i(t-\tau)-x_j(t-\tau))+\rho_{ij}\omega_{ij}(t)]+\alpha(t)\boldsymbol{v}_i(t),\\ \dot{\boldsymbol{v}}_i(t)=\boldsymbol{a}_0(t)-\gamma k\alpha(t)\sum\limits_{j=0,\ j\neq i}^{n}a_{ij}[(x_i(t-\tau)-x_j(t-\tau))+\rho_{ij}\omega_{ij}(t)].\end{cases}\tag{8}$$

It can be rewritten as

$$\begin{cases}\dot{x}_i(t) &= -k\alpha(t)\sum\limits_{j=1,j\neq i}^{n}[a_{ij}(x_i(t-\tau)-x_j(t-\tau))+a_{i0}(x_i(t-\tau)-x_0(t-\tau))]\\ &\quad -k\alpha(t)\sum\limits_{j=1,j\neq i}^{n}a_{ij}\rho_{ij}\omega_{ij}(t)-k\alpha(t)a_{i0}\rho_{i0}\omega_{i0}(t)+\alpha(t)\boldsymbol{v}_i(t),\\ \dot{\boldsymbol{v}}_i(t) &= \boldsymbol{a}_0(t)-\gamma k\alpha(t)\sum\limits_{j=1,j\neq i}^{n}[a_{ij}(x_i(t-\tau)-x_j(t-\tau))+a_{i0}(x_i(t-\tau)-x_0(t-\tau))]\\ &\quad -\gamma k\alpha(t)\sum\limits_{j=1,j\neq i}^{n}a_{ij}\rho_{ij}\omega_{ij}(t)-\gamma k\alpha(t)a_{i0}\rho_{i0}\omega_{i0}(t).\end{cases}\tag{9}$$

Denote $x(t)=[x_1^{\mathrm{T}}(t),x_2^{\mathrm{T}}(t),\ldots,x_n^{\mathrm{T}}(t)]^{\mathrm{T}}, \boldsymbol{v}(t)=[\boldsymbol{v}_1^{\mathrm{T}}(t),\boldsymbol{v}_2^{\mathrm{T}}(t),\ldots,\boldsymbol{v}_n^{\mathrm{T}}(t))^{\mathrm{T}}$. Then the dynamic system (9) can be rewritten in a compact form of a stochastic delay differential equation as:

$$\begin{cases}\mathrm{d}x(t)=[-k\alpha(t)Hx(t-\tau)+k\alpha(t)B1_nx_0(t-\tau)+\alpha(t)\boldsymbol{v}(t)]\,\mathrm{d}t-k\alpha(t)\Theta\,\mathrm{d}W(t),\\ \mathrm{d}\boldsymbol{v}(t)=[\boldsymbol{a}_01_n-\gamma k\alpha(t)Hx(t-\tau)+\gamma k\alpha(t)B1_nx_0(t-\tau)]\,\mathrm{d}t-\gamma k\alpha(t)\Theta\,\mathrm{d}W(t),\end{cases}\tag{10}$$

where the $W(t)=[W_0,W_1,W_2,\ldots,W_n]^{\mathrm{T}}$ is an (n^2+n)-dimensional independent Brownian motion vector with $W_0=[\omega_{10},\omega_{20},\ldots,\omega_{n0}]^{\mathrm{T}}$ and $W_i=[\omega_{1i},\omega_{2i},\ldots,\omega_{ni}]^{\mathrm{T}}$; L and B denote the graph Laplacian matrix and the leader adjacency matrix, respectively, $H=L+B$, $\Theta_0=\mathrm{diag}\{\rho_{10},\ldots,\rho_{n0}\}$, $\Theta_i=\mathrm{diag}\{\rho_{i1},\ldots,\rho_{in}\}$, for $i\in I$, and the matrix $\Theta\in R^{n\times n(n+1)}$ is defined as follows:

$$\Theta=[B\Theta_0,\mathrm{diag}\{a(1,\cdot)\Theta_1,\ldots,a(n,\cdot)\Theta_n\}].\tag{11}$$

Define two error state vectors of the position and velocity, respectively $\begin{cases}\bar{x}=x-x_0\mathbf{1}_n,\\ \bar{\boldsymbol{v}}=\boldsymbol{v}-\boldsymbol{v}_0\mathbf{1}_n.\end{cases}$ Based on the spectrum properties, $L\mathbf{1}_n=0$ and then it follows $-H\bar{x}=-Hx+B\mathbf{1}_nx_0$. For system (11), we have

$$\begin{cases}\mathrm{d}\bar{x}(t)=[-k\alpha(t)H\bar{x}(t-\tau)+\alpha(t)\bar{\boldsymbol{v}}(t)]\,\mathrm{d}t-k\alpha(t)\Theta\,\mathrm{d}W(t),\\ \mathrm{d}\bar{\boldsymbol{v}}(t)=[-\gamma k\alpha(t)H\bar{x}(t-\tau)]\,\mathrm{d}t-\gamma k\alpha(t)\Theta\,\mathrm{d}W(t).\end{cases}\tag{12}$$

The system (12) can be rewritten in a compact form of a stochastic delay differential equation as the following

$$\mathrm{d}\varepsilon(t)=[\alpha(t)C\varepsilon(t)+\alpha(t)E\varepsilon(t-\tau)]\,\mathrm{d}t+\alpha(t)\tilde{\Omega}\,\mathrm{d}W(t),\tag{13}$$

where $\varepsilon=\begin{bmatrix}\bar{x}\\ \bar{\boldsymbol{v}}\end{bmatrix}$, $C=\begin{bmatrix}0 & I\\ 0 & 0\end{bmatrix}$, $\tilde{\Omega}=\begin{bmatrix}-k\Theta\\ -\gamma k\Theta\end{bmatrix}$, and $E=\begin{bmatrix}-kH & 0\\ -\gamma kH & 0\end{bmatrix}$.

The initial function $\varepsilon_{t_0}\in C([-\tau,0],R^{2n})$ for this equation is $\varepsilon_{t_0}(\theta)=\varepsilon(t_0+\theta)$ for $\theta\in[-\tau,0]$. We expand this initial data to $[-2\tau,0]$ by $\varepsilon_{t_0}(\theta)=\varepsilon(t_0-\tau)$ for $\theta\in[-2\tau,-\tau]$. With this expansion, we have $|\varepsilon_{t_0}|_{2\tau}=|\varepsilon_{t_0}|_{\tau}$. This kind of norm will be involved actually in the argument for the main result.

Note that system (13) is a stochastic differential equation similar to the familiar Langevin's equation. There is no equilibrium to this equation, thus only boundedness of the solutions is expectable for this equation. Our main result will shows this soon.

Theorem 1 Suppose that Assumption 1 holds, and the leader vertex 0 is globally reachable in the topology graph $\bar{G}$. If there exist the feedback gain k and the time delay τ such that

$$\begin{cases} k > \dfrac{\lambda_{\max}(\bar{P})}{2\gamma(1-\gamma^2)}, \\ \tau < \dfrac{\lambda_{\min}(Q)}{(|N_1|+|N_2|)}, \\ (2\tau\mu)^2\rho\,\mathrm{e}^{2\tau\rho\mu} < \rho, \end{cases} \tag{14}$$

where

$$\begin{aligned} N_1 &= PECP^{-1}C^{\mathrm{T}}E^{\mathrm{T}}P, \\ N_2 &= PE^2P^{-1}(E^2)^{\mathrm{T}}P, \\ \rho &= \frac{\lambda_{\min}(Q)-\tau(|N_1|+|N_2|)}{\lambda_{\min}(P)}, \\ \rho &= n(1+\gamma^2)k^2\max\|\Theta\|^2\lambda_{\max}(P), \end{aligned}$$

$P=\begin{bmatrix} \bar{P} & -\gamma\bar{P} \\ -\gamma\bar{P} & \bar{P} \end{bmatrix}$, and $\bar{P}$ is a symmetric positive-definite solution to the Lyapunov matrix equation

$$H^{\mathrm{T}}\bar{P}+\bar{P}H = I_n,$$

then the system (1) can reach bounded consensus in mean square with the protocol (4), that is to say, the follower agents can track the leader practically.

Proof Consider a Lyapunov function candidate $V(\varepsilon(t)) = \varepsilon^{\mathrm{T}}(t)P\varepsilon(t)$. By Definition 1, we need only to show the boundedness of the mean square error $\mathbb{E}|\varepsilon(t)|^2$. Obviously we have $V(\varepsilon(t)) \geq \lambda_{\min}(P)|\varepsilon(t)|^2$. By the given Eq. (13) we have

$$\begin{aligned} \varepsilon(t-\tau) &= \varepsilon(t) - \int_{-\tau}^{0} \mathrm{d}\varepsilon(t+s) \\ &= \varepsilon(t) - \int_{-\tau}^{0} \alpha(t+s)C\varepsilon(t+s)\,\mathrm{d}s \\ &\quad - \int_{-\tau}^{0} \alpha(t+s)E\varepsilon(t+s-\tau)\,\mathrm{d}s \\ &\quad - \alpha(t+s)\int_{-\tau}^{0} \tilde{\Omega}\,\mathrm{d}W(t+s) \\ &= \varepsilon(t) - \int_{-\tau}^{0} \alpha(t+s)C\varepsilon(t+s)\,\mathrm{d}s \\ &\quad - \int_{-2\tau}^{-\tau} \alpha(t+s+\tau)E\varepsilon(t+s)\,\mathrm{d}s \\ &\quad - \alpha(t+s)\tilde{\Omega}\int_{-\tau}^{0} \mathrm{d}W(t+s). \end{aligned}$$

Taking the stochastic derivative of the function $V(\varepsilon(t))$ yields that

$$\begin{aligned} &\mathcal{L}V(\varepsilon(t))\big|_{(13)} \\ &= \alpha(t)\varepsilon^{\mathrm{T}}(t)[P(C+E)+(C+E)^{\mathrm{T}}P]\varepsilon(t) \\ &\quad -2\alpha(t)\varepsilon^{\mathrm{T}}(t)PEC\int_{-\tau}^{0} \alpha(t+s)\varepsilon(t+s)\,\mathrm{d}s \\ &\quad -2\alpha(t)\varepsilon^{\mathrm{T}}(t)PEE\int_{-2\tau}^{-\tau} \alpha(t+s+\tau)\varepsilon(t+s)\,\mathrm{d}s \\ &\quad -2\alpha(t)\alpha(t+s)\varepsilon^{\mathrm{T}}(t)PE\tilde{\Omega} \\ &\quad \times \int_{-\tau}^{0} \mathrm{d}W(t+s) + \alpha^2(t)\mathrm{Tr}(P\tilde{\Omega}\tilde{\Omega}^{\mathrm{T}}). \end{aligned} \tag{15}$$

Actually we have $\int_{-\tau}^{0}\mathrm{d}W(t+s) = \int_{t-\tau}^{t}\mathrm{d}W(u) = W(t) - W(t-\tau)$. Denote $F = C+E$, then we have

$$PF+F^{\mathrm{T}}P := -Q = -\begin{pmatrix} k(1-\gamma^2)I_n & -\bar{P} \\ -\bar{P} & 2\gamma\bar{P} \end{pmatrix}.$$

According to the Schur complement lemma, under the condition (14), the matrix Q is positive definite. Letting $\Phi = P^{-1}$, by the elementary inequality $2a^{\mathrm{T}}b \leq \varepsilon a^{\mathrm{T}}\Phi a + \varepsilon^{-1}b^{\mathrm{T}}\Phi^{-1}b\,(\varepsilon>0)$ and with $\alpha(t)\leq\mu$, we have

$$\begin{aligned} &-2\alpha(t)\varepsilon^{\mathrm{T}}(t)PEC\int_{-\tau}^{0} \alpha(t+s)\varepsilon(t+s)\,\mathrm{d}s \\ &\leq \tau\alpha(t)\varepsilon^{\mathrm{T}}(t)PECP^{-1}C^{\mathrm{T}}E^{\mathrm{T}}P\varepsilon(t) \\ &\quad +\mu\int_{-\tau}^{0} \alpha^2(t+s)V(\varepsilon(t+s))\,\mathrm{d}s, \end{aligned} \tag{16}$$

$$\begin{aligned} &-2\alpha(t)\varepsilon^{\mathrm{T}}(t)PEE\int_{-2\tau}^{-\tau} \alpha(t+s+\tau)\varepsilon(t+s)\,\mathrm{d}s \\ &\leq \tau\alpha(t)\varepsilon^{\mathrm{T}}(t)PE^2P^{-1}(E^2)^{\mathrm{T}}P\varepsilon(t) \\ &\quad +\mu\int_{-\tau}^{0} \alpha^2(t+s)V(\varepsilon(t+s-\tau))\,\mathrm{d}s, \end{aligned} \tag{17}$$

and $\alpha^2(t)\mathrm{Tr}(P\tilde{\Omega}\tilde{\Omega}^{\mathrm{T}}) \leq \rho\alpha^2(t)$. Based on the above analysis, then we have

$$\begin{aligned} &\mathcal{L}V(\varepsilon(t))\big|_{(13)} \\ &\leq -\alpha(t)\varepsilon^{\mathrm{T}}(t)Q\varepsilon(t) + \tau\alpha(t)\varepsilon^{\mathrm{T}}(t)PECP^{-1}C^{\mathrm{T}}E^{\mathrm{T}}P\varepsilon(t) \\ &\quad +\mu\int_{-\tau}^{0} \alpha^2(t+s)V(\varepsilon(t+s))\,\mathrm{d}s \\ &\quad +\tau\alpha(t)\varepsilon^{\mathrm{T}}(t)PE^2P^{-1}(E^2)^{\mathrm{T}}P\varepsilon(t) \\ &\quad +\mu\int_{-\tau}^{0} \alpha^2(t+s)V(\varepsilon(t+s-\tau))\,\mathrm{d}s \\ &\quad -2\alpha(t)\alpha(t+s)\varepsilon^{\mathrm{T}}(t)PE\tilde{\Omega}[W(t)-W(t-\tau)] + \rho\alpha^2(t) \\ &\leq -\rho\alpha(t)V(\varepsilon(t)) + \rho\alpha^2(t) + \mu\int_{-\tau}^{0} \alpha^2(t+s)V(\varepsilon(t+s))\,\mathrm{d}s \\ &\quad +\mu\int_{-\tau}^{0} \alpha^2(t+s)V(\varepsilon(t+s-\tau))\,\mathrm{d}s \\ &\quad -2\alpha(t)\alpha(t+s)\varepsilon^{\mathrm{T}}(t)PE\tilde{\Omega}[W(t)-W(t-\tau)]. \end{aligned} \tag{18}$$

Define the new Lyapunov function

$$U(t,\varepsilon) = \exp\left[\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\right]V(\varepsilon),\ t\in R^+, \varepsilon\in R^{2n},$$

then we can further derive that

$$\begin{aligned} &\mathcal{L}U(t,\varepsilon(t))\big|_{(13)} \\ &\leq \mathrm{e}^{\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s}\Big[\rho\alpha^2(t) + \mu\int_{-\tau}^{0} \alpha^2(t+s)V(\varepsilon(t+s))\,\mathrm{d}s \\ &\quad +\mu\int_{-\tau}^{0} \alpha^2(t+s)V(\varepsilon(t+s-\tau))\,\mathrm{d}s \\ &\quad -2\alpha(t)\alpha(t+s)\varepsilon^{\mathrm{T}}(t)PE\tilde{\Omega}[W(t)-W(t-\tau)]\Big], \end{aligned} \tag{19}$$

thus by the familiar Dynkin formula and the fact $\mathbb{E}[W(t)-W(t-\tau)]=0$, we then obtain

$$
\begin{aligned}
&\mathbb{E}U(t,\varepsilon(t))-\mathbb{E}U(t_0,\varepsilon(t_0))=\mathbb{E}\int_{t_0}^{t}\mathcal{L}U(s,\varepsilon(s))\mathrm{d}s\\
&\le \int_{t_0}^{t}\Big[\exp\Big(\rho\int_{t_0}^{s}\alpha(r)\mathrm{d}r\Big)\Big(\rho\alpha^2(s)+\mu\int_{-\tau}^{0}\alpha^2(s+\theta)\mathbb{E}V(\varepsilon(s+\theta))\mathrm{d}\theta+\mu\int_{-\tau}^{0}\alpha^2(t+s)\mathbb{E}V(\varepsilon(t+s-\tau))\mathrm{d}\theta\Big)\Big]\mathrm{d}s\\
&\le H(t)+\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\int_{-\tau}^{0}\alpha^2(s+\theta)\mathbb{E}U(s+\theta,\varepsilon(s+\theta))\mathrm{d}\theta\mathrm{d}s\\
&\quad+\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\int_{-\tau}^{0}\alpha^2(t+\theta)\mathbb{E}U(s+\theta-\tau,\varepsilon(s+\theta-\tau))\mathrm{d}\theta\mathrm{d}s\\
&\le H(t)+\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\Big(\int_{-2\tau}^{0}\alpha^2(s+l)\mathrm{d}l\sup_{-2\tau\le\theta\le0}\mathbb{E}U(s+\theta,\varepsilon(s+\theta))\Big)\mathrm{d}s\\
&\quad+\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\Big(\int_{-2\tau}^{0}\alpha^2(s+l)\mathrm{d}l\sup_{-2\tau\le\theta\le0}\mathbb{E}U(s+\theta,\varepsilon(s+\theta))\Big)\mathrm{d}s\\
&= H(t)+2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\Big(\int_{-2\tau}^{0}\alpha^2(s+l)\mathrm{d}l\sup_{-2\tau\le\theta\le0}\mathbb{E}U(s+\theta,\varepsilon(s+\theta))\Big)\mathrm{d}s,
\end{aligned}\tag{20}
$$

where

$$
\begin{aligned}
H(t)&=\rho\int_{t_0}^{t}\exp\Big(\rho\int_{t_0}^{s}\alpha(r)\mathrm{d}r\Big)\alpha^2(s)\mathrm{d}s\le\rho\mu\int_{t_0}^{t}\exp\Big(\rho\int_{t_0}^{s}\alpha(r)\mathrm{d}r\Big)\alpha(s)\mathrm{d}s\\
&=\frac{\rho\mu}{\rho}\Big[\exp\Big(\rho\int_{t_0}^{t}\alpha(r)\mathrm{d}r\Big)-1\Big]\le\tilde{H}(t)=\frac{\rho\mu}{\rho}\exp\Big(\rho\int_{t_0}^{t}\alpha(r)\mathrm{d}r\Big).
\end{aligned}\tag{21}
$$

This then leads to

$$
\begin{aligned}
&\sup_{-2\tau\le\theta\le0}\mathbb{E}U(t+\theta,\varepsilon(t+\theta))\\
&\le\mathbb{E}U(t_0,\varepsilon(t_0))+H(t)+2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\Big(\int_{-2\tau}^{0}\alpha^2(s+l)\mathrm{d}l\sup_{-2\tau\le\theta\le0}\mathbb{E}U(s+\theta,\varepsilon(s+\theta))\Big)\mathrm{d}s,
\end{aligned}
$$

Here we notice that $\sup_{-2\tau\le\theta\le0}\mathbb{E}U(t+\theta,\varepsilon(t+\theta))$ makes sense for $t\ge t_0$ due to the expansion of the initial function mentioned above. Define an auxiliary function

$$
\phi(t)=\mu\int_{t_0}^{t}\Big(\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\int_{-2\tau}^{0}\alpha^2(s+l)\mathrm{d}l\Big)\sup_{-2\tau\le\theta\le0}\mathbb{E}U(s+\theta,\varepsilon(s+\theta))\mathrm{d}s,
$$

then we have

$$
\sup_{-2\tau\le\theta\le0}\mathbb{E}U(t+\theta,\varepsilon(t+\theta))\le\mathbb{E}U(t_0,\varepsilon(t_0))+H(t)+2\phi(t),\quad\phi(t_0)=0,\tag{22}
$$

and in turn we can calculate and estimate that

$$
\begin{aligned}
\phi'(t)&=\mu\Big(\exp\Big(\rho\int_{t-2\tau}^{t}\alpha(r)\mathrm{d}r\Big)\int_{-2\tau}^{0}\alpha^2(t+l)\mathrm{d}l\Big)\sup_{-2\tau\le\theta\le0}\mathbb{E}U(t+\theta,\varepsilon(t+\theta))\\
&\le\mu\Big(\exp\Big(\rho\int_{t-2\tau}^{t}\alpha(r)\mathrm{d}r\Big)\int_{-2\tau}^{0}\alpha^2(t+l)\mathrm{d}l\Big)\big[\lambda_{\max}(P)\mathbb{E}|\varepsilon_{t_0}|_\tau^2+H(t)+2\phi(t)\big].
\end{aligned}
$$

where $\mathbb{E}U(t_0,\varepsilon(t_0))\le\lambda_{\max}(P)\mathbb{E}|\varepsilon_{t_0}|_\tau^2$ is applied. The inequality can be rewritten as

$$
\begin{aligned}
&\Big[\exp\Big(-2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\int_{-2\tau}^{0}\alpha^2(s+l)\mathrm{d}l\mathrm{d}s\Big)\phi(t)\Big]'\\
&\le\Big[\exp\Big(-2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\int_{-2\tau}^{0}\alpha^2(s+l)\mathrm{d}l\mathrm{d}s\Big)\Big]\big[\lambda_{\max}(P)\mathbb{E}|\varepsilon_0|^2+H(t)\big],
\end{aligned}
$$

which can be solved as

$$
\begin{aligned}
&\exp\Big(-2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\int_{-2\tau}^{0}\alpha^2(s+l)\mathrm{d}l\mathrm{d}s\Big)\phi(t)\\
&\le\int_{t_0}^{t}\Big[\exp\Big(-2\mu\int_{t_0}^{v}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\mathrm{d}r\Big)\int_{-2\tau}^{0}\alpha^2(s+l)\mathrm{d}l\mathrm{d}s\Big)\Big]\big[\lambda_{\max}(P)\mathbb{E}|\varepsilon_0|^2+H(v)\big]\mathrm{d}v,
\end{aligned}
$$

or say

$$\phi(t) \le \exp\Big(2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{-2\tau}^{0}\alpha^2(s+l)\,\mathrm{d}l\,\mathrm{d}s\Big) \times\int_{t_0}^{t}\Big[\exp\Big(-2\mu\int_{t_0}^{v}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{-2\tau}^{0}\alpha^2(s+l)\,\mathrm{d}l\,\mathrm{d}s\Big)\Big]\big[\lambda_{\max}(P)\mathbb{E}|\varepsilon_{t_0}|_\tau^2+H(v)\big]\,\mathrm{d}v, \tag{23}$$

Then by $V(\varepsilon)\ge\lambda_{\min}(P)|\varepsilon|^2$, the definition of $U(t,\varepsilon)$ and

$$\begin{aligned}\mathbb{E}U(t,\varepsilon(t)) &\le \sup_{-2\tau\le\theta\le 0}\mathbb{E}U(t+\theta,\varepsilon(t+\theta))\\ &\le \mathbb{E}|\varepsilon_{t_0}|_\tau^2+\tilde{H}(t)+2\phi(t),\end{aligned}$$

combined with $\alpha(t)\le\mu$, we can obtain

$$\mathbb{E}|\varepsilon(t)|^2 \le \exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\big[\mathbb{E}|\varepsilon_{t_0}|_\tau^2 +\tilde{H}(t)+2\phi(t)\big]/\lambda_{\min}(P), \tag{24}$$

or say $\mathbb{E}|\varepsilon(t)|^2\le\eta_1(t)\mathbb{E}|\varepsilon_{t_0}|_\tau^2+\eta_2(t)+\eta_3(t)$, where

$$\eta_1(t) = \exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)/\lambda_{\min}(P), \tag{25}$$

$$\begin{aligned}\eta_2(t) &= \rho\mu\exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\\ &\quad\times\int_{t_0}^{t}\exp\Big(\rho\int_{t_0}^{s}\alpha(r)\,\mathrm{d}r\Big)\alpha(s)\,\mathrm{d}s/\lambda_{\min}(P)\\ &\le \rho\mu/(\rho\lambda_{\min}(P)),\end{aligned} \tag{26}$$

by a simple calculation

$$\begin{aligned}\eta_3(t) &= \exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\\ &\quad\times\exp\Big(2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\\ &\quad\times\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\\ &\quad\times\int_{t_0}^{t}\Big[\exp\Big(-2\mu\int_{t_0}^{v}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\\ &\quad\times\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\Big]\\ &\quad\times\big[\lambda_{\max}(P)E|\varepsilon_{t_0}|_\tau^2+H(v)\big]\,\mathrm{d}v/\lambda_{\min}(P).\end{aligned} \tag{27}$$

For convenience in further analysis, we denote $\eta_3(t)=\eta_{31}(t)+\eta_{32}(t)$, where

$$\begin{aligned}\eta_{31}(t) &= \exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\exp\Big(2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\\ &\quad\times\int_{t_0}^{t}\Big[\exp\Big(-2\mu\int_{t_0}^{v}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\Big]\lambda_{\max}(P)E|\varepsilon_{t_0}|_\tau^2\,\mathrm{d}v/\lambda_{\min}(P),\\ \eta_{32}(t) &= \exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\exp\Big(2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\\ &\quad\times\int_{t_0}^{t}\Big[\exp\Big(-2\mu\int_{t_0}^{v}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\Big]H(v)\,\mathrm{d}v/\lambda_{\min}(P).\end{aligned}$$

Under our assumptions, we have $\lim_{t\to+\infty}\eta_1(t)=0$. Now we estimate $\limsup_{t\to+\infty}\eta_{32}(t)$ and $\limsup_{t\to+\infty}\eta_{31}(t)$ which will lead to the estimate $\limsup_{t\to+\infty}\eta_3(t)$. Assume that $\mathbb{E}|\varepsilon_{t_0}|_\tau^2>0$.

Obviously if $\int_{t_0}^{\infty}\alpha^2(s)\,\mathrm{d}s=0$, then we $\alpha(t)=0$ almost everywhere, which is a trivial case, and it is unnecessary to consider this case. Now we denote $\int_{t_0}^{\infty}\alpha^2(s)\,\mathrm{d}s=h_1(>0)$, then there is an instant $T>t_0$ such that $\int_{t_0}^{t}\alpha^2(s)\,\mathrm{d}s\ge h_1/2$ for $t\ge T$. Accordingly, $H(t)\ge\rho h_1/2=:h$.

At the same time, with $0\le\alpha\le\mu$, exchanging the order of the iterated integrals, we have

$$\begin{aligned}&\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\\ &\le \mu\,\mathrm{e}^{2\tau\rho\mu}\int_{t_0-2\tau}^{t}\int_{t_0\vee l}^{t\wedge(l+2\tau)}\alpha(l)\,\mathrm{d}s\,\mathrm{d}l\\ &\le \mu\,\mathrm{e}^{2\tau\rho\mu}\int_{t_0-2\tau}^{t}\int_{l}^{(l+2\tau)}\alpha(l)\,\mathrm{d}s\,\mathrm{d}l\\ &= 2\tau\mu\,\mathrm{e}^{2\tau\rho\mu}\int_{t_0-2\tau}^{t}\alpha(l)\,\mathrm{d}l.\end{aligned}$$

Denote

$$\limsup_{t\to+\infty}\exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big) \times\exp\Big(4\tau\mu\,\mathrm{e}^{2\tau\rho\mu}\int_{t_0-2\tau}^{t}\alpha(l)\,\mathrm{d}l\Big)=\varpi,$$

which exists obviously, then we have

$$\limsup_{t\to+\infty}\exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big) \times\exp\Big(2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\le\varpi.$$

By the L'Hôpital's rule, we can easily show that

$$\begin{aligned}\varpi &\le \limsup_{t\to+\infty}\exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\exp\Big(4\tau\mu\,\mathrm{e}^{2\tau\rho\mu}\int_{t_0-2\tau}^{t}\alpha(l)\,\mathrm{d}l\Big)\\ &= \limsup_{t\to+\infty}\frac{\exp\Big(4\tau\mu\,\mathrm{e}^{2\tau\rho\mu}\int_{t_0-2\tau}^{t}\alpha(l)\,\mathrm{d}l\Big)4\tau\mu\,\mathrm{e}^{2\tau\rho\mu}\alpha^2(t)}{\rho\,\mathrm{e}^{\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s}\alpha(t)} = (4\tau\mathrm{e}^{2\tau\rho\mu}/\rho)\varpi,\end{aligned}$$

which leads to $\varpi = 0$ due to $4\tau\mathrm{e}^{2\tau\rho\mu}/\rho < 1$ by the choice of τ. Thus we have

$$\exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\exp\Big(2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big) < 1,$$

for sufficient large t. We assume that this holds just for $t \ge T$.

By all these and the notation $\mathrm{cond}(P) = \lambda_{\max}(P)/\lambda_{\min}(P)$ we can derive that

$$\begin{aligned}&\lambda_{\max}(P)\mathbb{E}|\varepsilon_{t_0}|_\tau^2\eta_{32}(t) - h\eta_{31}(t)\\ \ge\ &\mathrm{cond}(P)\mathbb{E}|\varepsilon_{t_0}|_\tau^2\exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\exp\Big(2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\\ &\times\Big\{\Big[\int_{t_0}^{\mathrm{T}}\exp\Big(\int_{t_0}^{\mathrm{T}}\exp\Big(-2\mu\int_{t_0}^{v}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)[H(v)-h]\,\mathrm{d}v\Big\}\\ \ge\ &-h\,\mathrm{cond}(P)\mathbb{E}|\varepsilon_{t_0}|_\tau^2\exp\Big(-\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\exp\Big(2\mu\int_{t_0}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\\ &\times\int_{t_0}^{\mathrm{T}}\exp\Big(-2\mu\int_{t_0}^{v}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\,\mathrm{d}v \ge -\chi(T),\end{aligned}$$

where

$$\chi(T) = h\int_{t_0}^{\mathrm{T}}\exp\Big(-2\mu\int_{t_0}^{v}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\,\mathrm{d}v,$$

this leads to

$$\eta_{31}(t) \le h^*\eta_{32}(t) + \chi(T)/h,\ t \ge T, \tag{28}$$

where $h^* = \lambda_{\max}(P)\mathbb{E}|\varepsilon_{t_0}|_\tau^2/h$. Now, with the notation for $H(t)$, we rewrite $\eta_{32}(t)$ as

$$\eta_{32}(t) = \frac{\rho\int_{t_0}^{t}\Big[\int_{t_0}^{v}\exp\Big(\rho\int_{t_0}^{s}\alpha(r)\,\mathrm{d}r\Big)\alpha^2(s)\,\mathrm{d}s\exp\Big(2\mu\int_{v}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s\Big)\Big]\,\mathrm{d}v}{\exp\Big(\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\lambda_{\min}(P)}.$$

Similarly we also have

$$\mu\int_{v}^{t}\exp\Big(\rho\int_{s-2\tau}^{s}\alpha(r)\,\mathrm{d}r\Big)\int_{s-2\tau}^{s}\alpha^2(l)\,\mathrm{d}l\,\mathrm{d}s \le 4\tau\mu\,\mathrm{e}^{2\tau\rho\mu}\int_{v}^{t}\alpha(l)\,\mathrm{d}l,$$

then we can estimate $\eta_{32}(t) \le \bar{\eta}_{32}(t)$, where

$$\bar{\eta}_{32}(t) = \frac{\rho\int_{t_0}^{t}\Big[\int_{t_0}^{v}\exp\Big(\rho\int_{t_0}^{s}\alpha(r)\,\mathrm{d}r\Big)\alpha^2(s)\,\mathrm{d}s\exp\Big(4\tau\mu\,\mathrm{e}^{2\tau\rho\mu}\int_{v}^{t}\alpha(l)\,\mathrm{d}l\Big)\Big]\,\mathrm{d}v}{\exp\Big(\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\lambda_{\min}(P)}.$$

Denote $\limsup_{t\to+\infty}\bar{\eta}_{32}(t) = \nu$, then by the L'Hôpital's rule again, then we can derive that

$$\begin{aligned}\nu &= \limsup_{t\to+\infty}\frac{\rho\int_{t_0}^{t}\exp\Big(\rho\int_{t_0}^{s}\alpha(r)\,\mathrm{d}r\Big)\alpha^2(s)\,\mathrm{d}s}{\exp\Big(\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\alpha(t)\lambda_{\min}(P)} + \limsup_{t\to+\infty}\frac{\rho\int_{t_0}^{t}\Big[\int_{t_0}^{v}\exp\Big(\rho\int_{t_0}^{s}\alpha(r)\,\mathrm{d}r\Big)\alpha^2(s)\,\mathrm{d}s\exp\Big(4\tau\mu\,\mathrm{e}^{2\tau\rho\mu}\int_{v}^{t}\alpha(l)\,\mathrm{d}l\Big)\alpha(t)\Big]\,\mathrm{d}v}{\rho\exp\Big(\rho\int_{t_0}^{t}\alpha(s)\,\mathrm{d}s\Big)\alpha(t)\lambda_{\min}(P)}\\ &\le \frac{\rho\mu}{\rho\lambda_{\min}(P)} + 4\tau\mu\,\mathrm{e}^{2\tau\rho\mu}\nu/\rho,\end{aligned}$$

where the technique for estimating $H(t)$ is applied again for the first term, this estimate then immediately leads to $\nu \le \nu^*$, where

$$v^* = \frac{\rho\mu}{\lambda_{\min}(P)[\rho - 4\tau\mu\, e^{2\tau\rho\mu}]}, \tag{29}$$

namely we have $\limsup_{t\to+\infty} \bar{\eta}_{32}(t) \le v^*$, thus we can obtain $\limsup_{t\to+\infty} \eta_{32}(t) \le v^*$ and $\limsup_{t\to+\infty} \eta_{31}(t) \le h^* v^* + \chi(T)/h$. Finally we have $\limsup_{t\to+\infty} \eta_3(t) \le (h^*+1)v^* + \chi(T)/h$.

Based on the above analysis we finally gain the estimate

$$\begin{aligned} &\limsup_{t\to+\infty} \mathbb{E}|\varepsilon(t)|^2 \\ &\le \rho\mu/(\rho\lambda_{\min}(P)) + (h^*+1)v^* + \chi(T)/h, \end{aligned} \tag{30}$$

as required, this completes the proof.

Remark 4 Traditional research focuses on achieving consensus in infinite time. However, finite-time control technique has became one of the most interesting topic in control theory. Finite-time consensus means that the trajectories of states can be controlled to approach zero in a finite time and to keep them there then after. The convergence time can be shortened via finite-time control. In addition, as shown in Ref. [20], the finite-time control techniques have demonstrated better robustness and disturbance rejection properties. On the other hand, the finite-time control could be realized by non-smooth control techniques such as sliding mode control, switching control and adaptive state-feedback control. Thus, the proposed method could be further extended to finite-time control problem, and next we would carry out the research on this subject.

4. Simulation example

In this section a numerical example is presented to illustrate the proposed control protocol. Consider a leader-following multi-agent system consisting of a leader and 3 followers. Assume the undirected communication topology is fixed, which is described in Fig. 1.

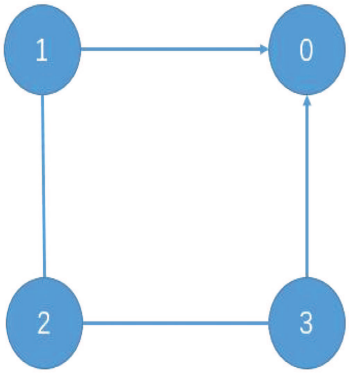


Fig. 1. (color online) The topology of a multi-agent system.

The Laplacian matrix L and the leader adjacency matrix B are given by

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 1 \end{bmatrix},$$

$$H = L + B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

The noises intensity is assumed to be $\rho_{ij} = 1$ when $a_{ij} = 1, i = 1,2,3, j = 0,1,2,3$, and the initial conditions are $x(t) = (2,1,-1)^{\mathrm{T}}, v(t) = (-2,-2,2)^{\mathrm{T}}, t \in [-\tau, 0]$. According to the equation $H^{\mathrm{T}}\bar{P} + \bar{P}H = I_n$, we can obtain the matrix

$$\bar{P} = \begin{bmatrix} 0.5379 & 0.5758 & 0.0227 \\ 0.5758 & 1.1667 & 0.0909 \\ 0.0227 & 0.0909 & 0.2500 \end{bmatrix}, \quad P = \begin{bmatrix} 0.5379 & 0.5758 & 0.0227 & -0.1076 & -0.1152 & -0.0045 \\ 0.5758 & 1.1667 & 0.0909 & -0.1152 & -0.2333 & -0.0182 \\ 0.0227 & 0.0909 & 0.2500 & -0.0045 & -0.0182 & -0.0500 \\ -0.1076 & -0.1152 & -0.0045 & 0.5379 & 0.5758 & 0.0227 \\ -0.1152 & -0.2333 & -0.0182 & 0.5758 & 1.1667 & 0.0909 \\ -0.0045 & -0.0182 & -0.0500 & 0.0227 & 0.0909 & 0.2500 \end{bmatrix}.$$

By the condition inequality (14), we can easily calculate the parameter $\lambda_{\max}(\bar{P}) = 1.5147$. Then the control gain $k > 2.6297$. In the control protocol (4) we choose $\alpha(t) = 1/(t+1)$, $\gamma = 0.8$, $k = 5$, $\mu = 1$, the time delay as $\tau = 10^{-4}$ s. In addition, We introduce the quantity $E(t) = \sqrt{\sum_{i=1}^{N} \|x_i(t) - x_0(t)\|^2/N}$, which is used to measure the quality of the bounded estimation of convergence. We can easily calculate the bounded of convergence $E(t) \le 0.3826$. Then we can choose the acceleration as $a_0(t) = \sin t$, the states errors are shown in Fig. 2 and states graph in Fig. 3, respectively. It shows that the stochastic bounded tracking consensus is reached in mean square in a noisy measurement with the designed protocol (4).

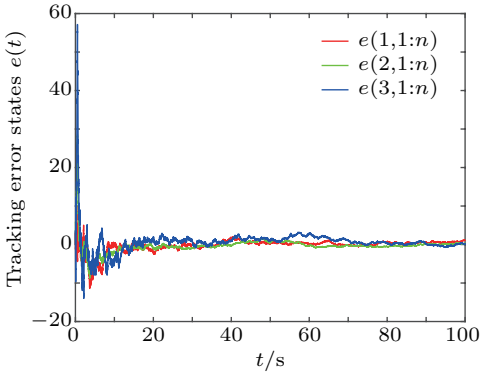


Fig. 2. (color online) The states errors of agents in multi-agent system.

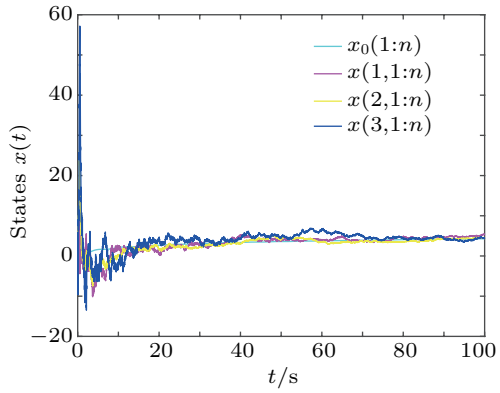


Fig. 3. (color online) The states of agents in multi-agent system.

5. Conclusions

By combining the tools of algebraic graph theory, matrix theory, and stochastic analysis, we have investigated the stochastic bounded tracking consensus of multi-agent systems with time-delays in a noisy environment and obtained the sufficient conditions on stochastic bounded consensus. It is illustrated that the stochastic bounded tracking consensus can be reached with the designed protocol under a fixed topology through a simulation example. The result can be extended to the switching information topologies cases. In the future , we can further introduce some control techniques such as pinning control, sample control, and so on, to get better convergence performance.

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