A note on the mass of Kerr–AdS black holes in the off-shell generalized ADT formalism*

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In this note, the off-shell generalized Abbott–Deser–Tekin (ADT) formalism is applied to explore the mass of Kerr–anti-de Sitter (Kerr–AdS) black holes in various dimensions within asymptotically rotating frames. The cases in four and five dimensions are explicitly investigated. It is demonstrated that the asymptotically rotating effect may make the charge non-integrable or unphysical when the asymptotic non-rotating timelike Killing vector associated with the charge is allowed to vary and the fluctuation of the metric is determined by the variation of all the mass and rotation parameters. To obtain a physically meaningful mass, it is proposed that one can let the non-rotating timelike Killing vector be fixed or perform calculations in the asymptotically static frame. The results further support that the ADT formalism is background-dependent.

Keywords: conserved charge, Kerr–AdS black hole, ADT formalism

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1. Introduction

Till now, exact rotating black hole solutions with a cosmological constant have been found in various dimensions within the context of Einstein gravity. In 1968, Carter first found a generalization of the four-dimensional (4D) rotating Kerr black hole with a cosmological constant.[1] Since this black hole has asymptotically de Sitter (dS) or anti-de Sitter (AdS) boundary conditions, it is usually called a Kerr–dS or Kerr–AdS black hole in the literature. Many years later, Hawking, Hunter and Taylor–Robinson found the five-dimensional (5D) generalization of the 4D Kerr–(A)dS black hole, as well as the solutions with just one nonzero angular momentum parameter in all dimensions.[2] In fact, the 5D Kerr–(A)dS black hole can also be regarded as a generalization of the 5D Ricci-flat rotating Myers–Perry black hole[3] including a cosmological constant. Subsequently, Gibbons, Lü, Page, and Pope further constructed the general Kerr–(A)dS black holes with arbitrary angular momenta in all higher dimensions,[4,5] which exactly satisfy the vacuum Einstein field equation with a cosmological constant. For the Kerr–AdS black holes, inspired by string theory, especially by the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, as well as black hole thermodynamics, much work has been done on their diverse aspects.[6–27]

Recently, in Ref. [28], relieving the restriction that the background spacetime satisfies the field equations, Kim, Kulkarni, and Yi proposed a quasi-local formalism of conserved charges within the framework of generic covariant pure gravity theories by constructing an off-shell ADT current to generalize the conventional on-shell Noether potential in the usual ADT formulation[29–32] to the off-shell level, as well as following the Barnich–Brandt–Compere (BBC) method[22,23,33,34] to incorporate a single parameter path in the space of solutions into their definition. Since the current and potential in the modified approach is off-shell, one may refer to it as the off-shell generalized ADT formalism. In contrast with the usual one, the off-shell generalization makes it more operable to derive the Noether potential from the corresponding current and the procedure of computation becomes more convenient to manipulate. Owing to these, it provides another fruitful way to evaluate the ADT charges for various theories of gravity. For several developments and applications of the off-shell generalized ADT formalism see the works.[35–43]

As usual, after obtaining the Kerr–AdS black hole solutions, it is of great necessity to identify their mass and angular momenta. Because of the asymptotically AdS structure, the usual Arnowitt–Deser–Misner (ADM) formalism, as well as the Komar integral, fails to produce their mass. So seeking other feasible approaches is desirable. Fortunately, some works[12–27] have succeeded in yielding the conserved charges of the Kerr–AdS black holes through a series of methods, such as the Ashtekar–Magnon–Das (AMD) formalism,[26,27] the (off-shell) ADT formulation, the BBC method, and so on. For example, in Ref. [21], the usual ADT formulation has been applied to obtain the mass of Kerr–AdS black holes in various dimensions. To realize this, the perturbation of the metric is

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set as the divergence between the metric and a fixed reference background, which is the spacetime got through letting the parameter associated with the mass in the original metric be zero. In Refs. [22] and [23], the same background was also adopted to calculate the conserved charges of the Kerr–AdS black holes via the BBC method. There the fluctuation of the metric only depends on the parameter related to the mass rather than all the solution parameters. Besides, the potential used to define the conserved charges coincides with that in the usual ADT formulation. Due to these, the BBC method adopted in Refs. [22] and [23] is essentially in accordance with the usual ADT one.

Notably, the usual ADT formulation and the BBC approach are background-dependent. Actually, in Refs. [21], [22], and [23] the reference spacetimes adopted to calculate the mass of Kerr–AdS black holes in all dimensions are non-rotating at infinity. Otherwise, both the two methods may fail to yield physical results if the timelike Killing vector associated with the mass is chosen as the usual one $\xi^{\mu}=-\delta^{\mu}_{t}$. For the off-shell generalized ADT formalism, we also wonder what will happen if it is used to deal with the mass of the Kerr–AdS black holes in an asymptotic rotating frame. On the other hand, in the light of the fact that the mass together with the angular momenta enters into the first law of thermodynamics for black holes as thermodynamical variables, the rotation parameters should be the members to determine the fluctuation of the metric for the Kerr-AdS black holes. However, in Refs. [22] and [23], only the mass parameter was regarded as a variable to fluctuate the metric, while the rotation parameters were fixed. In view of the above-mentioned issues, unlike in Refs. [21], [22], and [23], we shall utilize the off-shell generalized ADT formalism to explore the mass of the Kerr–AdS black holes in a more general manner. Namely, we do this under the conditions that the background spacetime is asymptotically rotating and the fluctuation of the metric is determined by the variation of all the mass and rotation parameters. The results demonstrate that the way that the Kerr–AdS black holes behave at infinity, rotating or not, plays a key role in determining whether the off-shell generalized ADT formalism can produce their physically meaningful mass.

The remainder of this work is arranged as follows. Sections 2 and 3 are devoted to investigating the off-shell generalized ADT mass of the Kerr–AdS black holes in four and five dimensions respectively. In Section 4, the mass of the Kerr–AdS black holes in arbitrary dimensions is calculated by generalizing the results in the 4D and 5D cases. The last section is our conclusions.

2. Mass of the 4D Kerr–AdS black hole via the off-shell generalized ADT formalism

As is well known, the theory of Einstein gravity in D dimensions is described by the Einstein–Hilbert Lagrangian

$$\mathscr{L}_{\rm EH}=\sqrt{-g}\big(R-2\Lambda\big), \tag{1}$$

with $\Lambda=-(D-1)(D-2)\ell^2/2$, where ℓ^{-1} is the radius of AdS spaces. The field equation for the gravitational field is

$$R_{\mu\nu}=-\frac{(D-1)(D-2)}{2}\ell^2 g_{\mu\nu}. \tag{2}$$

The 4D Kerr–AdS black hole[1] is an exact rotating solution with asymptotic AdS behavior of Eq. (2) in the case of $D=4$. In Boyer–Lindquist coordinates, the metric for the 4D Kerr–AdS black hole takes the form

$$\mathrm{d}s^2_{(4)}=-\frac{\Delta_{(4)}}{\Sigma_{(4)}}\Big[\mathrm{d}t-a\sin^2\theta\Big(\frac{\mathrm{d}\phi}{\Xi}-\omega_{\phi}\frac{\mathrm{d}t}{\Xi}\Big)\Big]^2+\frac{\Sigma_{(4)}}{\Delta_{(4)}}\mathrm{d}r^2+\frac{\Sigma_{(4)}}{F_{(4)}}\mathrm{d}\theta^2+\frac{F_{(4)}\sin^2\theta}{\Sigma_{(4)}}\Big[a\,\mathrm{d}t-(r^2+a^2)\Big(\frac{\mathrm{d}\phi}{\Xi}-\omega_{\phi}\frac{\mathrm{d}t}{\Xi}\Big)\Big]^2, \tag{3}$$

in which

$$\Delta_{(4)}=(r^2+a^2)(1+\ell^2r^2)-2mr,\quad \Sigma_{(4)}=r^2+a^2\cos^2\theta,\quad F_{(4)}=1-a^2\ell^2\cos^2\theta,\quad \Xi=1-\ell^2a^2, \tag{4}$$

and the constant $\omega_{\phi}=\omega_{\phi}(m,a,\ell)$, only depending on the integral parameters m and a, as well as the constant ℓ. Particularly, when $\omega_{\phi}=0$, the metric (3) becomes the usual form of the 4D Kerr–AdS black hole, which is asymptotic to AdS_4 in a rotating frame with the angular velocity $\Omega^{\infty}_{\phi}=-a\ell^2$. To guarantee that the black hole is static at infinity, observed relative to a frame that is non-rotating, one only needs to set $\omega_{\phi}=a\ell^2$.

We now go on to compute the mass of the 4D Kerr-AdS black hole via the off-shell generalized ADT method proposed in Ref. [28]. With the choice of a Killing vector ξ^{μ}, the definition of the ADT conserved charge related to the Einstein–Hilbert Lagrangian (1) is read off as

$$\delta Q_{\rm c}=\frac{1}{16\pi(D-2)!}\int_{\partial\Sigma}\sqrt{-g}\mathscr{Q}^{\mu\nu}_{ADT}\varepsilon_{\mu\nu\mu_1\mu_2\cdots\mu_{(D-2)}}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{(D-2)}}, \tag{5}$$

where the quantity $\varepsilon_{\mu\nu\mu_1\mu_2\cdots\mu_{(D-2)}}$ is the totally antisymmetric Levi–Civita tensor density, which is defined through the equation

$\varepsilon_{\mu_1\mu_2\cdots\mu_D} = D!\delta^0_{[\mu_1}\delta^1_{\mu_2}\cdots\delta^{D-1}_{\mu_D]}$, and the off-shell ADT potential is defined by

$$\mathscr{Q}^{\mu\nu}_{\rm ADT} = Q^{\mu\nu}_{\rm ADT} + \nabla^{[\mu}\delta\xi^{\nu]}, \quad Q^{\mu\nu}_{\rm ADT} = \xi_\sigma\nabla^{[\mu}h^{\nu]\sigma} - h^{\sigma[\mu}\nabla_\sigma\xi^{\nu]} + \frac{1}{2}h\nabla^{[\mu}\xi^{\nu]} - \xi^{[\mu}\nabla_\sigma h^{\nu]\sigma} + \xi^{[\mu}\nabla^{\nu]}h, \tag{6}$$

in which $h_{\mu\nu} = \delta g_{\mu\nu}$, $h = g^{\mu\nu}h_{\mu\nu}$, and $Q^{\mu\nu}_{\rm ADT}$ is the conventional (off-shell) ADT potential. The term with $\delta\xi^\mu$ comes from the variation of the off-shell Noether potential in accordance with the generalized off-shell ADT potential in Ref. [37], whose contribution to the off-shell ADT current is

$$\nabla_\nu\nabla^{[\mu}\delta\xi^{\nu]} = R^\mu_\nu\delta\xi^\nu + (1/2)\mathscr{L}_\xi\Theta^\mu,$$

where the surface term $\Theta^\mu = 2g^{\mu[\rho}\nabla_\nu h^{\nu]}_\rho$. Let us pay attention to the behavior of the charge in the case where the Killing vector ξ^μ is assumed to be fixed, that is $\delta\xi^\mu = 0$. In such a case, it has been shown that the charge $Q_{\rm c}$ associated with the linear combination of two Killing vectors preserves the linearity property in Appendix A. However, for the other cases where $\delta\xi^\mu \neq 0$, the linear property of the charge fails because of the appearance of terms proportional to the Komar integral.

For the 4D Kerr–AdS black hole described by the metric (3), the fluctuation of the spacetime $h_{\mu\nu}$ is determined by the infinitesimal change of both the parameters (m,a) rather than the single mass parameter m like in Refs. [22] and [23], that is

$$m \to m + {\rm d}m, \quad a \to a + {\rm d}a. \tag{7}$$

Under such conditions, we calculate the charge associated with the 4D timelike Killing $\xi^\mu_{(t)} = (-1,0,0,0)$, which is rotating at radial infinity. The (t,r) component of the off-shell ADT potential is computed as

$$\begin{aligned}&\sqrt{-g}\mathscr{Q}^{\rm tr}_{\rm ADT}[\xi^\mu_{(t)}] = \Upsilon_{(4)}r + \mathscr{O}\Big(\frac{1}{r}\Big) + \sin\theta(2\Xi + 3a\omega_\phi\sin^2\theta)\frac{{\rm d}m}{\Xi^2} + 3m\sin\theta[2a\ell^2\Xi + (4-3\Xi)\omega_\phi\sin^2\theta]\frac{{\rm d}a}{\Xi^3},\\ &\Upsilon_{(4)} = -a\ell^2\sin\theta(1-3\cos^2\theta)\frac{{\rm d}a}{\Xi}.\end{aligned} \tag{8}$$

Substituting the above equation into the formula (5) for the conserved charge, together with the condition that the $\Upsilon_{(4)}$ term makes no contribution to the mass since $\int_0^\pi \Upsilon_{(4)}\,{\rm d}\theta = 0$, we have

$${\rm d}Q_{\rm c}[\xi^\mu_{(t)}] = \frac{1}{\Xi^3}\Big\{\Xi\,{\rm d}m + 4ma\ell^2\,{\rm d}a + \big(\omega_\phi - a\ell^2\big)\big[a\Xi\,{\rm d}m + m(4-3\Xi)\,{\rm d}a\big]\Big\}. \tag{9}$$

A straightforward computation then shows that

$${\rm d}Q_{\rm c}[\xi^\mu_{(t)}] = {\rm d}M_{(4)} + \big(\omega_\phi + \Omega^\infty_\phi\big)\,{\rm d}J_{(4)}, \tag{10}$$

in which, the quantities $M_{(4)}$ and $J_{(4)}$ are read off as

$$M_{(4)} = \frac{m}{\Xi^2}, \quad J_{(4)} = \frac{ma}{\Xi^2}, \tag{11}$$

respectively. They are the usual mass and angular momentum of the 4D Kerr–AdS black hole presented in the literature.[15–23] On the other hand, the charge $Q_c[\xi^\mu_{(\phi)}]$ associated with the spacelike Killing vector $\xi^\mu_{(\phi)} = (0,0,0,1)$ is computed as

$${\rm d}Q_{\rm c}[\xi^\mu_{(\phi)}] = {\rm d}J_{(4)}, \tag{12}$$

from which one can obtain the angular momentum $J_{(4)}$. In fact, it essentially results from the Komar integral in Eq. (A2) in Appendix A.

Since the 4D Kerr black hole is rotating at infinity, the Killing vector corresponding to its mass should be chosen as the asymptotic non-rotating timelike Killing vector $\hat{\xi}^\mu_{(t)} = \xi^\mu_{(t)} - \big(\omega_\phi + \Omega^\infty_\phi\big)\xi^\mu_{(\phi)}$, which is just the linear combination of the Killing vector $\xi^\mu_{(t)}$ and $\xi^\mu_{(\phi)}$. With the help of Eq. (A1) in Appendix A, we have

$$\begin{aligned}{\rm d}\widetilde{M}_{(4)} = {\rm d}M_{(4)} &\\ - J_{(4)}\Big(\frac{\partial\big(\omega_\phi + \Omega^\infty_\phi\big)}{\partial m}\,{\rm d}m + \frac{\partial\big(\omega_\phi + \Omega^\infty_\phi\big)}{\partial a}\,{\rm d}a\Big),\end{aligned} \tag{13}$$

where $Q_{\rm c}[\hat{\xi}^\mu_{(t)}] = \widetilde{M}_{(4)}$. Obviously, for a general ω_ϕ, the second term on the right-hand side of Eq. (13), arising from the variation of the Killing vector $\hat{\xi}^\mu_{(t)}$, may make $\widetilde{M}_{(4)}$ be non-integrable in the case where both the parameters m and a are variables when the angular velocity $(\omega_\phi + \Omega^\infty_\phi) \neq 0$. In contrast to this, in the case where the metric perturbation $h_{\mu\nu}$ merely depends on the change of the mass parameter m, like in Refs. [22] and [23], $\widetilde{M}_{(4)}$ is integrable. Nevertheless, it is not "physically meaningful" unless $(\omega_\phi + \Omega^\infty_\phi)$ vanishes or ω_ϕ is independent of the mass parameter. Therefore, in order to guarantee that the mass $\widetilde{M}_{(4)} = M_{(4)}$, a rather effective way is to let the angular velocity at infinity vanish, namely, $\omega_\phi = -\Omega^\infty_\phi = a\ell^2$, or to fix the Killing vector $\hat{\xi}^\mu_{(t)}$, although such conditions are not required when the off-shell generalized

ADT method is applied to compute the angular momentum of the 4D Kerr-AdS black hole.

3. Mass of the 5D Kerr–AdS black hole

The 5D Kerr-AdS black hole, which behaves like an asymptotic AdS_5 space and possesses two independent rotations, is an exact solution of the field equation (2) in five dimensions. This black hole was first constructed in Ref. [2], and it can be seen as a special case of the general solutions in arbitrary higher dimensions, found in Refs. [4] and [5]. For the sake of convenience on our analysis, the metric for the 5D Kerr–AdS black hole takes the form

$$\begin{aligned} \mathrm{d}s^2_{(5)} =& -\frac{\Delta_{(5)}}{\Sigma_{(5)}}\Big[\mathrm{d}t - a(1-x^2)\mathrm{d}\hat{\phi} - bx^2\mathrm{d}\hat{\psi}\Big]^2 + \frac{\Sigma_{(5)}}{\Delta_{(5)}}\mathrm{d}r^2 + \frac{\Sigma_{(5)}}{F_{(5)}}\frac{\mathrm{d}x^2}{1-x^2} \\ &+ \frac{1+\ell^2 r^2}{r^2\Sigma_{(5)}}\Big[ab\,\mathrm{d}t - b(1-x^2)(r^2+a^2)\mathrm{d}\hat{\phi} - ax^2(r^2+b^2)\mathrm{d}\hat{\psi}\Big]^2 \\ &+ \frac{F_{(5)}}{\Sigma_{(5)}}(1-x^2)\Big[a\,\mathrm{d}t - (r^2+a^2)\mathrm{d}\hat{\phi}\Big]^2 + \frac{F_{(5)}}{\Sigma_{(5)}}x^2\Big[b\,\mathrm{d}t - (r^2+b^2)\mathrm{d}\hat{\psi}\Big]^2, \end{aligned} \tag{14}$$

where

$$\begin{aligned} &\Delta_{(5)} = \frac{(r^2+a^2)(r^2+b^2)(1+\ell^2 r^2)}{r^2} - 2m, \quad \Sigma_{(5)} = r^2 + a^2x^2 + b^2(1-x^2), \quad F_{(5)} = 1 - a^2\ell^2x^2 - b^2\ell^2(1-x^2), \\ &\Xi_a = 1 - a^2\ell^2, \quad \Xi_b = 1 - b^2\ell^2, \quad \mathrm{d}\hat{\phi} = \frac{\mathrm{d}\phi}{\Xi_a} - \omega_\phi\frac{\mathrm{d}t}{\Xi_a}, \quad \mathrm{d}\hat{\psi} = \frac{\mathrm{d}\psi}{\Xi_b} - \omega_\psi\frac{\mathrm{d}t}{\Xi_b}. \end{aligned} \tag{15}$$

In Eq. (14), both the coordinates ϕ and ψ range from $[0, 2\pi]$, while x takes the value $[0, 1]$. Significantly, when ω_ϕ and ω_ψ, which depend on the four parameters (m, a, b, ℓ), disappear and the coordinate x is substituted by a new one θ through the relation $x = \cos\theta$, the metric (14) returns to the usual form of the 5D Kerr–AdS black hole in the literature, whose angular velocities along the ϕ and ψ directions are $\Omega^\infty_\phi = -a\ell^2$ and $\Omega^\infty_\psi = -b\ell^2$ respectively at infinity. Unlike this, in the more general cases $\omega_\phi, \omega_\psi \neq 0$, the angular velocities of the metric (14) at infinity become $\omega_\phi + \Omega^\infty_\phi$ and $\omega_\psi + \Omega^\infty_\psi$.

In order to calculate the off-shell generalized ADT mass of the 5D Kerr–AdS black hole (14), the fluctuation of the metric is determined by the infinitesimal variation of the parameters (m, a, b) as

$$m \to m + \mathrm{d}m, \quad a \to a + \mathrm{d}a, \quad b \to b + \mathrm{d}b, \tag{16}$$

respectively. As before, we first consider the conserved charge associated with the Killing vector $\xi^\mu_{(5t)} = (-1, 0, 0, 0, 0)$. In the light of the off-shell ADT potential $\mathscr{Q}^{\mu\nu}_{\mathrm{ADT}}$ in Eq. (6), a complex calculation gives the following expression for the (t, r) component of the potential

$$\begin{aligned} \sqrt{-g}\mathscr{Q}^{\mathrm{tr}}_{\mathrm{ADT}}[\xi^\mu_{(5t)}] =& \frac{1}{\Xi_a\Xi_b}\big[3x\mathrm{d}m - 2b(2+\Xi_b-\Xi_a)\big(2x^3-x\big)\mathrm{d}b\big] + 2X_{(5)} + 4\omega_\phi\big(x-x^3\big)Y_{(5)} + 2X_{(5)}(a\leftrightarrow b) \\ &+ 4\omega_\psi x^3 Y_{(5)}(a\leftrightarrow b) + \Upsilon_{(5)}r^2 + \mathscr{O}\Big(\frac{1}{r^2}\Big), \end{aligned} \tag{17}$$

in which

$$\begin{aligned} X_{(5)} &= \frac{ax\mathrm{d}a}{\Xi_a^2\Xi_b}\big[3\Xi_a(\Xi_a-\Xi_b)x^4 + 2\Xi_a(1+\Xi_b-\Xi_a)x^2 - \Xi_a + 4m\ell^2\big], \\ Y_{(5)} &= \frac{1}{\Xi_a^3\Xi_b^2}\big[a\Xi_a\Xi_b\mathrm{d}m + m\Xi_b(4-3\Xi_a)\mathrm{d}a + 2abm\Xi_a\ell^2\mathrm{d}b\big], \quad \Upsilon_{(5)} = \frac{2\ell^2}{\Xi_a\Xi_b}\big(2x^3-x\big)\big(a\mathrm{d}a - b\mathrm{d}b\big). \end{aligned} \tag{18}$$

In the above equation, the quantity $\Upsilon_{(5)}$ has the property $\int_0^1 \Upsilon_{(5)}\mathrm{d}x = 0$, which implies that the contribution to the mass from the first term in Eq. (17) can be neglected. Another avenue to cancel the contribution from the r^2 term is to let the perturbation of the metric be independent of the rotation parameters a and b, like in Refs. [22] and [23]. This holds for the cancelation of the contribution from the r term in Eq. (8) as well. Further making use of the definition (5) for the off-shell ADT conserved charges, one obtains

$$\begin{aligned} \mathrm{d}Q_{\mathrm{c}}[\xi^\mu_{(5t)}] =& \mathrm{d}M_{(5)} + \big(\omega_\phi + \Omega^\infty_\phi\big)\mathrm{d}J_{(5\phi)} \\ &+ \big(\omega_\psi + \Omega^\infty_\psi\big)\mathrm{d}J_{(5\psi)}, \end{aligned} \tag{19}$$

where $M_{(5)}$, $J_{(5\phi)}$, and $J_{(5\psi)}$ are the mass and angular momenta along the ϕ and ψ directions obtained through other methods in the literature,[18–23] which are read off as

$$M_{(5)}=\frac{\pi}{4}\frac{m}{\Xi_a^2\Xi_b^2}(2\Xi_a+2\Xi_b-\Xi_a\Xi_b),$$
$$J_{(5\phi)}=\frac{\pi}{2}\frac{ma}{\Xi_a^2\Xi_b},\quad J_{(5\psi)}=\frac{\pi}{2}\frac{mb}{\Xi_a\Xi_b^2}. \tag{20}$$

By using the off-shell generalized ADT formula (5), the angular momenta $J_{(5\phi)}$ and $J_{(5\psi)}$, associated with the Killing vectors $\xi^{\mu}_{(5\phi)}=\delta^{\mu}_{\phi}$ and $\xi^{\mu}_{(5\psi)}=\delta^{\mu}_{\psi}$ respectively, can also be obtained through

$$\mathrm{d}Q_{\mathrm{c}}[\xi^{\mu}_{(5\phi)}]=\mathrm{d}J_{(5\phi)},\quad \mathrm{d}Q_{\mathrm{c}}[\xi^{\mu}_{(5\psi)}]=\mathrm{d}J_{(5\psi)}. \tag{21}$$

It was explicitly proved in Ref. [19] that the conserved charges $M_{(5)}$, $J_{(5\phi)}$, and $J_{(5\psi)}$ strictly satisfy the first law of thermodynamics. What is more, by making use of Eqs. (19), (20), and (A1), the off-shell ADT charge of the 5D Kerr–AdS black hole $Q_{\mathrm{c}}[\hat{\xi}^{\mu}_{(5t)}]$, where the asymptotic non-rotating timelike Killing vector $\hat{\xi}^{\mu}_{(5t)}=\xi^{\mu}_{(t)}-(\omega_{\phi}+\Omega^{\infty}_{\phi})\xi^{\mu}_{(5\phi)}-(\omega_{\psi}+\Omega^{\infty}_{\psi})\xi^{\mu}_{(5\psi)}$, is computed as

$$\begin{aligned}\mathrm{d}\widetilde{M}_{(5)}=\mathrm{d}M_{(5)}&-J_{(5\phi)}\left(\frac{\partial(\omega_{\phi}+\Omega^{\infty}_{\phi})}{\partial m}\mathrm{d}m+\frac{\partial(\omega_{\phi}+\Omega^{\infty}_{\phi})}{\partial a}\mathrm{d}a+\frac{\partial(\omega_{\phi}+\Omega^{\infty}_{\phi})}{\partial b}\mathrm{d}b\right)\\&-J_{(5\psi)}\left(\frac{\partial(\omega_{\psi}+\Omega^{\infty}_{\psi})}{\partial m}\mathrm{d}m+\frac{\partial(\omega_{\psi}+\Omega^{\infty}_{\psi})}{\partial a}\mathrm{d}a+\frac{\partial(\omega_{\psi}+\Omega^{\infty}_{\psi})}{\partial b}\mathrm{d}b\right),\end{aligned} \tag{22}$$

in which, $\widetilde{M}_{(5)}=Q_{\mathrm{c}}[\hat{\xi}^{\mu}_{(5t)}]$. Note that equation (22) is similar to Eq. (13) for the 4D Kerr–AdS black hole. As before, in order to make the quantity $\widetilde{M}_{(5)}$ coincide with the physically meaningful mass $M_{(5)}$, a rather simple way is to set $\omega_{\phi}=-\Omega^{\infty}_{\phi}$ and $\omega_{\psi}=-\Omega^{\infty}_{\psi}$, that is, the angular velocities of the 5D Kerr–AdS black hole vanish at infinity.

A remark is in order here. In the above discussions, we have demonstrated that it is a better choice to evaluate the mass of the 4D and 5D Kerr–AdS black holes in an asymptotically non-rotating frame. However, equation (A3) in Appendix A allows the possibility to compute their mass in a rotating frame at infinity when the asymptotic angular velocity is independent of the mass parameter. For instance, to gain the mass $M_{(4)}$ for the 4D Kerr–AdS black hole (3) with $\omega_{\phi}=0$, one can set that the variation of the metric only depends on that of the single parameter m and the Killing vector corresponding to the mass is chosen as the asymptotic non-rotating timelike Killing vector $\hat{\xi}^{\mu}_{(t)}=\xi^{\mu}_{(t)}+a\ell^2\xi^{\mu}_{(\phi)}=(-1,0,0,a\ell^2)$. The former makes the charge $Q_{\mathrm{c}}[\xi^{\mu}_{(t)}]$ be integrable. Its integral yields $Q_{\mathrm{c}}[\xi^{\mu}_{(t)}]=M_{(4)}-a\ell^2J_{(4)}$, while $Q_{\mathrm{c}}[\xi^{\mu}_{(\phi)}]=J_{(4)}$. Thus the linear combination of $Q_{\mathrm{c}}[\xi^{\mu}_{(t)}]$ and $Q_{\mathrm{c}}[\xi^{\mu}_{(\phi)}]$ gives rise to $Q_{\mathrm{c}}[\hat{\xi}^{\mu}_{(t)}]=Q_{\mathrm{c}}[\xi^{\mu}_{(t)}]+a\ell^2Q_{\mathrm{c}}[\xi^{\mu}_{(\phi)}]=M_{(4)}$. On the other hand, in the case where the parameters $\omega_{\phi},\omega_{\psi}=0$ within the metric form (14) for the 5D Kerr–AdS black hole, if the fluctuation of the metric is still determined by the parameter m rather than the ones (m,a,b), and the timelike Killing vector is set as $\hat{\xi}^{\mu}_{(5t)}=\xi^{\mu}_{(5t)}+a\ell^2\xi^{\mu}_{(5\phi)}+b\ell^2\xi^{\mu}_{(5\psi)}$, one obtains $M_{(5)}$. Besides, in the more general cases, if the Killing vectors are fixed, the terms proportional to the angular momenta in Eqs. (13) and (22) vanish, yielding the physical mass of the 4D and 5D Kerr–AdS black holes in the general asymptotically rotating frames.

4. Mass of Kerr–AdS black holes in arbitrary dimensions

In the present section, we deal with the off-shell generalized ADT mass for the general Kerr–AdS black holes in D $(D\geq 4)$ dimensions by generalizing the analysis for the 4D and 5D ones. The general Kerr–AdS black holes in $D=(2N+1)+\varepsilon$ dimensions, where $\varepsilon=1$ when D is even and $\varepsilon=0$ when D is odd, were constructed in Refs. [4] and [5]. They possess N independent rotations in N orthogonal 2-planes, characterized by the parameters a_i's $(1\leq i\leq N)$ and the azimuthal angles ϕ_i's. The metric for the D-dimensional Kerr–AdS black hole is read off as

$$\begin{aligned}\mathrm{d}s^2_{(D)}&=\mathrm{d}\bar{s}^2_{(D)}+\frac{2mU}{V(V-2m)}\mathrm{d}r^2+\frac{2m}{U}\left(W\mathrm{d}t-\sum_{i=1}^{N}a_i\mu_i^2\frac{\mathrm{d}\hat{\phi}_i}{\Xi_i}\right)^2,\\\mathrm{d}\bar{s}^2_{(D)}&=-W(1+\ell^2r^2)\mathrm{d}t^2+\sum_{i=1}^{N}\mu_i^2(r^2+a_i^2)\frac{\mathrm{d}\hat{\phi}_i^2}{\Xi_i}+\sum_{i=1}^{N+\varepsilon}(r^2+a_i^2)\frac{\mathrm{d}\mu_i^2}{\Xi_i}-\frac{\ell^2}{W(1+\ell^2r^2)}\left(\sum_{i=1}^{N+\varepsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i\mathrm{d}\mu_i\right)^2+\frac{U}{V}\mathrm{d}r^2,\end{aligned} \tag{23}$$

in which

$$U=\frac{r^2V}{1+\ell^2r^2}\sum_{i=1}^{N+\varepsilon}\frac{\mu_i^2}{r^2+a_i^2},\quad V=(r^{\varepsilon-2}+\ell^2r^{\varepsilon})\prod_{i=1}^{N}(r^2+a_i^2),\quad W=\sum_{i=1}^{N+\varepsilon}\frac{\mu_i^2}{\Xi_i},\quad \mathrm{d}\hat{\phi}_i=\mathrm{d}\phi_i-\omega_{(i)}(m,a_j,\ell)\mathrm{d}t,\quad \Xi_i=1-a_i^2\ell^2, \tag{24}$$

and the coordinates μ_i's obey the constraint $\sum_{i=1}^{N+\varepsilon}\mu_i^2=1$. The metric (23) describes rotating black holes in an asymptotically rotating frame with angular velocities $\Omega_{(i)}^{\infty}=\omega_{(i)}$. It becomes the one in Eq. (4.2) of Ref. [19] when the parameters $\omega_{(i)}$'s vanish, that is, the Kerr–AdS black holes are non-rotating at infinity.

We turn our attention to evaluating the off-shell generalized ADT mass of the Kerr–AdS black holes in D dimensions. As is shown in both the 4D and 5D cases, we choose the asymptotic non-rotating timelike Killing vector associated with the mass as the one $\hat{\xi}^{\mu}_{(Dt)}=\xi^{\mu}_{(Dt)}-\sum_i\Omega^{\infty}_{(i)}\xi^{\mu}_{(i)}$, where the timelike Killing vector $\xi^{\mu}_{(Dt)}=-\delta^{\mu}_{t}$ and the spacelike Killing vectors $\xi^{\mu}_{(i)}=\delta^{\mu}_{\phi_i}$, while the perturbations of the metric rely on the infinitesimal changes of all the parameters $(m,a_1,\ldots,a_N)$ through

$$m\to m+\mathrm{d}m,\quad a_i\to a_i+\mathrm{d}a_i\quad(i=1,\ldots,N). \tag{25}$$

Under these conditions, it is proposed that the variation of the off-shell generalized ADT charge $Q_{\mathrm{c}}[\hat{\xi}^{\mu}_{(Dt)}]$ for the D-dimensional Kerr–AdS black hole described by Eq. (23) takes the following form

$$\mathrm{d}Q_{\mathrm{c}}[\hat{\xi}^{\mu}_{(Dt)}]=\mathrm{d}M_{(D)}-\sum_{i=1}^{N}\left(\frac{\partial\Omega^{\infty}_{(i)}}{\partial m}\mathrm{d}m+\sum_{j=1}^{N}\frac{\partial\Omega^{\infty}_{(i)}}{\partial a_j}\mathrm{d}a_j\right)J_{(i)}, \tag{26}$$

where

$$M_{(D)}=\frac{\mathscr{A}_{D-2}}{4\pi}\frac{m}{\prod_j\Xi_j}\left(\sum_{i=1}^{N}\frac{1}{\Xi_i}-\frac{1-\varepsilon}{2}\right),$$
$$J_{(i)}=\frac{\mathscr{A}_{D-2}}{4\pi}\frac{ma_i}{\Xi_i\prod_j\Xi_j}. \tag{27}$$

In the above equation, $\mathscr{A}_{D-2}$ is the volume of the unit $(D-2)$ sphere. $M_{(D)}$ and $J_{(i)}$ are the physical mass and angular momenta in the literature.[18–23] $J_{(i)}$'s, which can result from the Komar integrals, coincide with the off-shell generalized ADT charges $Q_{\mathrm{c}}[\xi^{\mu}_{(i)}]$. It should be emphasized that the following equation

$$\mathrm{d}Q_{\mathrm{c}}[\xi^{\mu}_{(Dt)}]=\mathrm{d}M_{(D)}+\sum_{i=1}^{N}\Omega^{\infty}_{(i)}\mathrm{d}J_{(i)} \tag{28}$$

is utilized in order to obtain Eq. (26). In fact, due to Eq. (A1) in Appendix A, both equations (26) and (28) are equivalent.

Equation (26) can be regarded as the generalization of Eqs. (13) and (22) in D dimensions. By adopting Eq. (B2) in Appendix B to calculate the (t,r) component of the off-shell ADT potential, this equation has been checked to hold in $D\leq 7$ cases. Furthermore, to make $Q_{\mathrm{c}}[\hat{\xi}^{\mu}_{(Dt)}]=M_{(D)}$, one has to cancel the contribution arising from the variation of the asymptotic velocities $\Omega^{\infty}_{(i)}$'s. To realize this, a rather simple method is to perform the coordinate transformations $\phi_i\to\phi_i+\omega_{(i)}t$ to keep the Kerr–AdS black holes static at infinity. This is to say, as long as the symmetry related to the mass is generated by the usual timelike Killing vector $\xi^{\mu}_{(Dt)}$ and the fluctuations of the metric depend on the variation of all the mass and rotation parameters, only under the condition that these black holes are asymptotically non-rotating, can the off-shell generalized ADT formalism yield their physical mass.

For the Kerr–AdS black holes in the asymptotic rotating frame with the angular velocities $\Omega^{\infty}_{(i)}$'s irrelevant to the mass parameter m, when the perturbations of the metric are only dependent on the variation of this parameter, together with the relevant Killing vector given by the asymptotic non-rotating timelike one $\hat{\xi}^{\mu}_{(Dt)}$, one is able to obtain the mass $Q_{\mathrm{c}}[\hat{\xi}^{\mu}_{(Dt)}]=M_{(D)}$ according to Eq. (26).

Besides, if the Killing vector ξ^{μ} in Eq. (6) is supposed to be fixed, that is $\delta\xi^{\mu}=0$, giving rise to that the conventional ADT potential $Q^{\mu\nu}_{\mathrm{ADT}}$ rather than $\mathscr{Q}^{\mu\nu}_{\mathrm{ADT}}$ enters into the definition of the conserved charge Q_{c}, one observes that the terms with angular momenta in Eq. (26) disappear, yielding $\mathrm{d}Q_{\mathrm{c}}[\hat{\xi}^{\mu}_{(Dt)}]=\mathrm{d}M_{(D)}$. Its integral gives the mass of the D-dimensional Kerr–AdS black holes with general asymptotic angular velocities, which is consistent with the AMD mass.[19,26–28]

5. Summary

In this paper, we have made use of the off-shell generalized ADT formalism to compute the mass of the asymptotically-rotating Kerr–AdS black holes in all dimensions. Without the requirement to fix the Kiling vector, for 4D and 5D Kerr–AdS black holes in a general asymptotic rotating frame, when the timelike Killing vectors associated with the mass are chosen as the asymptotic non-rotating timelike one and the perturbations of the metric are determined by the variation of all the mass and angular momentum parameters, the conserved charges take the forms in Eqs. (13) and (22) respectively. A similar equation (26) holds for the Kerr–AdS black holes in diverse dimensions. All the equations show that the charges are non-integrable or unphysical because of the appearance of the general asymptotic angular velocities. Thus, in order to obtain charges coinciding with the physically meaningful mass in the literature, the spacetime has to be non-rotating at infinity. In this sense, all the results support that the (off-shell) ADT formalism depends on the reference background.

By comparison, in an asymptotic rotating frame with angular velocities that are only dependent of the rotation parameters, one can obtain the physical mass of the Kerr–AdS black

holes, however, it is required that the fluctuations of the metric merely rely on the mass parameter and the related Killing vector is set as the asymptotic non-rotating timelike one. Besides, in the more general asymptotic rotating frame, one can still get the physical mass if the asymptotic non-rotating timelike Killing vector is assumed to be fixed. In the light of the above, we suggest that one had better perform calculations in an asymptotic non-rotating frame when the off-shell generalized ADT formalism is applied to compute the mass of asymptotically AdS black holes. This also holds for the usual ADT formulation and the BBC method since they are essentially equivalent to the off-shell generalized ADT formalism for the Einstein gravity theory.

It should be emphasized that the cosmological constant is fixed in our calculations so that it is not involved in the perturbation of the metric. Otherwise, the charges of the Kerr–AdS black holes are non-integrable. To overcome this, the off-shell generalized ADT formalism has to be modified by reevaluating the contribution from the cosmological constant.[38] Accordingly, the mass of the Kerr–AdS black holes should be reconsidered with the varying cosmological constant.

Appendix A: The linear combination of the off-shell ADT charges

Suppose that there exist two Killing vectors $\xi^{\mu}_{(1)}$ and $\xi^{\mu}_{(2)}$, which correspond to the conserved charges $Q_{\rm c}[\xi^{\mu}_{(1)}]$ and $Q_{\rm c}[\xi^{\mu}_{(2)}]$ respectively. In terms of the formula (5), one sees that the variation of the conserved charge $Q_{\rm c}[c_1\xi^{\mu}_{(1)}+c_2\xi^{\mu}_{(2)}]$ associated with the linear combination of the two Killing vectors $c_1\xi^{\mu}_{(1)}+c_2\xi^{\mu}_{(2)}$ takes the form

$$\delta Q_{\rm c}[c_1\xi^{\mu}_{(1)}+c_2\xi^{\mu}_{(2)}]=c_1\delta Q_{\rm c}[\xi^{\mu}_{(1)}]+c_2\delta Q_{\rm c}[\xi^{\mu}_{(2)}]+(\delta c_1)\mathscr{Q}_{\rm K}[\xi^{\mu}_{(1)}]+(\delta c_2)\mathscr{Q}_{\rm K}[\xi^{\mu}_{(2)}], \tag{A1}$$

where the charge $\mathscr{Q}_{\rm K}$ is defined through the Komar integral

$$\mathscr{Q}_{\rm K}=\frac{1}{16\pi(D-2)!}\int_{\partial\Sigma}\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}\varepsilon_{\mu\nu\mu_1\mu_2\cdots\mu_{(D-2)}}\,{\rm d}x^{\mu_1}\wedge\cdots\wedge{\rm d}x^{\mu_{(D-2)}}. \tag{A2}$$

For another equivalent form of the Komar integral that can be conveniently extended to the higher-derivative gravity theories (see Ref. [44]). If the two constants (c_1,c_2) satisfy the restrictions $\delta c_1=0$ and $\delta c_2=0$, or more generally, the Killing vector ξ^{μ} in Eq. (6) is fixed, the charge $Q_{\rm c}[c_1\xi^{\mu}_{(1)}+c_2\xi^{\mu}_{(2)}]$ preserves the property of the linearity, namely,

$$Q_{\rm c}[c_1\xi^{\mu}_{(1)}+c_2\xi^{\mu}_{(2)}]=c_1Q_{\rm c}[\xi^{\mu}_{(1)}]+c_2Q_{\rm c}[\xi^{\mu}_{(2)}]. \tag{A3}$$

Appendix B: The (t,r) component of the off-shell ADT potential associated with the Killing vector $\xi^{\mu}=-\delta^{\mu}_{t}$

In this appendix, we shall present the explicit form of the (t,r) component of the off-shell ADT potential for a general metric ansatz that covers all known stationary and axisymmetric black hole solutions in D dimensions when the timelike Killing vector is set as $\xi^{\mu}=-\delta^{\mu}_{t}$.

For a black hole in D dimensions, there can exist $N=(D-1-\varepsilon)/2$ independent rotations, where $\varepsilon=1$ when D is even and $\varepsilon=0$ when D is odd. They correspond to N azimuthal directions ϕ^i's. In the coordinate system (t,r,θ^a,ϕ^i), where θ^a's denote the $(D-N-2)$ latitudinal angles, the general metric ansatz describing all known stationary and axisymmetric black hole solutions can be expressed as the following form

$$\begin{aligned}{\rm d}s^2=&-B\big({\rm d}t+Y_i{\rm d}\phi^i\big)^2+F\,{\rm d}r^2\\&+\tilde{g}_{ab}\,{\rm d}\theta^a\,{\rm d}\theta^b+\hat{g}_{ij}\,{\rm d}\phi^i\,{\rm d}\phi^j,\end{aligned} \tag{B1}$$

in which all the functions merely depend on the coordinate r and θ^a's. In our notation, the indices a,b,c range from 1 to $(D-N-2)$ while the indices $i,j,k=1,\ldots,N$.

By letting the Killing vector $\xi^{\mu}=-\delta^{\mu}_{t}$, in terms of the off-shell ADT potential in Eq. (6), its (t,r) component for the metric (B1), defined by $\mathscr{Q}=\mathscr{Q}^{\rm tr}_{\rm ADT}[\xi^{\mu}]$, is given by

$$\begin{aligned}\mathscr{Q}=&\frac{1}{4BF^2}\big[2F\delta(\alpha B)-\alpha\delta(BF)+2FB^2\dot{Y}^i\delta Y_i+\tilde{P}+\hat{P}\big],\\ \tilde{P}=&F\big(\dot{B}+\alpha B\big)\tilde{g}^{ab}\tilde{h}_{ab}-2BF\tilde{g}^{ab}\dot{\tilde{h}}_{ab}-B\dot{\tilde{g}}^{ab}\delta\big(F\tilde{g}_{ab}\big),\\ \hat{P}=&F\big(\dot{B}+\alpha B\big)\hat{g}^{ij}\hat{h}_{ij}-2BF\hat{g}^{ij}\dot{\hat{h}}_{ij}-B\dot{\hat{g}}^{ij}\delta\big(F\hat{g}_{ij}\big).\end{aligned} \tag{B2}$$

In the above equation,

$$\tilde{h}_{ab}=\delta\tilde{g}_{ab},\quad \hat{h}_{ij}=\delta\hat{g}_{ij},\quad \dot{Y}^i=\hat{g}^{ij}\dot{Y}_j,\quad \alpha=B\dot{Y}^iY_i, \tag{B3}$$

and the dot " $\cdot$ " denotes the partial derivative with respect to the coordinate r, for instance, $\dot{Y}_i=\partial_rY_i$, $\dot{\hat{g}}^{ij}=\partial_r\hat{g}^{ij}$, and so on. Equation (B2) simplifies the calculations of the ADT potential quite drastically and it can be widely applied to the Einstein gravity theory coupled with matter fields or not.

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