Soliton and rogue wave solutions of two-component nonlinear Schrödinger equation coupled to the Boussinesq equation*

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The nonlinear Schrödinger (NLS) equation and Boussinesq equation are two very important integrable equations. They have widely physical applications. In this paper, we investigate a nonlinear system, which is the two-component NLS equation coupled to the Boussinesq equation. We obtain the bright–bright, bright–dark, and dark–dark soliton solutions to the nonlinear system. We discuss the collision between two solitons. We observe that the collision of bright–bright soliton is inelastic and two solitons oscillating periodically can happen in the two parallel-traveling bright–bright or bright–dark soliton solution. The general breather and rogue wave solutions are also given. Our results show again that there are more abundant dynamical properties for multi-component nonlinear systems.

Keywords: multi-component NLS–Boussinesq equation, soliton solution, rogue wave solution

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1. Introduction

As is well known, the nonlinear Schrödinger equation is a classical integrable equation. It can arise in diverse areas of physics, such as water wave, nonlinear optics, biophysics, Bose–Einstein condensates, and plasma physics.[1–5] There are a lot of studies about it, like the N-soliton, breather, and rogue wave solutions.[6–9] It is found that the nonlinear Schrödinger equation coupled to some other equations can yield new nonlinear equations,[10–13] one of which is the coupled NLS–Boussinesq system,

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} - Nu = 0,\\ &\alpha(N_{xxxx} - 3(N^2)_{xx}) - \beta N_{tt} + \gamma N_{xx} + \delta(|u|^2)_{xx} = 0, \end{aligned} \tag{1}$$

where α, β, γ, and δ are real constants, and u is the complex function, and N is the real function. If $\alpha = \beta = 0$, equation (1) reduces to an NLS equation. Equation (1) is Painlevé integrable when $\beta = 3\alpha$.[9,10] In the magnetized plasma, when the propagation speed of the low-frequency wave is near the magnetosonic speed, u and N in system (1) are respectively the envelop of Langmuir wave (high-frequency wave response of the electrons) and the amplitude of the bidirectional propagation ion–acoustic wave (low-frequency wave response of the ions).[8] The bright N-soliton solution, Homoclinic orbit (Akhmediev breather) solution, a general breather and rogue wave solution of Eq. (1) were studied previously in Refs. [12], [14]–[16].

Recently, the investigation of multi-component nonlinear system has received much attention in some areas, that is, the vector NLS equation,[17–23] the coupled derivative NLS equation,[24] the multi-component Yajima–Oikawa system,[25] and so on. One has found that the multi-component nonlinear systems have indeed some novel dynamical behaviors, e.g., the two-component NLS (CNLS) equation, whose nonlinear localized wave solutions are more complex than the ones of NLS equation. It has been shown that the bright–bright solitons for the focusing CNLS equation undergo shape changing collisions with energy redistribution.[17–19] Only the focusing and mixed CNLS have rogue wave solutions,[20,21] and the de-focusing CNLS does not have bright–bright solitons.[20] Those inspire us to investigate the vector NLS–Boussinesq equation. In this paper, we focus mainly on the following two-component nonlinear Schrödinger equations coupled to the Boussinesq equation:

$$\begin{aligned} &\mathrm{i}u_{jt} + u_{jxx} - Nu_j = 0, \quad (j = 1,2),\\ &\alpha(N_{xxxx} - 3(N^2)_{xx}) - \beta N_{tt} + \gamma N_{xx}\\ &+ (\delta_1|u_1|^2 + \delta_2|u_2|^2)_{xx} = 0, \quad \delta_j = \pm 1. \end{aligned} \tag{2}$$

Our main aim is to investigate whether equation (2) has more interesting properties than Eq. (1). We will derive the bright–bright, bright–dark, and dark–dark soltion solutions for Eq. (2). We will study the collision between the two solitons, and show that the collision between bright–bright solitons is inelastic in most of the cases. One interesting phenomenon will be described, i.e., the two solitons oscillating periodically can occur in the two parallel-traveling bright–bright or bright–dark soliton solutions. The general breather including Ma breather, Akhmediev breather, and rogue wave solution will be given.

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2. Soliton solutions to Eq. (2)

Set the transformations $u_j = g_j/f, j = 1,2$, and $N = -2(\ln f)_{xx}$, where g_j are complex functions, but f is real. Then equation (2) is written as the following bilinear form:

$$\begin{aligned}&(\mathrm{i}D_t + D_x^2)g_j \cdot f = 0, \quad j = 1,2,\\ &(\alpha D_x^4 - \beta D_t^2 + \gamma D_x^2 + \lambda) f\cdot f = \delta_1 g_1\cdot g_1^* + \delta_2 g_2 \cdot g_2^*, \end{aligned} \tag{3}$$

where λ is an integration constant to be determined, $*$ denotes the complex conjugate.

2.1. Bright–bright soliton solution

We restrict the power series expansions of f, g_j about small parameter ε as

$$g_m = \sum_{j=0}^{+\infty} \varepsilon^{2j+1} g_{2j+1}^{(m)}, m = 1,2, f = 1 + \sum_{j=0}^{+\infty} \varepsilon^{2j} f_{2j}. \tag{4}$$

In the following, we set $\lambda = 0$, $\eta_j = P_j x + \mathrm{i}P_j^2 t + \eta_j^0$, P_j, η_j^0 are arbitrary complex parameters, and denote $F(l, j^*)$ by

$$F(l,j^*) = \frac{\delta_1 a_l^{(1)} a_j^{(1)*} + \delta_2 a_l^{(2)} a_j^{(2)*}}{2(P_l + P_j^*)^2(\alpha(P_l + P_j^*)^2 + \beta(P_l - P_j^*)^2 + \gamma)}. \tag{5}$$

Substituting expansion (4) into Eq. (3) and solving the linear equations of the same power of ε, one can obtain the one bright–bright soliton solution,

$$\begin{aligned} u_j &= \frac{a_1^{(j)} \mathrm{e}^{\eta_1}}{1 + F(1,1^*)\mathrm{e}^{\eta_1 + \eta_1^*}}\\ &= \frac{a_1^{(j)} \mathrm{e}^{\mathrm{i}\eta_{1\mathrm{I}} - \tau_{11}}}{2} \mathrm{sech}(\eta_{1\mathrm{R}} + \tau_{11}), \quad j = 1,2,\\ N &= -2P_{1\mathrm{R}}^2 \mathrm{sech}^2(\eta_{1\mathrm{R}} + \tau_{11}), \end{aligned} \tag{6}$$

where $\mathrm{e}^{2\tau_{11}} = F(1,1^*)$, $a_1^{(j)}$ are arbitrary complex parameters, and the subscripts R and I denote the real and imaginary parts respectively. Either of u_j is a bright soliton wave, whose traveling speed is $P_{1\mathrm{I}}$. $|u_j|$ reach their maximum values $|a_1^{(j)}|/2\mathrm{e}^{\tau_{11}}$ at the line $\eta_{1\mathrm{R}} + \tau_{11} = 0$, respectively.

When $\beta = 3\alpha$, there exist bright–bright two-soliton solutions $u_j = g_j/f, j = 1,2$, and $N = -2(\ln f)_{xx}$, where

$$\begin{aligned} g_m &= a_1^{(m)}\mathrm{e}^{\eta_1} + a_2^{(m)}\mathrm{e}^{\eta_2} + G^{(m)}(1,2,1^*)\mathrm{e}^{\eta_1+\eta_2+\eta_1^*}\\ &\quad + G^{(m)}(1,2,2^*)\mathrm{e}^{\eta_1+\eta_2+\eta_2^*}, \quad m = 1,2,\\ f &= 1 + F(1,1^*)\mathrm{e}^{\eta_1+\eta_1^*} + F(1,2^*)\mathrm{e}^{\eta_1+\eta_2^*}\\ &\quad + F(2,1^*)\mathrm{e}^{\eta_2+\eta_1^*} + F(2,2^*)\mathrm{e}^{\eta_2+\eta_2^*}\\ &\quad + F(1,1^*,2,2^*)\mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} G^{(m)}(1,2,j^*) &= (P_2 - P_1)\left(\frac{a_2^{(m)}F(1,j^*)}{P_2 + P_j^*} - \frac{a_1^{(m)}F(2,j^*)}{P_1 + P_j^*}\right),\\ &\quad m = 1,2,\\ F(1,1^*,2,2^*) &= |P_1 - P_2|^2\left(\frac{F(1,1^*)F(2,2^*)}{(P_1 + P_2^*)(P_2 + P_1^*)}\right.\\ &\quad \left. - \frac{F(1,2^*)F(2,1^*)}{(P_1 + P_1^*)(P_2 + P_2^*)}\right). \end{aligned} \tag{8}$$

To obtain the nonsingular solution, the denominator f needs to be nonzero. The expression of f can be rewritten as

$$\begin{aligned} f &= 2\mathrm{e}^{\eta_{1\mathrm{R}}+\eta_{2\mathrm{R}}}[\mathrm{e}^{\tau_{11}+\tau_{22}}\cosh(\eta_{1\mathrm{R}} - \eta_{2\mathrm{R}} + \tau_{11} - \tau_{22})\\ &\quad + |F(1,2^*)|\cos(\eta_{1\mathrm{I}} - \eta_{2\mathrm{I}} + \theta_{12})\\ &\quad + \mathrm{e}^{\tau_{1122}}\cosh(\eta_{1\mathrm{R}} + \eta_{2\mathrm{R}} + \tau_{1122})], \end{aligned}$$

where $\mathrm{e}^{2\tau_{1122}} = F(1,1^*,2,2^*), |F(1,2^*)|\mathrm{e}^{\mathrm{i}\theta_{12}} = F(1,2^*)$. Thus when $\mathrm{e}^{\tau_{11}+\tau_{22}} + \mathrm{e}^{\tau_{1122}} > |F(1,2^*)|$, the solution is nonsingular.

Next, we will analyze the asymptotic behavior of the bright–bright 2-soliton.

(i) Along the line $\eta_{1\mathrm{R}} = 0$, and supposing $P_{2\mathrm{R}}(P_{2\mathrm{I}} - P_{1\mathrm{I}})t \to +\infty$, we have

$$\begin{aligned} u_j &\sim \frac{a_1^{(j)}\mathrm{e}^{\mathrm{i}\eta_{1\mathrm{I}} - \tau_{11}}}{2}\mathrm{sech}(\eta_{1\mathrm{R}} + \tau_{11}), \quad j = 1,2,\\ N &\sim -2P_{1\mathrm{R}}^2\mathrm{sech}^2(\eta_{1\mathrm{R}} + \tau_{11}). \end{aligned} \tag{9}$$

If $P_{2\mathrm{R}}(P_{2\mathrm{I}} - P_{1\mathrm{I}})t \to -\infty$, we have

$$\begin{aligned} u_j &\sim \frac{G^{(j)}(1,2,2^*)\mathrm{e}^{\mathrm{i}\eta_{1\mathrm{I}} - \tau_{22} - \tau_{1122}}}{2}\mathrm{sech}(\eta_{1\mathrm{R}} + \tau_{1122} - \tau_{22}),\\ &\quad j = 1,2,\\ N &\sim -2P_{1\mathrm{R}}^2\mathrm{sech}^2(\eta_{1\mathrm{R}} + \tau_{1122} - \tau_{22}). \end{aligned} \tag{10}$$

(ii) Along the line $\eta_{2\mathrm{R}} = 0$, and supposing $P_{1\mathrm{R}}(P_{1\mathrm{I}} - P_{2\mathrm{I}})t \to +\infty$, we have

$$\begin{aligned} u_j &\sim \frac{a_2^{(j)}\mathrm{e}^{\mathrm{i}\eta_{2\mathrm{I}} - \tau_{22}}}{2}\mathrm{sech}(\eta_{2\mathrm{R}} + \tau_{22}), \quad j = 1,2,\\ N &\sim -2P_{2\mathrm{R}}^2\mathrm{sech}^2(\eta_{2\mathrm{R}} + \tau_{22}). \end{aligned} \tag{11}$$

If $P_{1\mathrm{R}}(P_{1\mathrm{I}} - P_{2\mathrm{I}})t \to -\infty$, we have

$$\begin{aligned} u_j &\sim \frac{G^{(j)}(1,2,1^*)\mathrm{e}^{\mathrm{i}\eta_{2\mathrm{I}} - \tau_{11} - \tau_{1122}}}{2}\mathrm{sech}(\eta_{2\mathrm{R}} + \tau_{1122} - \tau_{11}),\\ N &\sim -2P_{2\mathrm{R}}^2\mathrm{sech}^2(\eta_{2\mathrm{R}} + \tau_{1122} - \tau_{11}). \end{aligned} \tag{12}$$

It is obvious that if $a_j^{(2)} = 0, j = 1,2$, then $u_2 = 0$, the solution reduces to two bright–bright solitons of Eq. (1). If $\alpha = \beta = 0$, u_1 and u_2 in Eqs. (6) and (7) are just the bright–bright one- and two-soliton solutions of the coupled NLS system. For the bright–bright two-soliton solutions of Eq. (2), we find that the collisions of the two bright solitons in u_1 and u_2 are inelastic, provided that $a_1^{(1)}a_2^{(2)} \neq a_2^{(1)}a_1^{(2)}$. There exists an exchange of energies between the two solitons during the collision. The phenomenon is also observed in the other two-component nonlinear systems.[25–27] Figures 1(a) and 1(b) show the inelastic collision. But figures 1(d)–1(e) describe the elastic collision with $a_1^{(1)}a_2^{(2)} = a_2^{(1)}a_1^{(2)}$. If one of $a_j^{(m)}$ is zero, we can observe

an interesting phenomenon that the energy of one soliton is completely absorbed by the other one when they collide (see Fig. 2(b) where $a_2^2 = 0$). In Figs. 2(d) and 2(e), we can see that if $a_1^{(1)} = a_2^{(2)} = 0$ or $a_2^{(1)} = a_1^{(2)} = 0$, only one soliton with a shift can be observed in both components u_1 and u_2. All the energy of one soliton is gathered in u_1, the other one is gathered in u_2, and the collision is elastic.

It is interesting to note that when $P_{1\mathrm{I}} = P_{2\mathrm{I}}$, the two solitons travel along a parallel line. If $|P_{1\mathrm{R}}| \neq |P_{2\mathrm{R}}|$, the two solitons oscillate periodically (see Fig. 3). The periodic phenomenon is caused by the periodicities of the real and imaginary parts of both solitons. Since the phase of the soliton is determined by $\mathrm{e}^{\eta_{j\mathrm{I}}}$, and $\eta_{1\mathrm{I}} - \eta_{2\mathrm{I}} = (P_{1\mathrm{R}}^2 - P_{2\mathrm{R}}^2)t$, the collision period T can be expressed as $T = |2\pi/(P_{1\mathrm{R}}^2 - P_{2\mathrm{R}}^2)|$.

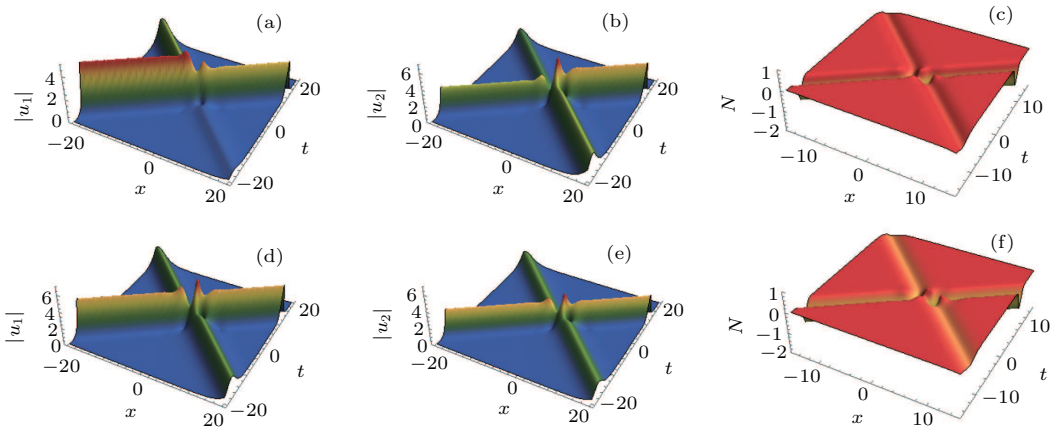


Fig. 1. (color online) $\alpha = 1$, $\beta = 3$, $\gamma = -12$, $\delta_1 = -1$, $\delta_2 = -1$, $P_1 = -1 + 0.7\mathrm{i}$, $P_2 = 0.5 - 0.5\mathrm{i}$, $\eta_1^0 = \eta_2^0 = 0$. (a)–(c) $a_1^{(1)} = 2\mathrm{i}$, $a_2^{(1)} = 3\mathrm{i}$, $a_1^{(2)} = 1 + \mathrm{i}$, $a_2^{(2)} = 2 - \mathrm{i}$. (d)–(f) $a_1^{(1)} = 2\mathrm{i}$, $a_2^{(1)} = 3\mathrm{i}$, $a_1^{(2)} = 1 + \mathrm{i}$, $a_2^{(2)} = (3 + 3\mathrm{i})/2$.

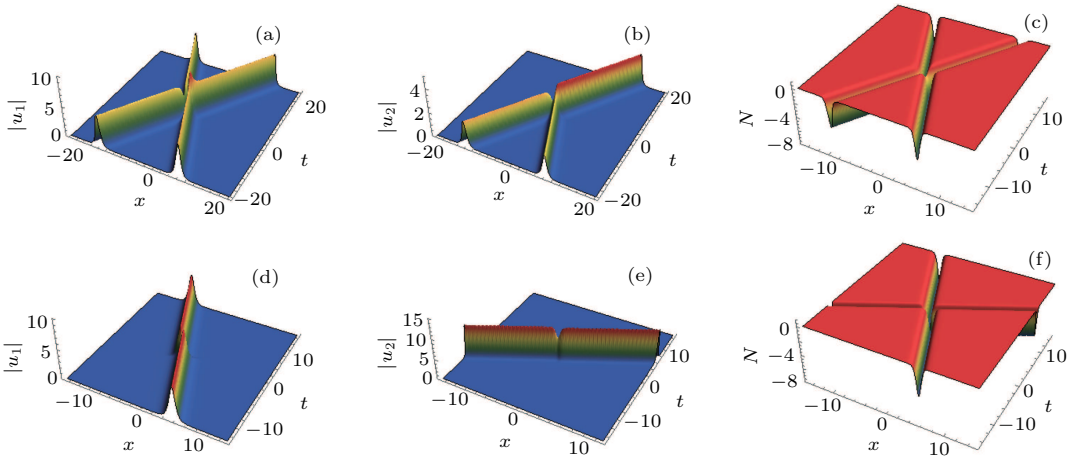


Fig. 2. (color online) $\alpha = 1$, $\beta = 3$, $\gamma = -12$, $\delta_1 = 1$, $\delta_2 = -1$, $\eta_1^0 = 0$, $\eta_2^0 = 0$, $\eta_1^0 = \eta_2^0 = 0$. (a)–(c) $P_1 = 2 + 0.3\mathrm{i}$, $P_2 = -2 - 0.2\mathrm{i}$, $a_1^{(1)} = 2$, $a_2^{(1)} = \mathrm{i}$, $a_1^{(2)} = \mathrm{i}$, $a_2^{(2)} = 0$. (d)–(f) $P_1 = 2 + \mathrm{i}$, $P_2 = -2 - 0.2\mathrm{i}$, $a_1^{(1)} = 0$, $a_2^{(1)} = \mathrm{i}$, $a_1^{(2)} = \mathrm{i}$, $a_2^{(2)} = 0$.

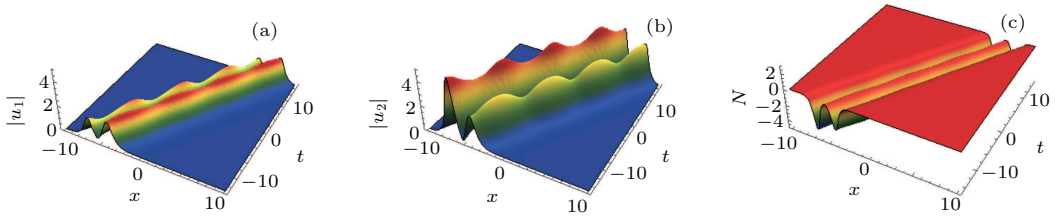


Fig. 3. (color online) $\alpha = 1$, $\beta = 3$, $\gamma = -12$, $\delta_1 = 1$, $\delta_2 = 1$, $\eta_1^0 = 0$, $\eta_2^0 = 0$, $P_1 = -1.8 + 0.2\mathrm{i}$, $P_2 = 2 + 0.2\mathrm{i}$, $\eta_1^0 = \eta_2^0 = 0$, $a_1^{(1)} = 2\mathrm{i}$, $a_2^{(1)} = \mathrm{i}$, $a_1^{(2)} = -2 - \mathrm{i}$, $a_2^{(2)} = 2 - \mathrm{i}$.

2.2. Bright–dark soliton solution

One can check that equation (2) has an exact solution $u_1 = 0$, $u_2 = \rho_2 \mathrm{e}^{\mathrm{i}(k_2 x - \omega_2 t)}$, $N = 0$, where k_2 and ω_2 are real and $\omega_2 = k_2^2$. We set a transformation $h = \rho_2 \mathrm{e}^{\mathrm{i}(k_2 x - \omega_2 t)}\tilde{h}$, and we still use h to denote $\tilde{h}$. Then the bilinear form Eq. (3) becomes

$$\begin{aligned}
&(\mathrm{i}D_t + D_x^2) g \cdot f = 0,\\
&(\mathrm{i}D_t + 2\mathrm{i}k_2 D_x + D_x^2) h \cdot f = 0,\\
&(\alpha D_x^4 - \beta D_t^2 + \gamma D_x^2 - \lambda) f \cdot f = g \cdot g^* + \delta_2 \rho_2^2 h \cdot h^*. \quad (13)
\end{aligned}$$

To obtain bright–dark soliton solutions, we suppose

$$g=\sum_{j=0}^{+\infty}\varepsilon^{2j+1}g_{2j+1},\quad h=1+\sum_{j=0}^{+\infty}\varepsilon^{2j}h_{2j},$$
$$f=1+\sum_{j=0}^{+\infty}\varepsilon^{2j}f_{2j}. \tag{14}$$

Substituting Eq. (14) into Eq.(13) and solving the linear equations of the same power of ε, one can obtain the one bright–dark soliton solution: $u_1=g/f$, $u_2=\rho_2\,\mathrm{e}^{\mathrm{i}(k_2x-\omega_2t)}h/f$, $N=-2(\ln f)_{xx}$, where $g=a_1\,\mathrm{e}^{\eta_1}$, $h=1+H(1,1^*)\,\mathrm{e}^{\eta_1+\eta_1^*}$, and $f=1+F(1,1^*)\,\mathrm{e}^{\eta_1+\eta_1^*}$, with

$$\lambda=-\delta_2\rho_2^2,\quad H(l,j^*)=-\frac{P_l-\mathrm{i}k_2}{P_j^*+\mathrm{i}k_2}F(l,j^*),$$
$$F(l,j^*)=\frac{\delta_1a_la_j^*}{2(P_l+P_j^*)^2\left(\alpha(P_l+P_j^*)^2+\beta(P_l-P_j^*)^2+\gamma+\dfrac{\delta_2\rho_2^2}{2(P_j^*+\mathrm{i}k_2)(P_l-\mathrm{i}k_2)}\right)}. \tag{15}$$

When $F(1,1^*)>0$, the solution is nonsingular and can be written as

$$u_1=\frac{a_1\,\mathrm{e}^{\mathrm{i}\eta_{1\mathrm{I}}-\tau_{11}}}{2}\mathrm{sech}(\eta_{1\mathrm{R}}+\tau_{11}),$$
$$u_2=\rho_2\,\mathrm{e}^{\mathrm{i}(k_2x-\omega_2t)}\left(\frac{1+\phi_{11}}{2}+\frac{\phi_{11}-1}{2}\tanh(\eta_{1\mathrm{R}}+\tau_{11})\right),$$
$$N=-2P_{1\mathrm{R}}^2\mathrm{sech}^2(\eta_{1\mathrm{R}}+\tau_{11}), \tag{16}$$

where $\mathrm{e}^{2\tau_{lj}}=F(l,j^*)$ and $\phi_{lj}=-(P_l-\mathrm{i}k_2)/(P_j^*+\mathrm{i}k_2)$. It is obvious that $|u_2|$ is always below the height ρ_2 of the background plane wave, and reaches its minimum value $|1+\phi_{11}|\rho_2/2$ at the line $\eta_{1\mathrm{R}}+\tau_{11}=0$.

When $\beta=3\alpha$, we obtain the bright–dark two-soliton $u_1=g/f$, $u_2=\rho_2\,\mathrm{e}^{\mathrm{i}(k_2x-\omega_2t)}h/f$, and $N=-2(\ln f)_{xx}$, where

$$\begin{aligned}
g=&\,(a_1\,\mathrm{e}^{\eta_1}+a_2\,\mathrm{e}^{\eta_2})+(G(1,2,1^*)\,\mathrm{e}^{\eta_1+\eta_2+\eta_1^*}\\
&+G(1,2,2^*)\,\mathrm{e}^{\eta_1+\eta_2+\eta_2^*}),\\
h=&\,1+(H(1,1^*)\,\mathrm{e}^{\eta_1+\eta_1^*}+H(1,2^*)\,\mathrm{e}^{\eta_1+\eta_2^*}\\
&+H(2,1^*)\,\mathrm{e}^{\eta_2+\eta_1^*}+H(2,2^*)\,\mathrm{e}^{\eta_2+\eta_2^*})\\
&+H(1,1^*,2,2^*)\,\mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*},\\
f=&\,1+(F(1,1^*)\,\mathrm{e}^{\eta_1+\eta_1^*}+F(1,2^*)\,\mathrm{e}^{\eta_1+\eta_2^*}\\
&+F(2,1^*)\,\mathrm{e}^{\eta_2+\eta_1^*}+F(2,2^*)\,\mathrm{e}^{\eta_2+\eta_2^*})\\
&+F(1,1^*,2,2^*)\,\mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*},
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
G(1,2,j^*)=&\,(P_2-P_1)\left(\frac{a_2F(1,j^*)}{P_2+P_j^*}-\frac{a_1F(2,j^*)}{P_1+P_j^*}\right),\\
F(1,1^*,2,2^*)=&\,|P_1-P_2|^2\left(\frac{F(1,1^*)F(2,2^*)}{(P_1+P_2^*)(P_2+P_1^*)}\right.\\
&\left.-\frac{F(1,2^*)F(2,1^*)}{(P_1+P_1^*)(P_2+P_2^*)}\right),\\
H(1,1^*,2,2^*)=&\,\frac{(P_1-\mathrm{i}k_2)(P_2-\mathrm{i}k_2)}{(P_1^*+\mathrm{i}k_2)(P_2^*+\mathrm{i}k_2)}F(1,1^*,2,2^*).
\end{aligned} \tag{18}$$

Next, we will analyze the asymptotic behavior of the bright–dark two-soliton.

i) Along the line $\eta_{1\mathrm{R}}=0$ and supposing $P_{2\mathrm{R}}(P_{2\mathrm{I}}-P_{1\mathrm{I}})t\to+\infty$, we have

$$u_1\sim\frac{a_1\,\mathrm{e}^{\mathrm{i}\eta_{1\mathrm{I}}-\tau_{11}}}{2}\mathrm{sech}(\eta_{1\mathrm{R}}+\tau_{11}),$$
$$u_2\sim\rho_2\,\mathrm{e}^{\mathrm{i}(k_2x-\omega_2t)}\left(\frac{1+\phi_{11}}{2}+\frac{\phi_{11}-1}{2}\tanh(\eta_{1\mathrm{R}}+\tau_{11})\right),$$
$$N\sim-2P_{1\mathrm{R}}^2\mathrm{sech}^2(\eta_{1\mathrm{R}}+\tau_{11}). \tag{19}$$

If $P_{2\mathrm{R}}(P_{2\mathrm{I}}-P_{1\mathrm{I}})t\to-\infty$,

$$\begin{aligned}
u_1\sim&\,\frac{G(1,2,2^*)\,\mathrm{e}^{\mathrm{i}\eta_{1\mathrm{I}}-\tau_{22}-\tau_{1122}}}{2}\mathrm{sech}(\eta_{1\mathrm{R}}+\tau_{1122}-\tau_{22}),\\
u_2\sim&\,\rho_2\,\mathrm{e}^{\mathrm{i}(k_2x-\omega_2t)}\frac{H(2,2^*)}{F(2,2^*)}\left(\frac{1+\phi_{11}}{2}\right.\\
&\left.+\frac{\phi_{11}-1}{2}\tanh(\eta_{1\mathrm{R}}+\tau_{1122}-\tau_{22})\right),\\
N\sim&\,-2P_{1\mathrm{R}}^2\mathrm{sech}^2(\eta_{1\mathrm{R}}+\tau_{1122}-\tau_{22}).
\end{aligned} \tag{20}$$

ii) Along the line $\eta_{2\mathrm{R}}=0$, if $P_{1\mathrm{R}}(P_{1\mathrm{I}}-P_{2\mathrm{I}})t\to+\infty$, we have

$$\begin{aligned}
u_1\sim&\,\frac{a_2\,\mathrm{e}^{\mathrm{i}\eta_{2\mathrm{I}}-\tau_{22}}}{2}\mathrm{sech}(\eta_{2\mathrm{R}}+\tau_{22}),\\
u_2\sim&\,\rho_2\,\mathrm{e}^{\mathrm{i}(k_2x-\omega_2t)}\left(\frac{1+\phi_{22}}{2}+\frac{\phi_{22}-1}{2}\tanh(\eta_{2\mathrm{R}}+\tau_{22})\right),\\
N\sim&\,-2P_{2\mathrm{R}}^2\mathrm{sech}^2(\eta_{2\mathrm{R}}+\tau_{22}).
\end{aligned} \tag{21}$$

If $P_{1\mathrm{R}}(P_{1\mathrm{I}}-P_{2\mathrm{I}})t\to-\infty$,

$$\begin{aligned}
u_1\sim&\,\frac{G(1,2,1^*)\,\mathrm{e}^{\mathrm{i}\eta_{2\mathrm{I}}-\tau_{11}-\tau_{1122}}}{2}\mathrm{sech}(\eta_{2\mathrm{R}}+\tau_{1122}-\tau_{11}),\\
u_2\sim&\,\rho_2\,\mathrm{e}^{\mathrm{i}(k_2x-\omega_2t)}\frac{H(1,1^*)}{F(1,1^*)}\left(\frac{1+\phi_{22}}{2}\right.\\
&\left.+\frac{\phi_{22}-1}{2}\tanh(\eta_{2\mathrm{R}}+\tau_{1122}-\tau_{11})\right),\\
N\sim&\,-2P_{2\mathrm{R}}^2\mathrm{sech}^2(\eta_{2\mathrm{R}}+\tau_{1122}-\tau_{11}).
\end{aligned} \tag{22}$$

One can observe that the intensities of the two solitons before and after interactions are the same (elastic) both for bright soliton in u_1 and dark soliton in u_2 (see Fig. 4). In the expression of u_1, the background parameter ρ_2 has some influences

on the amplitudes of the bright soliton. But by calculations, it is shown that the bright-component parameter a_j has no influence on the amplitude $H(1,1^*)/F(1,1^*)$ nor the phase shift $e^{\tau_{1122}-\tau_{11}-\tau_{22}}$ of the dark soliton during collisions. Thus, a_j parameters in the bright soliton solution do not affect the collision of the dark soliton. Like the bright–bright solitons, when $P_{1I} = P_{2I}$, the two solitons travel in parallel, and the periodical appearance of the two-soliton shape is shown in Fig. 5.

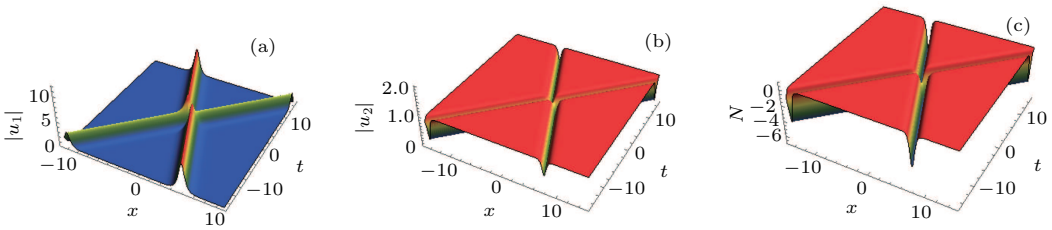


Fig. 4. (color online) Two solitons travel parallel, and the periodical appearance $\alpha = 1$, $\beta = 3$, $\gamma = -12$, $\delta_1 = 1$, $\delta_2 = 1$, $\rho_2 = 1$, $k_2 = 1$, $P_1 = 2+0.5\mathrm{i}$, $P_2 = 2-0.2\mathrm{i}$, $\eta_1^0 = \eta_2^0 = 0$, $a_1 = 2\mathrm{i}$, $a_2 = 3\mathrm{i}$.

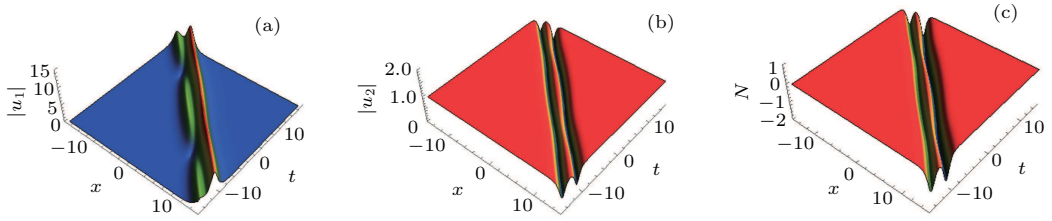


Fig. 5. (color online) $\alpha = 1$, $\beta = 3$, $\gamma = -12$, $\delta_1 = 1$, $\delta_2 = 1$, $\rho_2 = 1$, $k_2 = 1$, $P_1 = 1+0.5\mathrm{i}$, $P_2 = 1.2+0.5\mathrm{i}$, $\eta_1^0 = \eta_2^0 = 0$, $a_1 = 2\mathrm{i}$, $a_2 = 3\mathrm{i}$.

2.3. Dark–dark soltion solution

Equation (2) has the following plane wave solutions:

$$u_j = \rho_j e^{\mathrm{i}(k_j x - \omega_j t)}, \quad N = 0, \quad j = 1,2, \tag{23}$$

if $\omega_j - k_j^2 = 0, j = 1,2$. Using the transformations $u_j = \rho_j e^{\mathrm{i}(k_j x - \omega_j t)} g_j/f, j = 1,2$ and $N = -2(\ln f)_{xx}$, where g_j are complex functions, and f is a real function, system (2) is written as

$$\begin{aligned}&(\mathrm{i}D_t + 2\mathrm{i}k_j D_x + D_x^2) g_j \cdot f = 0, j = 1,2,\\ &(\gamma D_x^2 + \alpha D_x^4 - \beta D_t^2 - \lambda) f \cdot f = \delta_1 \rho_1^2 g_1 \cdot g_1^* + \delta_2 \rho_2^2 g_2 \cdot g_2^*.\end{aligned} \tag{24}$$

To obtain the dark–dark soliton solution, we suppose

$$g_m = 1 + \sum_{j=0}^{+\infty} \varepsilon^{2j} g_{2j}^{(m)}, \quad m = 1,2, \quad f = 1 + \sum_{j=0}^{+\infty} \varepsilon^{2j} f_{2j}. \tag{25}$$

Substituting these into Eq. (24) and solving the linear equations of the same power of ε, we obtain the one dark–dark soliton solution:

$$\begin{aligned}u_j &= \rho_j e^{\mathrm{i}(k_j x - \omega_j t)} \left(\frac{1 + a_1^{(j)}}{2} + \frac{a_1^{(j)} - 1}{2} \tanh\left(\frac{\eta_1}{2}\right)\right), \ j = 1,2\\ N &= -2P_1^2 \mathrm{sech}^2\left(\frac{\eta_1}{2}\right),\end{aligned} \tag{26}$$

where $\eta_j = P_j x - \Omega_j t + \eta_j^0$, P_j, Ω_j and η_j^0 are real, but $a_m^{(j)}, m = 1,2$ are complex parameters, and

$$\lambda = -\delta_1 \rho_1^2 - \delta_2 \rho_2^2, \quad \mu_j = \Omega_j - 2k_1 P_j, \quad \nu_j = \Omega_j - 2k_2 P_j,$$

$$a_j = -\frac{P_j^2 + \mathrm{i}\mu_j}{P_j^2 - \mathrm{i}\mu_j}, \quad b_j = -\frac{P_j^2 + \mathrm{i}\nu_j}{P_j^2 - \mathrm{i}\nu_j}. \tag{27}$$

The frequency Ω_j and wave number P_j satisfy the following equation:

$$\begin{aligned}&(\gamma P_j^2 + \alpha P_j^4 - \beta \Omega_j^2)(P_j^4 + (\Omega_j - 2k_1 P_j)^2)(P_j^4\\ &+ (\Omega_j - 2k_2 P_j)^2) + 2\delta_1 \rho_1^2 P_j^4 (P_j^4 + (\Omega_j - 2k_2 P_j)^2)\\ &+ 2\delta_2 \rho_2^2 P_j^4 (P_j^4 + (\Omega_j - 2k_1 P_j)^2) = 0.\end{aligned} \tag{28}$$

We can see that $|u_1|$ and $|u_2|$ are always smaller than the heights of their background plane ρ_1 and ρ_2, respectively. They reaches their minimal values $|1 + a_1^{(1)}|\rho_1/2$ and $|1 + a_1^{(2)}|\rho_2/2$ at line $\eta_1 = 0$.

When $\beta = 3\alpha$, the dark–dark two-soliton can be given by

$$\begin{aligned}u_j &= \rho_j e^{\mathrm{i}(k_j x - \omega_j t)} \left(\frac{1 + a_1^{(j)} e^{\eta_1} + a_2^{(j)} e^{\eta_2} + a_1^{(j)} a_2^{(j)} M e^{\eta_1 + \eta_2}}{1 + e^{\eta_1} + e^{\eta_2} + M e^{\eta_1 + \eta_2}} \right),\\ &\quad j = 1,2,\\ N &= -2[\ln(1 + e^{\eta_1} + e^{\eta_2} + M e^{\eta_1 + \eta_2})]_{xx},\end{aligned} \tag{29}$$

where

$$M = \frac{(P_2\Omega_1 - P_1\Omega_2)^2 + P_1^2 P_2^2 (P_1 - P_2)^2}{(P_2\Omega_1 - P_1\Omega_2)^2 + P_1^2 P_2^2 (P_1 + P_2)^2},$$

and Ω_j, P_j satisfy Eq. (28).

Now we discuss the asymptotic behavior of the dark–dark two-soliton.

(I) Along the line $\eta_1 = 0$, and supposing $(\Omega_2 - \frac{P_2}{P_1}\Omega_1)t \to +\infty$, we have

$$u_j \sim \rho_j e^{\mathrm{i}(k_j x - \omega_j t)} \left(\frac{1 + a_1^{(j)}}{2} + \frac{a_1^{(j)} - 1}{2} \tanh\left(\frac{\eta_1}{2}\right)\right), \quad j = 1,2,$$

$$N \sim -2P_1^2\mathrm{sech}^2(\eta_1/2). \tag{30}$$

If $(\Omega_2 - \frac{P_2}{P_1}\Omega_1)t \to -\infty$,

$$u_j \sim \rho_j a_2^{(j)} \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)}\left(\frac{1+a_1^{(j)}}{2} + \frac{a_1^{(j)}-1}{2} \times \tanh\left(\frac{\eta_1+\tau}{2}\right)\right), \quad j=1,2$$
$$N \sim -2P_1^2\mathrm{sech}^2\left(\frac{\eta_1+\tau}{2}\right). \tag{31}$$

(II) Along the line $\eta_2 = 0$, and $(\Omega_1 - \frac{P_1}{P_2}\Omega_2)t \to +\infty$, we have

$$u_j \sim \rho_j \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)}\left(\frac{1+a_2^{(j)}}{2} + \frac{a_2^{(j)}-1}{2}\tanh\left(\frac{\eta_2}{2}\right)\right), \quad j=1,2,$$
$$N \sim -2P_2^2\mathrm{sech}^2\left(\frac{\eta_2}{2}\right). \tag{32}$$

If $(\Omega_1 - \frac{P_1}{P_2}\Omega_2)t \to -\infty$,

$$u_j \sim \rho_j a_1^{(j)} \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)}\left(\frac{1+a_2^{(j)}}{2} + \frac{a_2^{(j)}-1}{2} \times \tanh(\frac{\eta_2+\tau}{2})\right), \quad j=1,2,$$
$$N \sim -2P_2^2\mathrm{sech}^2\left(\frac{\eta_2+\tau}{2}\right), \tag{33}$$

where $\mathrm{e}^{\tau} = M$. It is obvious that the collisions of the two dark–dark soliton solutions are elastic (see Figs. 6(a)–6(c)). But we should notice that the solutions u_1 and u_2 indeed interrelate with each other. The relationship (28) shows the fact that the background solution $u_j = \rho_j \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)}$ has influences on the amplitude, the wave velocity, and the phase of the soliton solution.

If the speed of wave $v = \Omega_1/P_1 = \Omega_2/P_2$, $P_1 + P_2 \neq 0$, and P_j $(j = 1,2)$ satisfy the equation,

$$(\gamma + \alpha P_j^2 - 3\alpha v^2)(P_j^2 + (v-2k_1)^2)(P_j^2 + (v-2k_2)^2) + 2\delta_1\rho_1^2(P_j^2 + (v-2k_2)^2) + 2\delta_2\rho_2^2(P_j^2 + (v-2k_1)^2) = 0, \tag{34}$$

then there are two parallel-traveling dark–dark solitons. Because η_1 and η_2 are both real, in this case there are not two periodically parallel-travelling dark–dark solitons, which is different from the case of two parallel-travelling bright–bright solitons (see Figs. 6(d)–6(f)).

As the application of Eq. (1) in an electron–ion plasma, equation (2) has physical application in the electron–ion plasma, where the high-frequency wave of the electrons has two types of electronic oscillations u_1 and u_2. The $|u_1|$ and $|u_2|$ represent the intensities of the waves, respectively. The term bright (dark) means that the intensity of the wave is always higher (lower) than the one of background plane wave. It should be pointed out that the components u_1 and u_2 in all of the solutions can degenerate to the solutions of the CNLS system by making $\alpha = \beta = 0$. The CNLS system has important applications in two-mode nonlinear optical fiber and Bose–Einstein condensate.

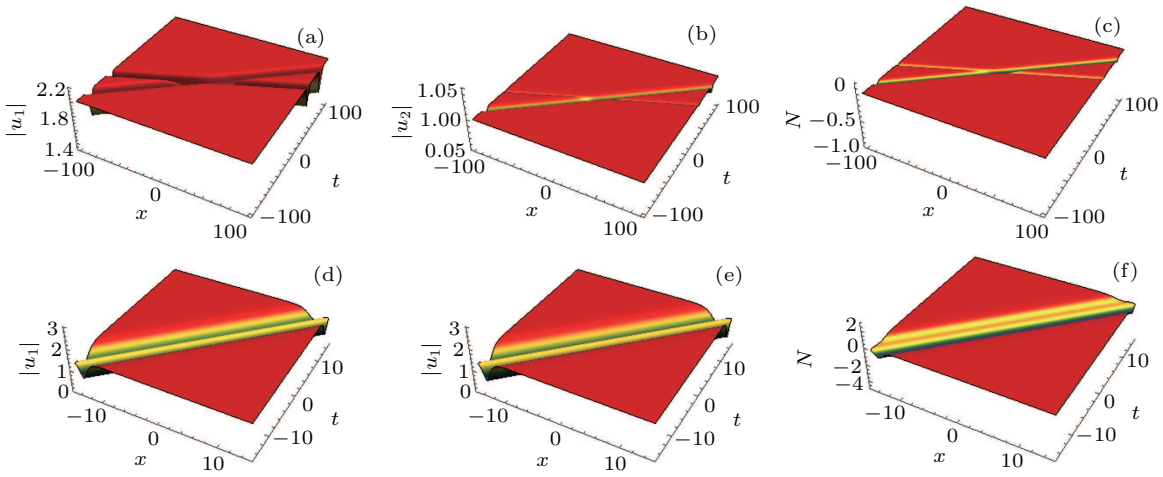


Fig. 6. (color online) $\alpha = 1$, $\beta = 3$, $\gamma = -6$, $\delta_1 = 1$, $\delta_2 = 1$, $\eta_1^0 = \eta_2^0 = 0$. (a)–(c) $\rho_1 = 1$, $\rho_2 = 1$, $k_1 = 1$, $k_2 = -1$, $P_1 = -0.3$, $P_2 = 0.5$, $\Omega_1 \approx -0.7463930$, $\Omega_2 \approx 0.6430507$. (d)–(f) $\rho_1 = \sqrt{10}/2$, $\rho_2 = \sqrt{10}/2$, $k_1 = 0$, $k_2 = 0.1$, $P_1 = \Omega_1 \approx -1.9647270$, $P_2 = \Omega_2 \approx 1.1583131$.

2.4. Breather and rogue wave solutions to Eq. (2)

The procedure of deriving the breather solution is similar to the case of dark–dark two-soliton. In the expression of dark–dark two-soliton solution Eq. (29), let P_j, Ω_j, η_j^0 be complex and $P_2 = P_1^*$, $\Omega_2 = \Omega_1^*$, then we will obtain the general breather solutions to Eq. (2). It is obvious that the condition $M > 1$ gives the nonsingular general breather solutions, which can be written in terms of trigonometric and hyperbolic functions as

$$u_j = \rho_j \mathrm{e}^{\mathrm{i}(k_j x-\omega_j t+\phi_1^{(j)})}\left(\frac{\sqrt{M}\cosh(\eta_{1\mathrm{R}}+\sigma+i\phi_1^{(j)})+\cos(\eta_{1\mathrm{I}}+i\phi_2^{(j)})}{\sqrt{M}\cosh(\eta_{1\mathrm{R}}+\sigma)+\cos(\eta_{1\mathrm{I}})}\right), \quad j=1,2,$$

$$N = -\frac{2(MP_{1\mathrm{R}}^2-P_{1\mathrm{I}}^2)+\frac{|P_1|^2}{2}(M\mathrm{e}^{\eta_{1\mathrm{R}}}\cos(\eta_{1\mathrm{I}}-2\theta)+\mathrm{e}^{-\eta_{1\mathrm{R}}}\cos(\eta_{1\mathrm{I}}+2\theta))}{(\sqrt{M}\cosh(\eta_{1\mathrm{R}}+\sigma)+\cos(\eta_{1\mathrm{I}}))^2}, \tag{35}$$

where $\sigma=\ln M/2$, $\phi_1^{(j)}=\arg(a_1^{(j)})$, $\phi_2^{(j)}=\ln|a_1^{(j)}|$, $j=1,2$, and $\theta=\arg(P_1)$. In Fig. 7, for different values of the parameters, the general breather corresponding to shape u_1 or u_2 has three or four extreme points in one period.

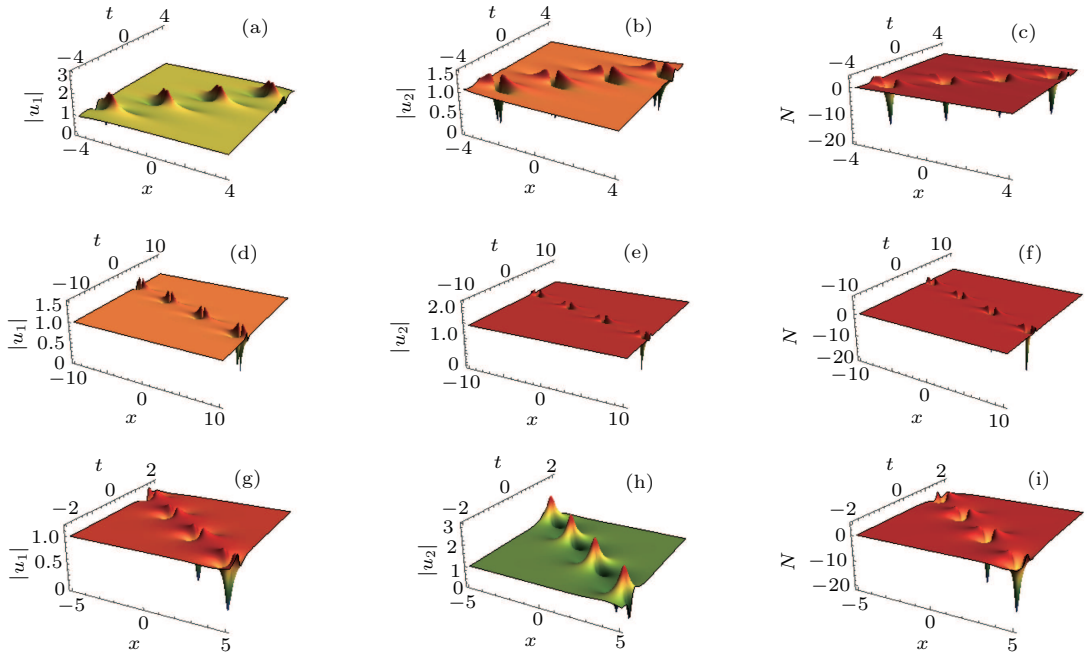


Fig. 7. (color online) General brather solutions, showing that $|u_1|$ and $|u_2|$ can have different shapes, while ϕ always present period-like dark rogue wave. (a)–(c) $\alpha=1$, $\beta=3$, $\gamma=-15$, $\delta_1=1$, $\delta_2=1$, $\rho_1=1$, $\rho_2=1$, $k_1=1$, $k_2=-2$, $P_1=-1-0.4\mathrm{i}$, $\Omega_1\approx -1.2078515+3.1351688\mathrm{i}$, and $\eta_1^0=0$. (d)–(f) $\alpha=1$, $\beta=3$, $\gamma=-15$, $\delta_1=-1$, $\delta_2=1$, $\rho_1=1$, $\rho_2=\sqrt{2}$, $k_1=2$, $k_2=-3$, $P_1=0.5+0.4\mathrm{i}$, $\Omega_1\approx -1.2557131+1.5833253\mathrm{i}$, and $\eta_1^0=0$. (g)–(i) $\alpha=1$, $\beta=3$, $\gamma=-15$, $\delta_1=-1$, $\delta_2=-1$, $\rho_1=1$, $\rho_2=1$, $k_1=3, k_2=-0.5, P_1=1+\mathrm{i}$, $\Omega_1\approx -3.0502228+3.2692445\mathrm{i}$, and $\eta_1^0=0$.

Considering the following equations:

$$(|u_1|^2)_x=0, \quad (|u_1|^2)_t=0,$$

then we have the stationary points,

$$(x,t)=\left(\frac{n\pi\Omega_{1\mathrm{R}}-\eta_{1\mathrm{I}}^0\Omega_{1\mathrm{R}}+(\eta_{1\mathrm{R}}^0+\sigma)\Omega_{1\mathrm{I}}}{P_{1\mathrm{I}}\Omega_{1\mathrm{R}}-P_{1\mathrm{R}}\Omega_{1\mathrm{I}}}, \frac{n\pi P_{1\mathrm{R}}-\eta_{1\mathrm{I}}^0 P_{1\mathrm{R}}+(\eta_{1\mathrm{R}}^0+\sigma)P_{1\mathrm{I}}}{P_{1\mathrm{I}}\Omega_{1\mathrm{R}}-P_{1\mathrm{R}}\Omega_{1\mathrm{I}}}\right), \quad n\in\mathbb{N}.$$

We classify these points as saddles or extreme points according to their second order derivatives. When $n=0$, if it is a saddle point, then the next saddle point is $n=2$. So the distance between these two points at x and t is respectively,

$$d(x)=\frac{2\Omega_{1\mathrm{R}}\pi}{P_{1\mathrm{I}}\Omega_{1\mathrm{R}}-P_{1\mathrm{R}}\Omega_{1\mathrm{I}}}, \quad d(t)=\frac{2P_{1\mathrm{R}}\pi}{P_{1\mathrm{I}}\Omega_{1\mathrm{R}}-P_{1\mathrm{R}}\Omega_{1\mathrm{I}}}.$$

The Akhmedev breather solution(see Figs. 8(a)–8(c)) which occurs due to the modulation instability[27] can be obtained by $P_{1\mathrm{R}}=0$. Under this condition, the general breather solution (29) is periodic at t and localized at x. If we restrict the imaginary part of the wave number $\Omega_{1\mathrm{R}}$ to 0, i.e., $\Omega_{1\mathrm{R}}=0$, then the general breather solution (29) is periodic at x and localized at t, which is the Ma breather (see Figs. 8(e)–8(f)).

As well know, the rogue wave can be derived from the general breather solution. Suppose $P_1=\varepsilon$, a complex small parameter, then we will have

$$\Omega_1=\varepsilon\tilde{\Omega}_1+\varepsilon^3\tilde{\Omega}_2+O(\varepsilon^3),$$

where $\tilde{\Omega}_1$ satisfies

$$2\delta_1\rho_1^2(\tilde{\Omega}_1-2k_2)^2+2\delta_2\rho_2^2(\tilde{\Omega}_1-2k_1)^2 +(\gamma-3\alpha\tilde{\Omega}_1^2)(\tilde{\Omega}_1-2k_1)^2(\tilde{\Omega}_1-2k_2)^2=0. \tag{36}$$

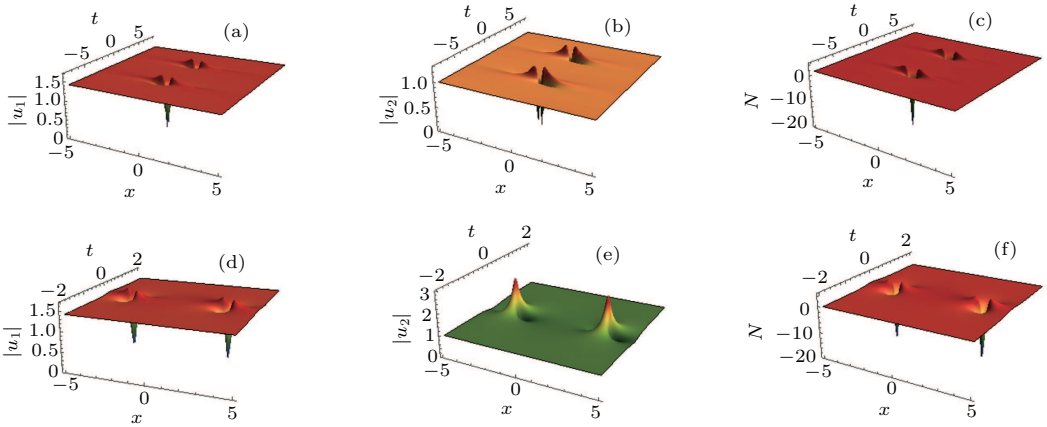


Fig. 8. (color online) (a)–(c) Akhmediev breather solutions, which are localized at x and periodic at t, with $\alpha=1$, $\beta=3$, $\gamma=-15$, $\delta_1=1$, $\delta_2=-1$, $\rho_1=\sqrt{2}$, $\rho_1=1$, $k_1=3$, $k_2=-2$, $P_1\approx-0.3169205-0.0007789\mathrm{i}$, $\Omega_1=-i$ and $\eta_1^0=0$. (d)–(f) Solutions are the Ma breather, which are localized at t and periodic at x, with $\alpha=1$, $\beta=3$, $\gamma=-15$, $\delta_1=1$, $\delta_2=-1$, $\rho_1=\sqrt{2}$, $\rho_2=1$, $k_1=3$, $k_2=-0.5$, $P_1=\mathrm{i}$, $\Omega_1\approx-3.2048558+0.0084137\mathrm{i}$, and $\eta_1^0=0$.

If $\eta_1^0=\mathrm{i}\pi$, then $\eta_1=\mathrm{i}\pi+\varepsilon(x-\tilde{\Omega}_1 t)+o(\varepsilon^2)$, and

$$\begin{aligned}
M &= 1-\frac{4|\varepsilon|^2}{(\tilde{\Omega}_1-\tilde{\Omega}_1^*)^2}+O(\varepsilon^2,\varepsilon^{*2},|\varepsilon|^2),\\
a_1^{(j)} &= 1+\frac{2\varepsilon}{\mathrm{i}(\tilde{\Omega}_1-2k_j)}-\frac{2\varepsilon^2}{(\tilde{\Omega}_1-2k_j)^2}+O(\varepsilon^2), a_2^{(j)}\\
&= 1+\frac{2\varepsilon^*}{\mathrm{i}(\tilde{\Omega}_1^*-2k_j)}-\frac{2\varepsilon^{*2}}{(\tilde{\Omega}_1^*-2k_j)^2}+O(\varepsilon^{*2}),\\
&\quad j=1,2.
\end{aligned}$$

Substituting the above formulae into the expressions of f, g_j, f_{xx}, f_x yields

$$\begin{aligned}
f &= |\varepsilon|^2\left(|x-\tilde{\Omega}_1 t|^2-\frac{4}{(\tilde{\Omega}_1-\tilde{\Omega}_1^*)^2}\right)\\
&\quad +O(\varepsilon^2,\varepsilon^{*2},|\varepsilon|^2),\\
f_{xx}f-f_x^2 &= -|\varepsilon|^4\left[(x-\tilde{\Omega}_1 t)^2+(x-\tilde{\Omega}_1^* t)^2+\frac{8}{(\tilde{\Omega}_1-\tilde{\Omega}_1^*)^2}\right]\\
&\quad +O(\varepsilon^{*j}\varepsilon^{4-j}),\\
g_j &= |\varepsilon|^2\left(|x-\tilde{\Omega}_1 t|^2+\frac{2(x-\tilde{\Omega}_1^* t)}{\mathrm{i}(\tilde{\Omega}_1-2k_j)}\right.\\
&\quad \left.+\frac{2(x-\tilde{\Omega}_1 t)}{\mathrm{i}(\tilde{\Omega}_1^*-2k_j)}-\frac{4}{(\tilde{\Omega}_1-\tilde{\Omega}_1^*)^2}-\frac{4}{|\tilde{\Omega}_1-2k_j|^2}\right)\\
&\quad +O(\varepsilon^2,\varepsilon^{*2},|\varepsilon|^2),\quad j=1,2.
\end{aligned}$$

Taking the limit $\varepsilon\to 0$, and letting $\tilde{\Omega}_1=m+\mathrm{i}n$, where m and n are real, the expression of the rational solution is written as

$$\begin{aligned}
u_j &= \rho_j \mathrm{e}^{\mathrm{i}(k_j x-\omega_j t+\phi_1^{(j)})}\left(1-4\frac{\mathrm{i}(m-2k_j)(x-mt)-\mathrm{i}n^2t+1}{((m-2k_j)^2+n^2)((x-mt)^2+n^2t^2+1/n^2)}\right),\quad j=1,2,\\
N &= \frac{4[(x-mt)^2-n^2t^2-1/n^2]}{((x-mt)^2+n^2t^2+1/n^2)^2}.
\end{aligned}\tag{37}$$

It should be noted that condition (36) is rewritten in the following form

$$\gamma-3\alpha(m^2-n^2)+2\delta_1\rho_1^2\frac{(m-2k_1)^2-n^2}{((m-2k_1)^2+n^2)^2}+2\delta_2\rho_2^2\frac{(m-2k_2)^2-n^2}{((m-2k_2)^2+n^2)^2}=0,\tag{38}$$

$$3\alpha m+2\delta_1\rho_1^2\frac{(m-2k_1)}{((m-2k_1)^2+n^2)^2}+2\delta_2\rho_2^2\frac{(m-2k_2)}{((m-2k_2)^2+n^2)^2}=0.\tag{39}$$

We thus see that for a given initial plane-wave, one can determine m, n from Eqs. (38) and (39), and then determine the rational solution (37). Furthermore, we can classify the rogue wave solution as four different types by the dynamical behavior. Solving the following equations

$$(|u_1|^2)_x=0,\quad (|u_1|^2)_t=0,$$

gives the stationary points,

$$\begin{aligned}
(x,t) &= (0,0),\quad (x,t)=\frac{A}{n^2+(m-2k_1)^2}(2k_1,1),\\
(x,t) &= \frac{B}{n(n^2+(m-2k_1)^2)}\left(m^2+n^2-2mk_1,m-2k_1\right),
\end{aligned}$$

where

$$\begin{aligned}
A &= \pm\sqrt{\frac{3(m-2k_1)^2}{n^2}-1},\\
B &= \pm\sqrt{3-\frac{(m-2k_1)^2}{n^2}}.
\end{aligned}$$

When $1/3 < (m-2k_1)^2/n^2 < 3$, there are four extreme points, otherwise there are three extreme points. The height of the central point is

$$|u_1(0,0)|^2 = \rho_1^2\left[1-\frac{4n^2}{n^2+(m-2k_1)^2}\right]^2.$$

So we can classify the rogue wave as four different types as follows.

I) If $(m-2k_1)^2/n^2 \geq 3$, there are only three stationary points. The central point is a minimal one, and it is lower than the background. In this case, a dark rogue wave is derived.

II) If $1 < (m-2k_1)^2/n^2 < 3$, there are five stationary points. The central point is a saddle one, and it is lower than the background, which is called four-petal rogue wave.

III) If $1/3 < (m-2k_1)^2/n^2 < 1$, there are five stationary points. The central point is a saddle one, and it is higher than the background. So a two-peak rogue wave is obtained.

IV) If $(m-2k_1)^2/n^2 \leq 1/3$, there are only three stationary points. The central point is a maximal one, and its height is at least twice higher than the background, which is called bright rogue wave.

While for Boussinesq equation $N(x,t)$ in Eq. (37), its structure is simple. The three stationary points for $N(x,t)$ are

$$(0,0), \left(\pm\frac{\sqrt{3}}{n},0\right),$$

and the value of $N(x,t)$ at the central(minimum) point $(0,0)$ is $-4n^2$, which is always lower then the background. The maximum points occur at $(\pm\sqrt{3}/n,0)$, and their values are both $n^2/2$. so the Boussinesq equation $N(x,t)$ in Eq. (37) has two small peaks and one deep hole in the center between two peaks, which is just the usual dark rogue wave.

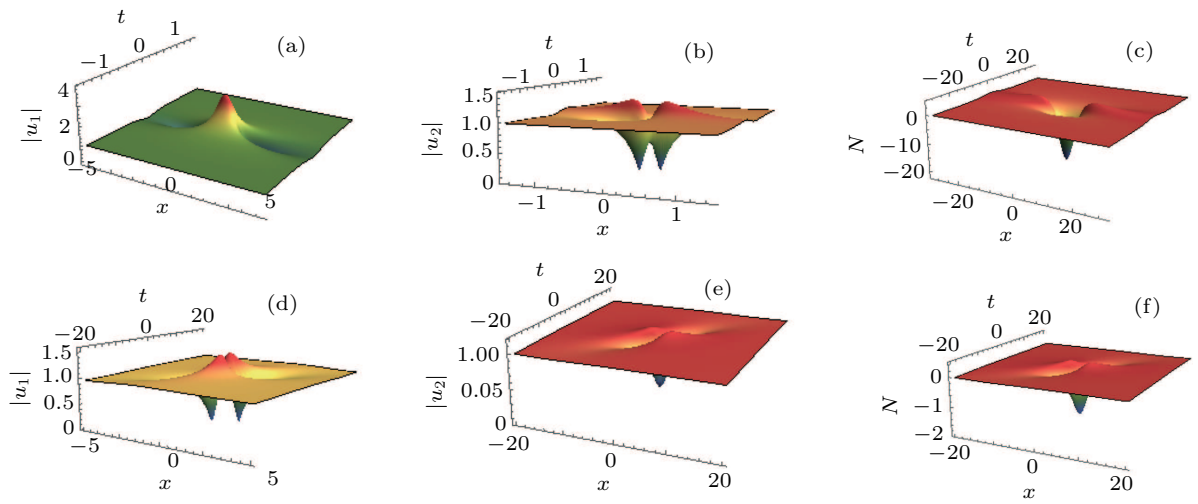


Fig. 9. (color online) $\delta_1=-1$, $\delta_2=-1$, $\rho_1=1$, $\rho_1=1$, $k_1=0$, $k_2=-2$. (a)–(c) u_1 is bright rogue wave, u_2 is a four-petal rogue wave. $\gamma=-15$, $\Omega_1\approx 0.003984461+3.152611999\mathrm{i}$. (d)–(f) u_1 is a four-petal rogue wave, u_2 is dark rogue wave, $\gamma=-10$, $\Omega_1\approx 0.6348057106+0.64629221881\mathrm{i}$.

Figure 9 shows the rogue wave with two sets of parameters. In Fig. 9(b) $|u_2|$ and in Fig. 9(d) $|u_1|$ are the rogue waves with four petals, in Fig. 9(a) $|u_1|$ is a bright rogue wave, in Fig. 9(e) $|u_2|$ is a dark rogue wave. Actually, for different values of δ_1 and δ_2, we can always find appropriate values for the left parameters to make the rogue waves of u_1 and u_2 are different combinations of bright, dark and four-petal rogue waves. It should be noted that the rogue wave solutions for NLS equation, coupled NLS equation, (2+1)-dimensional NLS equation have been investigated in Refs. [9], [28]–[35].

3. Conclusions

The bright–bright, bright–dark, and dark–dark soliton solutions to Eq. (2) are constructed. We observe that the collision between bright–bright solitons is inelastic and two solitons oscillating periodically can happen in the two parallel-traveling bright–bright or bright–dark soliton solutions. The general breather and rogue wave solutions are also given. Our results show again that there are more abundant dynamical properties for multi-component nonlinear systems.

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