# Local structure-preserving methods for the generalized Rosenau–RLW–KdV equation with power law nonlinearity*

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Local structure-preserving algorithms including multi-symplectic, local energy- and momentum-preserving schemes are proposed for the generalized Rosenau–RLW–KdV equation based on the multi-symplectic Hamiltonian formula of the equation. Each of the present algorithms holds a discrete conservation law in any time–space region. For the original problem subjected to appropriate boundary conditions, these algorithms will be globally conservative. Discrete fast Fourier transform makes a significant improvement to the computational efficiency of schemes. Numerical results show that the proposed algorithms have satisfactory performance in providing an accurate solution and preserving the discrete invariants.

**Keywords:** Rosenau-type equation, multi-symplectic conservation law, energy conservation law, structure-preserving algorithm

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## 1. Introduction

A nonlinear wave phenomenon is one of the important areas of scientific research. There are various mathematical models describing the wave behaviors. The well-known KdV equation was put forward as a model for small-amplitude long waves on the surface of water in a channel and the RLW equation was originally introduced to describe the behavior of the undular bore. Both KdV and RLW equations have been well studied theoretically and numerically.[1,2] Since the case of wave–wave and wave–wall interactions cannot be described by the KdV equation, the Rosenau equation was proposed, which describes the dynamic of a dense discrete system.[3] For the further consideration of nonlinear waves, the viscous terms $u_{xxx}$ and $u_{xxt}$ need to be included in the Rosenau equation respectively. The resulting equations are usually called Rosenau–KdV and –RLW equations, respectively. The initial boundary value problems have been studied numerically in the past few years.[4–9] In this paper, we are interested in developing numerical methods for the following generalized Rosenau–RLW–KdV (GR-RLW–KdV) equation with a power law nonlinearity

$$u_t + \kappa u_x - \sigma u_{xxt} + b u_{xxx} + \alpha u_{xxxxt} + \beta (u^p)_x = 0, \qquad (1)$$

subjected to an initial condition $u(x,0) = u_0(x),\ x \in [x_l, x_r]$ and periodic boundary conditions, where $p$ is a positive integer. The parameter $b = 0$ gives the generalized Rosenau–RLW (GR-RLW) equation and parameter $\sigma = 0$ reads the generalized Rosenau–KdV (GR-KdV) equation. Since the GR-RLW–KdV equation includes a power law nonlinearity, to simulate it well, the numerical method should have some excellent properties such as good accuracy in solution and long-term stability. To the best of our knowledge, there are few numerical methods for the equation in the literature.

Generally, the performance of a numerical method is affected by not only the accuracy of the method but also other factors such as the conservative approximation property of the method. Better solutions can be expected from numerical schemes which have effective conservative approximation properties rather than the ones which have nonconservative properties.[10–13] Some conservative methods have been proposed for the Rosenau-type equations.[5–8] The preserving discrete global conservation laws of these methods depend on the boundary conditions. That is, if the original problem does not have appropriate boundary conditions, these conservative methods cannot be applied to it. Actually, the GR-RLW–KdV equation admits local conservation laws, such as the multi-symplectic conservation law, local energy conservation law, and momentum conservation law. These conservation laws are independent of the boundary conditions and can produce richer information of the original system. Naturally, it is desired to develop numerical methods preserving the discrete version of the local conservation laws. Note that as the boundary conditions are appropriate, these methods will be globally conservative. The main purpose of this paper is to develop some local structure-preserving methods for the GR-RLW–KdV equation based on its multi-symplectic Hamiltonian formula, and carry out some numerical experiments.

To show the derivation of the schemes conveniently, we

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describe the computational grid and give some definitions of finite difference operators. Consider a mesh $x_l = x_0 < x_1 < ,\ldots, < x_{J-1} < x_J = x_r$ as a uniform partition of the solution domain $[x_l, x_r]$ by the knots $x_j$ with $h = x_{j+1} - x_j$, $j = 0, 1, \ldots, J$. Let $u_j^n$ be a numerical approximation to $u(x_j, n\tau)$ where $\tau$ is an increment in time. Define shift operators $T_x u_j^n = u_{j+1}^n$, $T_t u_j^n = u_j^{n+1}$ and finite difference operators

$$\begin{aligned}
\delta_t u_j^n &= \frac{1}{\tau}(u_j^{n+1} - u_j^n),\\
\delta_x u_j^n &= \frac{1}{h}(u_{j+1}^n - u_j^n),\\
A_x u_j^n &= \frac{1}{2}(u_j^n + u_{j+1}^n),\\
A_t u_j^n &= \frac{1}{2}(u_j^n + u_j^{n+1}).
\end{aligned}$$

For simplicity, in most of the equations we present, notation $u_j^n$ is replaced by $u$, and one of the indices is held constants, in which case, we drop it from the notation.

## 2. Structure-preserving algorithms

By introducing some intermediate variables, the GR-RLW–KdV equation can be written into a multi-symplectic form

$$Mz_t + Kz_x = \nabla S(z), \tag{2}$$

where

$$S(z) = \frac{1}{2\alpha}\phi r - uv - wv - \kappa u^2/2 - \beta u^{p+1}/(p+1)$$

is a scalar-valued smooth function, $z = (\psi, u, v, w, v, \phi, r) \in \mathbb{R}^7$, and the two skew-symmetric matrices $M = (m_{i,j})$, $K = (k_{i,j}) \in \mathbb{R}^{7\times 7}$ with non-zero entries are given by $m_{1,2} = -m_{2,1} = -1/2$, $m_{2,4} = -m_{4,2} = -\sigma/2$, $m_{4,6} = -m_{6,4} = -1/2$, $k_{1,3} = -k_{3,1} = 1$, $k_{2,4} = -k_{4,2} = b/2$, $k_{2,5} = -k_{5,2} = 1$, and $k_{4,7} = -k_{7,4} = -1/2$. The system (2) possesses three local conservation laws,[14] namely multi-symplectic conservation law (MSCL)

$$\partial_t \omega + \partial_x \chi = 0, \quad \omega = z dz \wedge M_+ \mathrm{d}z, \quad \chi = \mathrm{d}z \wedge K_+ \mathrm{d}z, \tag{3}$$

local energy conservation law (LECL)

$$\partial_t E + \partial_x F = 0, \quad E = S(z) + z_x^{\mathrm{T}} K_+ z, \quad F = -z_t^{\mathrm{T}} K_+ z, \tag{4}$$

and local momentum conservation law (LMCL)

$$\partial_t I + \partial_x G = 0, \quad G = S(z) + z_t^{\mathrm{T}} M_+ z, \quad I = -z_x^{\mathrm{T}} M_+ z, \tag{5}$$

where splitting matrices $M_+$ and $K_+$ satisfy

$$\begin{aligned}
M = M_+ + M_-, \quad & K = K_+ + K_-,\\
M_+^{\mathrm{T}} = -M_-, \quad & K_+^{\mathrm{T}} = -K_-.
\end{aligned} \tag{6}$$

It is obvious that the three conservation laws are local, i.e., they are independent of the boundary conditions. With appropriate boundary conditions such as periodic/homogeneous boundary conditions, integrating them over the spatial region gives global conservation laws. Since the local conservation laws are more essential than the global ones, we propose some methods preserving the discrete version of MSCL, LECL, and LMCL, respectively.

The concepts of the multi-symplectic structure and the multi-symplectic integrator were introduced in Ref. [14], which have wide applications.[15–19] Multi-symplectic (MS) box/Preissman integrator

$$M\delta_t A_x z + K\delta_x A_t z = \nabla S(A_t A_x z) \tag{7}$$

is very popular for the multi-symplectic Hamiltonian partial differential equation (PDE). Applying it to the multi-symplectic Hamiltonian formulation of the GR-RLW–KdV equation, and then eliminating the auxiliary variables gives a scheme

$$\delta_t A_x^4 u + \kappa A_t \delta_x A_x^3 u - \sigma \delta_t \delta_x^2 A_x^2 u + b A_t \delta_x^3 A_x u + \alpha \delta_t \delta_x^4 u + \beta \delta_x A_x^2 (A_t A_x u)^p = 0. \tag{8}$$

The scheme (8) possesses a discrete MSCL $\delta_t^+ \omega + \delta_x^+ \chi = 0$ where $\omega = \mathrm{d}A_x z \wedge M_+ \mathrm{d}A_x z$ and $\chi = \mathrm{d}A_t z \wedge K_+ \mathrm{d}A_t z$, which is a discrete version of the MSCL (3).

Next, we are interested in developing schemes possessing the discrete version of LECL (4) and LMCL (5) for the GR-RLW–KdV equation. We discretize the system (2) as follows:

$$M\delta_t A_x z + K\delta_x A_t z = \bar{\nabla}^t S(A_x T_t z, A_x z), \tag{9}$$

where $\bar{\nabla}^t$ represents a discrete gradient method in time. By some simple derivations, we prove that the scheme (9) preserves a discrete LECL

$$\begin{aligned}
&\delta_t \mathscr{E} + \delta_x \mathscr{F} = 0,\\
&\mathscr{E} = S(A_x z) + (\delta_x z)^{\mathrm{T}} K_+ A_x z,\\
&\mathscr{F} = -(\delta_t z)^{\mathrm{T}} K_+ A_t z.
\end{aligned} \tag{10}$$

Also, we make a discretization

$$M\delta_t A_x z + K\delta_x A_t z = \bar{\nabla}^x S(A_t T_x z, A_t z), \tag{11}$$

for the system (2) where $\bar{\nabla}^x$ is a discrete gradient method in space. The scheme has a discrete LMCL

$$\delta_t \mathscr{I} + \delta_x \mathscr{G} = 0,$$

$$\mathscr{G} = S(A_t z) + (\delta_t z)^{\mathrm{T}} M_+ A_t z,$$
$$\mathscr{I} = -(\delta_x z)^{\mathrm{T}} M_+ A_x z. \tag{12}$$

In this paper, the discrete gradient method is taken as the averaged vector field.[20] Applications of LECL (9) and LMCL (11) integrators to the multi-symplectic formulations of the GR-RLW–KdV equation give respectively the following local energy-preserving (LEP) scheme

$$\delta_t A_x^4 u + \kappa A_t \delta_x A_x^3 u - \sigma \delta_t \delta_x^2 A_x^2 u + b A_t \delta_x^3 A_x u + \alpha \delta_t \delta_x^4 u + \beta \delta_x A_x^2 f(A_x T_t u, A_x u) = 0, \tag{13}$$

and local momentum-preserving (LMP) scheme

$$\delta_t A_x^4 u + \kappa A_t \delta_x A_x^3 u - \sigma \delta_t \delta_x^2 A_x^2 u + b A_t \delta_x^3 A_x u + \alpha \delta_t \delta_x^4 u + \beta \delta_x A_x^2 f(A_t T_x u, A_t u) = 0, \tag{14}$$

where

$$f(s,r) = \frac{1}{p+1}(s^{p+1} - r^{p+1})/(s-r).$$

**Remark 1**

(i) The difference of the schemes (8), (13), and (14) is in the nonlinear term.

(ii) Different discrete gradient methods in scheme (9) or scheme (11) may lead to different local energy-/momentum-preserving methods.

## 3. Conservative properties and fast computation

Assume that the GR-RLW–KdV equation is imposed on periodic or homogeneous boundary conditions. Then the LEP and LMP schemes conserve discrete global energy and momentum conservation laws respectively, i.e., $\mathscr{E}^n = \mathscr{E}^{n-1} = \cdots = \mathscr{E}^0$ and $\mathscr{I}^n = \mathscr{I}^{n-1} = \cdots = \mathscr{I}^0$, where

$$\mathscr{E}^n = h \sum_{j=0}^{J-1} \left( \frac{\beta}{p+1}(A_x u)^{p+1} + \frac{\kappa}{2}(A_x u)^2 - \frac{b}{2}(\delta_x u)^2 \right),$$
$$\mathscr{I}^n = h \sum_{j=0}^{J-1} \left( (A_x u)^2 + \sigma(\delta_x u)^2 + \alpha(\delta_x^2 u_{j-1})^2 \right).$$

By summing the schemes (8), (13), and (14) over all index $j$s, we prove that the schemes have the same discrete mass conservative law $\mathscr{M}^n = \cdots = \mathscr{M}^0$ where $\mathscr{M}^n = h \sum_{j=0}^{J-1} u$.

For the GR-RLW–KdV equations with periodic boundary conditions, all the proposed structure-preserving algorithms can be written in a form of

$$Au^{n+1} = Bu^n + N(u^{n+1}, u^n), \tag{15}$$

where both $A$ and $B$ are constant circulant matrices. The nonlinear system (15) can be solved by a fix-point iteration method with initial iterative vector $u^{n+1,(0)} = u^n$, i.e.,

$$Au^{n+1,(k+1)} = Bu^n + N(u^{n+1,(k)}, u^n),\ k = 1, 2, \ldots. \tag{16}$$

Note that the constant circulant matrix $A$ can be represented as

$$A = \mathbb{F}^{-1} \Lambda \mathbb{F}, \tag{17}$$

where $\mathbb{F}$ is the matrix of discrete Fourier transform coefficients with entries given by $[\mathbb{F}]_{r,s} = (\mathrm{e}^{-\mathrm{i}2\pi/J})^{rs}$, $\mathrm{i} = \sqrt{-1}$, $[\mathbb{F}^{-1}]_{r,s} = (1/J)(\mathrm{e}^{\mathrm{i}2\pi/J})^{rs}$ and the diagonal matrix $\Lambda = \mathbb{F} f$ with $f = ([A]_{1,1}, [A]_{2,1}, \ldots, [A]_{J,1})^{\mathrm{T}}$. The property (17) is very helpful to improve the computational efficiency of equation (16), i.e., it can be solved explicitly.

## 4. Numerical results

In the following, we carry out some numerical experiments to show the performance of the proposed schemes in term of the accuracy of solution and preservation of the discrete invariants.

**Example 1:** (GR-RLW equation)

As the parameter $b = 0$, the GR-RLW–KdV equation degenerates to the GR-RLW equation which has an analytic solitary wave solution

$$u(x,t,x_0) = A \operatorname{sech}^{4/(p-1)}(B(x - Ct - x_0)),$$

where

$$A = \exp\left( \frac{1}{p+1} \ln \frac{(p+3)(3p+1)(p+1)}{2(p^2+3)(p^2+4p+7)} \right),$$
$$B = \frac{p-1}{\sqrt{4p^2+8p+20}},$$
$$C = \frac{p^4+4p^3+14p^2+20p+25}{p^4+4p^3+10p^2+12p+21},$$

and $x_0$ represents the initial center of the single solitary wave. Consider the problem with initial data $u_0(x) = u(x,0,0)$, $x \in [-60, 120]$, and periodic boundary conditions. The maximal errors in solution for different values of $p$ at time $T = 40$ with $\tau = 0.1$ and $h = 0.5$ are reported in Table 1. One can see that calculated numerical results of the present schemes are satisfactory and the result of the LMP scheme is smallest. Figure 1 shows the performance of the proposed schemes in preserving the discrete invariants as simulating the GR-RLW–KdV equation with $p = 16$. The results confirm that the MS scheme is mass-preserving, the LEP scheme is mass- and energy-preserving, and the LMP scheme is mass- and momentum-preserving.

**Table 1.** Comparison of maximal error norms for the GR-RLW equation with different values of $p$: $\tau = 0.1$ and $h = 0.5$.

| $p$ | MS/$10^{-3}$ | LEP/$10^{-3}$ | LMP/$10^{-3}$ | Ref. [6]/$10^{-2}$ |
|---|---|---|---|---|
| 4 | 1.58 | 1.54 | 1.42 | 2.79 |
| 8 | 5.24 | 5.13 | 2.83 | 2.95 |
| 16 | 6.21 | 6.11 | 3.84 | 2.25 |

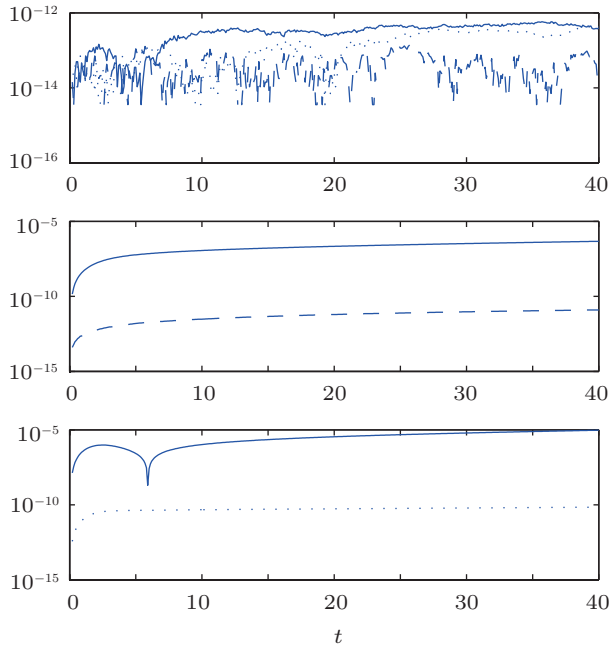



**Fig. 1.** (color online) The changes in discrete mass (top), energy (mid), and momentum (bottom). Solid line: the MS scheme; dashed line: the LEP scheme; dotted line: the LMP scheme.

**Example 2:** (GR-KdV and GR-RLW-KdV equations)

The GR-RLW–KdV equation possesses a solitary wave solution $u(x,t,x_0) = A\,\mathrm{sech}^{4/(p-1)}(B(x-Ct-x_0))$, where $x_0$ is an arbitrary constant,

$$A = \Big\{(D-\kappa\alpha(p^2+2p+5))(p+3)(3p+1)\cdot[16\beta(p+1)\big(-8\sigma\alpha b(p+1)^2+\alpha(D-\kappa\alpha(p^2+2p+5))(p^2+2p+5)\big)]^{-1}\Big\}^{1/(p-1)},$$

$$B = \frac{p-1}{p+1}\sqrt{\frac{D-\kappa\alpha(p^2+2p+5)}{32\alpha b}},$$

$$C = \frac{b(p-1)^2}{-\sigma(p-1)^2+4\alpha B^2(p^2+2p+5)},$$

$$D = \sqrt{\kappa^2\alpha^2(p^2+2p+5)^2+16\alpha b(b+\kappa\sigma)(p+1)^2}.$$

We consider first the parameter $\sigma = 0$. The GR-RLW–KdV equation reduces to the GR-KdV equation. Our simulations are done with $\kappa = 1$, $\alpha = 1$, $b = 1$, $\beta = 1/2$, initial data $u_0(x) = u(x,0,0)$, $x \in [-70,100]$, $\tau = 0.1$, and $h = 0.1$ up to $T = 40$. The results listed in Table 2 indicate that the proposed schemes provide satisfactory solutions and preserve the discrete invariants well throughout computations. Also, the results of the present schemes are better than those of the existing method.[7]

**Table 2.** The results for the simulation of the GR-KdV equation with different values of $p$.

| $p$ | Method | $L_\infty/10^{-4}$ | $\lvert\mathscr{M}^n-\mathscr{M}^0\rvert$ | $\lvert\mathscr{I}^n-\mathscr{I}^0\rvert$ | $\lvert\mathscr{E}^n-\mathscr{E}^0\rvert$ |
|---|---|---|---|---|---|
| 2 | MS | 2.39 | $7.9\times10^{-9}$ | $8.6\times10^{-8}$ | $8.5\times10^{-9}$ |
| | LEP | 2.38 | $7.9\times10^{-9}$ | $1.3\times10^{-8}$ | $3.1\times10^{-9}$ |
| | LMP | 2.38 | $7.9\times10^{-9}$ | $9.0\times10^{-9}$ | $8.9\times10^{-10}$ |
| | Ref. [7] | 0.188 | $1.2\times10^{-5}$ | – | $1.5\times10^{-4}$ |
| 16 | MS | 0.161 | $2.6\times10^{-8}$ | $4.0\times10^{-8}$ | $9.7\times10^{-8}$ |
| | LEP | 0.148 | $2.6\times10^{-8}$ | $1.6\times10^{-7}$ | $1.9\times10^{-8}$ |
| | LMP | 0.147 | $2.6\times10^{-8}$ | $1.1\times10^{-7}$ | $6.8\times10^{-9}$ |

Second, we solve the GR-RLW–KdV equation with $\kappa = 1$, $\sigma = 1$, $b = 1$, $\alpha = 1$, $\beta = 1/2$, various values of $p$, and initial data $u_0(x) = u(x,0,0)$ over region $[-50,50]$. The simulations are carried out with $\tau = 0.05$ and $h = 0.1$ up to $T = 30$. Figure 2 displays the variation of the maximal errors in solution as the value of $p$ increases. One can see that the error of every scheme grows as the value of $p$ increases. One can also see that the maximal error norms of the three schemes are almost the same for the case of $p = 2$, but as the value of $p$ increases, the difference in maximal error norms becomes much more apparent. To show the ability of the present schemes in preserving the discrete invariants, we simulate the problem with $p = 20$. The changes in mass, energy, and momentum are all in the scale of $1.0\times10^{-8}$ (the results are not shown here).

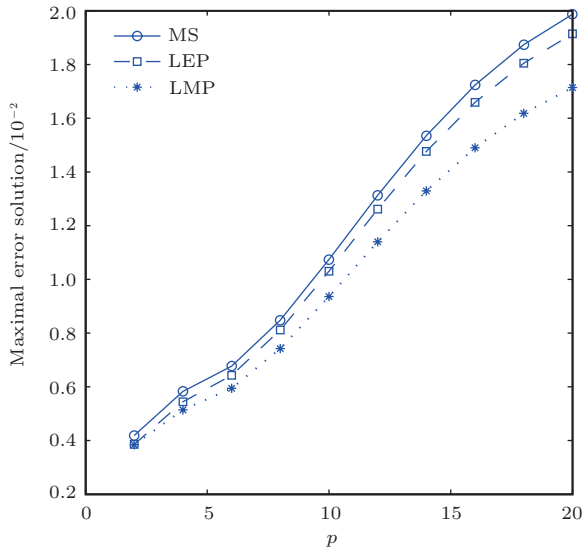



**Fig. 2.** (color online) The maximal error in solution at $T = 30$ versus the values of $p$.

## 5. Concluding remarks

The GR-RLW–KdV equation can be reformed into a multi-symplectic Hamiltonian formula, which admits MSCL, LECL, and LMCL. These conservation laws are local, i.e., they are independent of boundary conditions. For the problem with appropriate boundary conditions, one can obtain global conservation laws. Therefore, the local conservation

laws can produce more information of the original problem than the global ones. In this work, we proposed some local structure-preserving algorithms preserving the discrete version of MSCL, LECL, and LMCL respectively for the GR-RLW–KdV equation. With periodic/homogeneous boundary conditions, the present schemes are globally conservative. By the discrete fast Fourier transform, the computational efficiency of the proposed schemes is improved. Some numerical experiments are carried out to test the numerical performance of the present schemes.

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