Three-dimensional MHD flow over a shrinking sheet: Analytical solution and stability analysis

Sumaira Afzal[1,†], Saleem Asghar[1,2], and Adeel Ahmad[1]

[1] *Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Chak Shahzad, 44000 Islamabad, Pakistan*
[2] *Department of Mathematics, King Abdul Aziz University, Jeddah, Saudi Arabia*

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The magnetohydrodynamic (MHD) steady and unsteady axisymmetric flows of a viscous fluid over a two-dimensional shrinking sheet are addressed. The mathematical analysis is carried out in the presence of a large magnetic field. The steady state problem results in a singular perturbation problem having an infinite domain singularity. The secular term appearing in the solution is removed and a two-term uniformly valid solution is derived using the Lindstedt–Poincaré technique. This asymptotic solution is validated by comparing it with the numerical solution. The solution for the unsteady problem is also presented analytically in the asymptotic limit of large magnetic field. The results of velocity profile and skin friction are shown graphically to explore the physical features of the flow field. The stability analysis of the unsteady flow is made to validate the asymptotic solution.

Keywords: steady and unsteady magnetohydrodynamic flows, two-directional shrinking sheet, exact and asymptotic solutions, stability analysis

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1. Introduction

The study of fluid flow over a continuously stretching/shrinking sheet has received considerable attention due to its applications in different fields of engineering and industry. The underlying stretching velocities could be linear, polynomial, and exponential. In polymer industry, stretching of the surfaces has a definite impact on the quality of the sheet. A few examples of such processes are extrusion of polymers, aerodynamic extrusion of plastic sheeting, and condensation processes of metallic plates (see Altan *et al.*[1] and Fisher[2]). Crane[3] was first to find the exact analytical solution of the boundary layer flow of an incompressible viscous fluid over a continuously stretching sheet. Some further research of the stretching phenomena can be found in Refs. [4]–[13]. The shrinking of the sheet is a relatively new phenomenon in fluid flow with additional complexities. These complexities arise due to the additional vortices generated by the shrinking of the sheet towards a point. The study of shrinking phenomenon was initiated by Miklavcic and Wang.[14]

The magnetic field has been successfully employed to control convection for electrically conducting fluids such as high-temperature melts of metals, alloys, and semiconductors. Thus, magnetohydrodynamic (MHD) flows induced by the stretching sheet have been a subject of great importance. Sarpkaya[15] pointed out that the electrically conducting fluids such as nuclear fuel slurries, mercury amalgams, liquid metals, biological fluids, and certain lubrication oils and greases have many uses in the presence of a magnetic field. In the context of these applications, Pavlov[16] studied a steady flow due to stretching of the surface of an electrically conducting fluid in the presence of a uniform magnetic field and provided an accurate closed-form solution. Merkin and Kumaran[17] considered shrinking of the sheet for the unsteady MHD flow of viscous fluid.

Generally, stretching (shrinking) in one dimension is a two-dimensional boundary value problem (bvp); whereas, a bidirectional stretching (shrinking) gives rise to a three-dimensional bvp, the subject of the present study. Notwithstanding the physical importance, the analytic solution of the three-dimensional Navier–Stokes equations also has great intrinsic value.[18–20] In this backdrop, there are a number of observations which need to be discussed further.

From a mathematics standpoint, the steady state flow for a shrinking sheet does not exist unless a strong suction is applied to control the boundary layer development. We propose to achieve this end by the application of a large magnetic field for both steady and unsteady boundary driven flows for shrinking sheets. We understand that the exact analytical and asymptotic solutions are more informative in explaining physics of the fluid as compared to the numerical solutions. Such a solution is thus highly desirable in terms of its mathematical and scientific value. For the unsteady asymptotic solutions, it is appropriate to carry out the stability analysis of the velocity field; that is even more important for the shrinking phenomenon.

Taking stock of the preceding discussion, we present an approximate asymptotic analytical solution for a steady MHD

†Corresponding author. E-mail: sumairaafzal66@yahoo.com

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viscous flow over a two-dimensional shrinking sheet. The difficulty in this attempt comes through the appearance of a singular term in the governing equation, which is taken care of by using the Lindstet–Poincaré technique to present a uniformly valid solution. The asymptotic limit of the magnetic field yields interesting physics of the MHD flow along with useful information on the momentum boundary layer, the velocity field, and the skin friction coefficient. The application of the strong magnetic field helps in controlling the boundary layer thickness as a viable alternative to the suction velocity. The time evolution of the behavior of the velocity field is also analyzed by non-linear stability analysis, showing that the solution remains stable as the unsteadiness evolves in time. The anticipated observations will be helpful in industry and useful in providing a deeper understanding of the flow physics under the influence of a large magnetic field for shrinking problems.

2. Formulation of the problem

A three-dimensional unsteady laminar flow of an electrically conducting viscous fluid subject to a constant transverse magnetic field applied in the z direction over a bidirectional shrinking sheet is considered (Fig. 1). The governing equations of the problem are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} =\frac{-1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\sigma\beta_0^2}{\rho}u, \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} =\frac{-1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\sigma\beta_0^2}{\rho}v, \tag{3}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} =\frac{-1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right), \tag{4}$$

where p is the fluid pressure, ρ is the fluid density, (u,v,w) are the velocity components along the (x,y,z) directions, and σ and ν are the electrical conductivity and the kinematic viscosity, respectively. The induced magnetic field is ignored and there are no electric field influences. The surface lying at $z=0$ is shrinking in two directions along the x-axis and y-axis with linear velocities, which yields a three-dimensional boundary value problem. The boundary conditions for the linear two-way shrinking are given by

$$\begin{aligned} &u=-ax,\ \ v=-ay,\ \ w=0\ \ \text{at}\ \ z=0,\\ &u\to 0,\ \ v\to 0,\ \ \ \ \text{as}\ \ z\to\infty. \end{aligned} \tag{5}$$

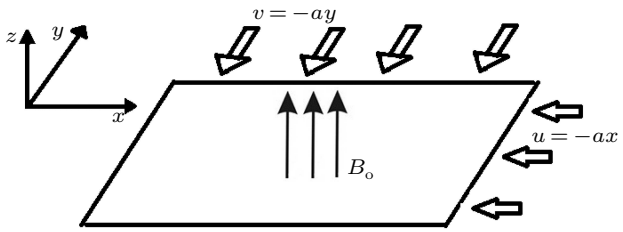


Fig. 1. Diagram of the physical system.

2.1. The steady problem

For the steady problem, we define the similarity variable as

$$\eta=\sqrt{\frac{a}{\nu}}z, \tag{6}$$

and introduce a dimensionless variable f such that the velocity components are expressed as

$$u=axf'(\eta),\ \ v=ayf'(\eta),\ \ w=-2\sqrt{a\nu}f(\eta). \tag{7}$$

After substituting expressions (6) and (7) into Eqs. (1)–(4), we arrive at the following nonlinear differential equation:

$$f'''-Mf'-(f')^2+2ff''=0, \tag{8}$$

where primes denote the derivative with respect to η and $M=\sigma\beta_0^2/(\rho a)$ is the magnetic parameter. We note that the same equation follows if we non-dimensionalize the governing equations and use the boundary layer approximations. The boundary conditions (5) in the new variable take the form

$$\begin{aligned} &f=0,\ \ f'=-1,\ \ \text{at}\ \ \eta=0,\\ &f'\to 0,\ \ \text{as}\ \ \eta\to\infty. \end{aligned} \tag{9}$$

Large M asymptotic limit For large M, i.e, $M\gg 1$, by setting $\varepsilon=\frac{1}{M}\ll 1$, equation (8) becomes

$$f'+\varepsilon\left(f'''+(f')^2-2ff''\right)=0. \tag{10}$$

We observe that in $\mathrm{O}(\varepsilon)$ analysis, the small parameter appears with the highest order derivative in Eq. (10). To proceed further, we introduce the following stretching transformation:[17]

$$\tilde{f}=\frac{f}{\sqrt{\varepsilon}},\quad \tilde{\eta}=\frac{\eta}{\sqrt{\varepsilon}}, \tag{11}$$

in the differential equation (10), which then takes the form

$$\tilde{f}'''-\tilde{f}'+\varepsilon(2\tilde{f}\tilde{f}''-(\tilde{f}')^2)=0. \tag{12}$$

The boundary conditions (9) become

$$\begin{aligned} &\tilde{f}=0,\ \ \tilde{f}'=-1,\quad \text{at}\quad \tilde{\eta}=0,\\ &\tilde{f}'\to 0,\quad \text{as}\quad \tilde{\eta}\to\infty. \end{aligned} \tag{13}$$

We assume an expansion of the form

$$\tilde{f}=f_0+\varepsilon f_1+\mathrm{O}(\varepsilon^2). \tag{14}$$

By substituting Eq. (14) in Eq. (12), the leading order system in O(1) is given as

$$f_0''' - f_0' = 0, \tag{15}$$

with the boundary conditions

$$\begin{aligned} &f_0 = 0, \ \ f_0' = -1, \quad \text{at} \quad \tilde{\eta} = 0, \\ &f_0' \to 0, \quad \text{as} \quad \tilde{\eta} \to \infty. \end{aligned} \tag{16}$$

The exact solution of the above system is

$$f_0(\tilde{\eta}) = -1 + \mathrm{e}^{-\tilde{\eta}}. \tag{17}$$

The system of $O(\varepsilon)$ can be written as

$$f_1''' - f_1' - (f_0')^2 + 2 f_0 f_0'' = 0, \tag{18}$$

$$\begin{aligned} &f_0 = 0, \ \ f_0' = 0, \quad \text{at} \quad \tilde{\eta} = 0, \\ &f_0' \to 0, \quad \text{as} \quad \tilde{\eta} \to \infty. \end{aligned} \tag{19}$$

This system allows an exact solution of the form

$$f_1 = \frac{1}{6}\left[\mathrm{e}^{-2\tilde{\eta}} + (6\tilde{\eta} + 4)\,\mathrm{e}^{-\tilde{\eta}} - 5\right]. \tag{20}$$

From Eq. (14), the two-term perturbation solution is given by

$$\tilde{f}_{2\mathrm{T}} = -1 + \mathrm{e}^{-\tilde{\eta}} + \frac{\varepsilon}{6}\left[\mathrm{e}^{-2\tilde{\eta}} + (6\tilde{\eta} + 4)\,\mathrm{e}^{-\tilde{\eta}} - 5\right] + O(\varepsilon^2). \tag{21}$$

We observe that the solution (21) is not uniformly valid for all $\tilde{\eta}$ because of the infinite domain singularity of the form $\varepsilon\tilde{\eta}\,\mathrm{e}^{-\tilde{\eta}}$ and the region of non-uniformity is $\tilde{\eta} = O\left(\sqrt{M}\right)$. To remove the secular term and make expression (21) uniformly valid for all $\tilde{\eta}$, we use the Lindstedt–Poincarè technique. We introduce a strained coordinate τ such that

$$\tau = \tilde{\eta}(1 + \epsilon w_1 + \epsilon^2 w_2 + \cdots), \tag{22}$$

and express

$$f(\tau, \epsilon) = f_0(\tau) + \varepsilon f_1(\tau) + O\left(\varepsilon^2\right). \tag{23}$$

Using Eqs. (22) and (23) in Eq. (12), we obtain the following two-term uniformly valid solution:

$$\begin{aligned} f_{2\mathrm{T}} \ = \ & -1 + \mathrm{e}^{-\tau} + \varepsilon\left(\frac{1}{6}\left(\mathrm{e}^{-2\tau} - 2\mathrm{e}^{-\tau} + 1\right) - \left(1 + \mathrm{e}^{-\tau}\right)\right) \\ & + O\left(\varepsilon^2\right), \end{aligned} \tag{24}$$

where the strained coordinate (22) turns out to be

$$\tau = \tilde{\eta}\left(1 - \varepsilon + O(\varepsilon^2)\right). \tag{25}$$

Equation (24) shows the boundary layer behavior of the velocity field with boundary layer thickness and displacement thickness given by

$$\delta \sim \frac{1.5}{\sqrt{M}}\sqrt{\frac{\nu}{a}}, \tag{26}$$

$$\delta_1 = \frac{2.5991}{\sqrt{M}}\sqrt{\frac{\nu}{a}}. \tag{27}$$

Equations (26) and (27) suggest that the boundary layer thickness and displacement decrease with stretching index a and magnetic parameter M.

2.2. The unsteady problem

For the unsteady case, we define the similarity transformations

$$\begin{aligned} &u = axf'(\eta, t), \ \ v = ayf'(\eta, t), \\ &w = -2\sqrt{a\nu}f(\eta, t), \ \ \bar{t} = at. \end{aligned} \tag{28}$$

Using Eq. (28) in Eqs. (2)–(3), the initial and boundary value problem takes the form

$$f''' - Mf' - (f')^2 + 2ff'' = \frac{\partial f'}{\partial \bar{t}}, \tag{29}$$

$$\begin{aligned} &f(\eta, 0) = 0, \ \ (0 \le \eta \le \infty), \\ &f(0, \bar{t}) = 0, \ \ f'(0, \bar{t}) = -1, \\ &f'(\eta, \bar{t}) \to 0, \quad \text{as} \quad \eta \to \infty \ \ (\bar{t} > 0). \end{aligned} \tag{30}$$

By setting $\varepsilon = 1/M \ll 1$ and using the scaling transformations

$$\tilde{f} = \frac{f}{\sqrt{\varepsilon}}, \ \ \tilde{\eta} = \frac{\eta}{\sqrt{\varepsilon}}, \ \ \tilde{t} = \frac{\bar{t}}{\varepsilon}, \tag{31}$$

equation (29) becomes

$$\tilde{f}''' - \tilde{f}' + \varepsilon\left(2\tilde{f}\tilde{f}'' - (\tilde{f}')^2\right) = \frac{\partial \tilde{f}'}{\partial \tilde{t}}. \tag{32}$$

Since $\varepsilon \ll 1$, the dominant terms in Eq. (32) are

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{\eta}^2} - \tilde{u} = \frac{\partial \tilde{u}}{\partial \tilde{t}}, \tag{33}$$

with boundary conditions

$$\begin{aligned} &\tilde{u} = 0, \quad \text{at} \quad \tilde{t} = 0 \ \ (0 \le \tilde{\eta} \le \infty), \\ &\tilde{u} = -1, \quad \text{at} \quad \tilde{\eta} = 0, \\ &\tilde{u} \to 0, \quad \text{as} \quad \tilde{\eta} \to \infty \ \ (\tilde{t} > 0), \end{aligned} \tag{34}$$

where $\tilde{u} = \partial \tilde{f}/\partial \tilde{\eta}$. The solution of the above system can be written in the following integral form:

$$\tilde{u} = -\mathrm{e}^{-\tilde{\eta}} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{(-1-r)\tilde{t}}\sin\left(\sqrt{r}\tilde{\eta}\right)}{1 + r}\,\mathrm{d}r. \tag{35}$$

Equation (35) shows that the steady state is approached exponentially in $\tilde{t}$. The expression for the skin friction is given by

$$\tau_{\mathrm{w}} = \sqrt{M} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{(-1-r)M\bar{t}}\sqrt{r}}{1 + r}\,\mathrm{d}r. \tag{36}$$

The integral in Eq. (36) can be evaluated asymptotically giving the following expression for the skin friction:

$$\tau_{\mathrm{w}} \sim \sqrt{M} + \frac{\mathrm{e}^{-M\bar{t}}}{2\sqrt{\pi}(M\bar{t})^{3/2}} + \cdots, \quad \bar{t} \to \infty. \tag{37}$$

From expressions (35) and (36), it can be seen that the perturbation to the steady state is of $O(e^{-M\bar{t}})$ for large M and $\bar{t}$, which shows that the steady state is approached more rapidly as M increases. This observation is shown in Fig. 2.

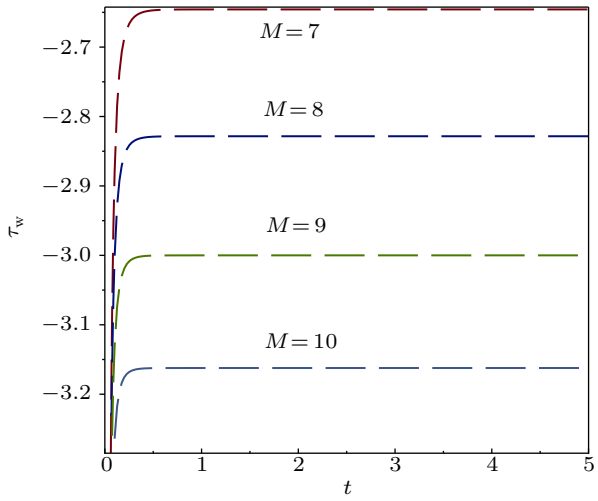


Fig. 2. (color online) Skin friction τ_w. The steady state is approached more rapidly as M increases.

2.3. Stability analysis

For the asymptotic solution, there is always a need to have a stability analysis to ensure that there is no permanent departure from the basic steady state solution as the time evolves. To discuss the stability of the solution of the boundary value problem (29) and (30), we define

$$\bar{f}(\eta,t) = f(\eta) + \varepsilon w(\eta,t), \tag{38}$$

where $w(\eta,t)$ is a small perturbation to the steady state solution, ε is a small parameter, and $f(\eta)$ satisfies the steady boundary value problem (8) and (9). By inserting Eq. (38) into Eq. (29), the perturbed field satisfies the equation

$$w''' + 2fw'' - (2f' + M)w' + 2f''w = \frac{\partial w'}{\partial t}, \tag{39}$$

with the boundary conditions

$$\begin{aligned} &w(\eta,0) = 0, \quad (0 \le \eta \le \infty), \\ &w(0,t) = 0, \quad w'(0,t) = 0, \\ &w'(\eta,t) \to 0, \quad \text{as} \quad \eta \to \infty \quad (t > 0). \end{aligned} \tag{40}$$

To determine the stability, we seek a solution of the form

$$w(\eta,t) = e^{-\lambda t} g(\eta), \tag{41}$$

where the parameter λ controls the growth of the solution as the time evolves. Using Eq. (41) in Eqs. (39) and (40) yields the following eigenvalue problem:

$$\begin{aligned} &g''' + 2fg'' + (\lambda - M - 2f')g' + 2f''g = 0, \\ &g(0) = 0, \quad g'(0) = 0, \quad g'(\infty) = 0, \end{aligned} \tag{42}$$

where $f(\eta)$ is given by Eq. (21). The sign of the smallest eigenvalue will determine the stability or otherwise of the solution. Using a numerical procedure, we can find an infinite number of eigenvalues in ascending order $\lambda_1 < \lambda_2 < \lambda_3 \cdots$ for each value of magnetic parameter M. The computation of these eigenvalues is given in Table 1. This reveals that the smallest eigenvalues for each M are positive and real. This concludes that the steady solution is linearly stable. We further observe that λ_1 increases with the increase of the magnetic parameter. Thus, the steady state reaches faster with an increasing M as guaranteed by Eq. (41).

Table 1. Smallest eigenvalues for different magnetic parameter M.

M	λ_1
5	6.727841
10	12.061701
13	15.158494
15	17.203931
18	20.254509
20	22.280140

3. Results and discussion

A uniformly valid asymptotic solution is derived for the steady state boundary value problem given by Eqs. (8) and (9) for the large magnetic field limit. The numerical solution is presented using fourth order Runge–Kutta in combination with the shooting method. The numerical and analytical results of the velocity profile for large magnetic number M are plotted in Fig. 3. The two solutions approach each other as M increases and an excellent match is reached for M as large as 15. It is also observed that the velocity increases with magnetic field M. Figure 3 shows that the boundary layer thickness decreases with an increasing M. This corroborates with the expression of boundary layer thickness established in Eq. (26).

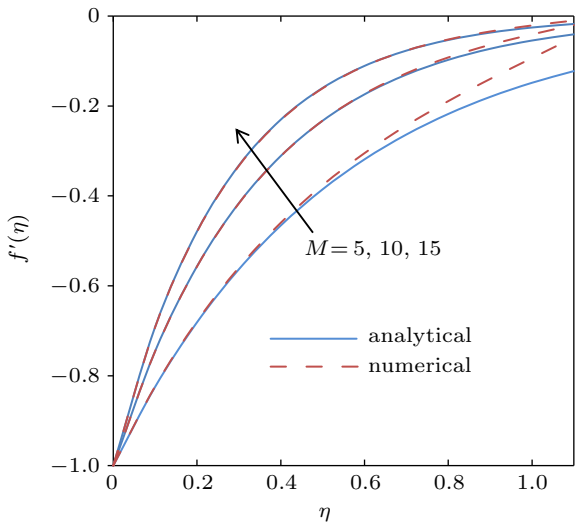


Fig. 3. (color online) Comparison of analytical and numerical results of velocity profile for large magnetic parameter M.

Figure 4 gives the skin friction calculated both analytically and numerically. The two results are a perfect match and show that the skin friction increases with M. This helps to delay the boundary layer separation which in turn reduces the drag. This is one of the most phenomenal implications of the present study.

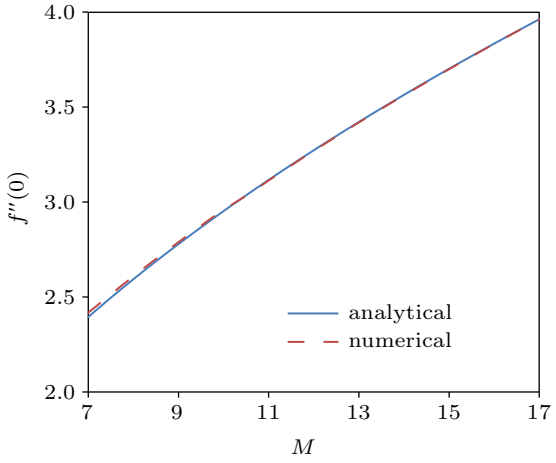


Fig. 4. (color online) Comparison of numerical and analytical results for skin friction against M.

We emphasize that the unsteady (35) and the steady (21) solutions must match each other for the large time limit if the solution is stable. Since our solution is stable, we expect and show this matching for the skin friction in Table 2. The matching is very sound even at time $t = 1$. We further note that the percentage error between steady and unsteady solutions of skin friction reduces significantly with increasing M.

Table 2. Percentage error between steady and unsteady solutions of skin friction for different M.

M	Unsteady flow ($t = 1$) $f''(0)$	Steady flow $f''(0)$	Error/%
15	3.8729	3.7008	4.4444
20	4.4721	4.3230	3.3333
50	7.0710	6.9767	1.3333
100	10.0	9.9333	0.6666
200	14.1421	14.0949	0.3333

4. Conclusion

The three-dimensional steady and unsteady (MHD) flows of a viscous fluid over a bi-directional axisymmetric shrinking sheet are examined. The effects of a strong magnetic field on the velocity field and the skin friction are investigated analytically for both steady and unsteady flows. First, the steady state asymptotic solution is developed using the large magnetic field limit. The two-term perturbation solution contains a secular term (infinite domain singularity) that is removed using the Lindsted–Poincaré technique to obtain a uniformly valid solution. The numerical solution for all M is also presented for the sake of comparison with the analytical solution. The analytic solution so obtained has an obvious advantage over the numerical solution that the existence and thickness of the boundary layer is shown explicitly. The asymptotic limits of large magnetic field for which analytical and numerical solutions of velocity field and skin friction coefficient start matching each other are found to be $M = 15$ and $M = 11$, respectively.

The analytical solution for the transient unsteady flow due to bidirectional shrinking is discussed. By keeping the dominant term for large M, an exact analytical solution is presented. It is observed that the unsteady solution reaches the steady state at $t = 1$ when M is as large as 15. The role of increasing magnetic field in reducing the time from unsteady to steady solution is quite significant. This observation is made for the physically important phenomena of skin friction coefficient (see Table 2). The stability of the approximate asymptotic results is also investigated. The solution is found to be stable in time for all M. Importantly, the stability is found to be reached earlier for large M.

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