Methods of reduction for Lagrange systems on time scales with nabla derivatives*

Shi-Xin Jin(金世欣)[1] and Yi Zhang(张毅)[1,2,†]

[1] School of Science, Nanjing University of Science and Technology, Nanjing 210009, China
[2] College of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, China

(Received 16 August 2016; revised manuscript received 12 December 2016; published online 22 December 2016)

The Routh and Whittaker methods of reduction for Lagrange system on time scales with nabla derivatives are studied. The equations of motion for Lagrange system on time scales are established, and their cyclic integrals and generalized energy integrals are given. The Routh functions and Whittaker functions of Lagrange system are constructed, and the order of differential equations of motion for the system are reduced by using the cyclic integrals or the generalized energy integrals with nabla derivatives. The results show that the reduced Routh equations and Whittaker equations hold the form of Lagrnage equations with nabla derivatives. Finally, two examples are given to illustrate the application of the results.

Keywords: reduction of dynamical system, cyclic integral, energy integral, time scales

PACS: 45.20.Jj, 89.75.Da, 71.10.Pm **DOI:** 10.1088/1674-1056/26/1/014501

1. Introduction

Time scale is an arbitrary nonempty closed subset of the set of real numbers. It unifies and extends the theories of continuous and discrete cases.[1–6] The theories of calculus and dynamic equations on time scales are applied to many different fields, such as physics, economics, optimal control, and so on. The theories of time scales was introduced by Hilger.[7] In 1988, Hilger[8,9] studied the calculus on time scales, in which the difference and differential equations were unified and extended.

In 2001, the book about dynamic equations on time scales was published by Bohner and Peterson,[10] the definitions and propositions of calculus on time scales were introduced, and the first-order linear equations and the second-order linear equations were established on time scales, respectively. The calculus of variations on time scales was initiated by Bohner using the delta derivative and delta integral.[11] The Euler–Lagrange equations of a first-order variational problem with delta derivative were derived. Atici, Biles, and Lebedinsky[4] discussed the theories of integral on time scales, which have been applied in the fields of economics. The Euler–Lagrange equations were obtained for a first-order variational problem on time scales involving nabla derivatives instead of delta ones. Martins and Torres[12] extended the variational problem with delta derivative to the case with nabla derivative, and the necessary optimality condition of the Euler–Lagrange equations for higher order variational problem with nabla derivative was proved by using a new and more general fundamental lemma of the calculus of variations on time scales. Later, Torres[13] further discussed the variational problems both with delta and nabla derivatives, in which the variational problems of the discrete, the quantum, and the continuous were unified and extended with delta and nabla derivatives.

In 2008, Bartosiewicz and Torres[14] studied the Noether theorem of Lagrange systems on time scales. The Noether theorem of the systems on time scales were given based on the invariance with transforming time and without transforming time. Martins and Torres[15] discussed the Noether symmetry theorem of variational problems with nabla derivatives, they proved a Noether-type symmetry theorem and a Dubois–Reymond necessary optimality condition for the calculus of variational with nabla derivative. Malinowska and Ammi[16] extended the Noeher theorem on time scale to control problems, and the Noether theorem for optimal control problems defined on delta derivative was proved. Meanwhile, the Noether symmetries of nonholonomic systems on time scales were discussed by Fu and his coworkers.[17] Zhang and Song[18] extended the Noether symmetries of mechanical systems to Birkhoffian systems on time scales. Some results of the variational problems and Noether symmetries of mechanical systems on time scales have been obtained,[19–22] while the methods of reduction for the systems on time scales have not been investigated yet in the literature.

While for the holonomic mechanical systems, the Routh function is constructed to eliminate the circlic coordinates by using the the cyclic integrals, and the Whittaker function is constructed to decrease the number of generalized coordinates by using the generalized energy integrals. The equations of motion for the mechanical systems are reduced, the reduced equations are still hold the form of equations of motion for the

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11572212 and 11272227) and the Innovation Program for Graduate Student of Jiangsu Province, China (Grant No. KYLX16-0414).

†Corresponding author. E-mail: zhy@mail.usts.edu.cn

© 2017 Chinese Physical Society and IOP Publishing Ltd

systems, it is the famous Routh[23] and Whittaker[24] methods of reduction in mechanics systems. In the past twenty years, the Routh and Whittaker methods of reduction in mechanics systems have been extended, and a series of important research results have been obtained.[25–30] Up to mow, the research of the methods for order reduction is limited to the standard mechanical systems, but it has not been related to the mechanical systems on time scales.

In this paper, we mainly study the Routh and Whittaker reduction methods by using the cyclic integrals and generalized energy integrals of Lagrange systems on the time scale. The structure of this paper is as follows. In Section 2, the definitions and propositions on time scales are given. The cyclic integral of Lagrange system with nabla derivative is defined in Section 3. The Routh method of reduction for Lagrange system with nabla derivatives is derived in Section 4. In Section 5, the generalized energy integral of Lagrange system with nabla derivative is derived. The Whittaker method of reduction for Lagrange system with nabla derivative is studied in Section 6. In Section 7, two examples are given to illustrate the application of the results.

2. Definitions and propositions of the calculus on time scales

In this section, we review some basic definitions and properties of calculus on time scales used in the following sections briefly. Detailed discussions and proofs can be found in Ref. [10].

A time scale $\mathbb{T}$ is a nonempty closed subset of the set of real numbers $\mathbb{R}$. Let $\mathbb{T}$ be an arbitrary time scale, their denote $\sigma:\mathbb{T}\to\mathbb{T}$ is the forward jump operator

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}, \tag{1}$$

and $\rho:\mathbb{T}\to\mathbb{T}$ is the backward jump operator

$$\rho(t)=\sup\{s\in\mathbb{T}:s<t\}, \tag{2}$$

where $\inf\varnothing=\sup\mathbb{T}$, $\sup\varnothing=\inf\mathbb{T}$.

A point $t\in\mathbb{T}$ is said to be right dense, right scattered, left dense, right scatter, if $\sigma(t)=t$, $\sigma(t)>t$, $\rho(t)=t$, and $\rho(t)<t$, respectively. A point t is isolated if $\rho(t)<t<\sigma(t)$, and t is dense if $\rho(t)=t=\sigma(t)$. The (backward) graininess function $\upsilon:\mathbb{T}\to[0,+\infty)$ is defined by

$$\upsilon(t)=t-\rho(t). \tag{3}$$

In order to introduce the definition of a nabla derivative, a set $\mathbb{T}_k$ is defined, which is derived from $\mathbb{T}$ as follows: if $\mathbb{T}$ has a right scattered minimum m, then $\mathbb{T}_k=\mathbb{T}\backslash\{m\}$, otherwise $\mathbb{T}_k=\mathbb{T}$.

Let function $f:\mathbb{T}\to\mathbb{R}$, then for all $t\in\mathbb{T}_k$, $f^\rho:\mathbb{T}_k\to\mathbb{R}$ is defined by

$$f^\rho(t)=f(\rho(t)), \tag{4}$$

i.e., $f^\rho=f\circ\rho$.

Let function $f:\mathbb{T}\to\mathbb{R}$, $t\in\mathbb{T}_k$, if for all $\varepsilon>0$, there exists a neighborhood $\mathbb{U}$ of t (i.e., $\mathbb{U}=(t-\delta,t+\delta)\cap\mathbb{T}$), such that

$$|f(\rho(t))-f(s)-f^\nabla(t)(\rho(t)-s)|\le\varepsilon|\rho(t)-s|, \tag{5}$$

then $f^\nabla(t)$ is called the nabla derivative of f at t. Usually, denote $f^\nabla(t)$ by $\frac{\nabla}{\nabla t}f(t)$. If for all $t\in\mathbb{T}_k$, there exists $f^\nabla(t)$, then f is the nabla derivative at $t\in\mathbb{T}_k$.

A function $f:\mathbb{T}\to\mathbb{R}$ is called ld-continuous if it is continuous at the left dense points and its right sided limits exists (finite) at all right dense points in $\mathbb{T}$. The set of all ld-continuous functions $f:\mathbb{T}\to\mathbb{R}$ is denoted by $C_{\rm ld}=C_{\rm ld}(\mathbb{T},\mathbb{R})$, and the set of all nabla differentiable functions with ld-continuous derivative given by $C^1_{\rm ld}=C^1_{\rm ld}(\mathbb{T},\mathbb{R})$.

A function $F:\mathbb{T}\to\mathbb{R}$ is called a nabla antiderivative of $f:\mathbb{T}\to\mathbb{R}$, if for all $t\in\mathbb{T}_k$, provided $F^\nabla(t)=f(t)$. In this case the nabla integral of f from a to b $(a,b\in\mathbb{T})$ is defined by $\int_a^b f(t)\nabla t=F(b)-F(a)$.

The propositions of calculus on time scales are as follows:

$$\begin{aligned}(fg)^\nabla(t)&=f(t)g^\nabla(t)+f^\nabla(t)g^\rho(t)\\&=f^\rho(t)g^\nabla(t)+f^\nabla(t)g(t),\end{aligned} \tag{6}$$

$$f^\rho(t)=f(t)-\upsilon(t)f^\nabla(t), \tag{7}$$

$$f^{\rho\nabla}(t)=(1-\upsilon^\nabla)f^{\nabla\rho}. \tag{8}$$

Formula (6) is the integration by parts on time scales.

Lemma 1[12] Let $f\in C_{\rm ld}.f:[a,b]\to\mathbb{R}$, if $\int_a^b f(t)\eta^\nabla(t)\nabla t=0$ for all $\eta\in C^1_{\rm ld}$ and $\eta(a)=\eta(b)=0$, then $f(t)=c$, for some $c\in\mathbb{R}$.

Proposition 1[12] Suppose that $a,b\in\mathbb{T},a<b$, and $f\in C_{\rm ld}([a,b],\mathbb{R})$, such that $f\ge0$ on $[a,b]$. If $\int_a^b f(t)\nabla t=0$, then $f(t)=0$.

Remark The proposition 1 cannot conclude that $f\equiv0$ on $t\in[a,b]$. For example, consider $\mathbb{T}=\{1,2,3,4,5\}$ and $f=1$ if $t=1$ and $f=0$ otherwise. Clearly, f is continuous and $f\ge0$ on $t\in[a,b]$. We have $\int_1^5 f(t)\nabla t=\sum_{t=2}^5 f(t)=0$, but $f\ne0$ on $\mathbb{T}$.

3. Cyclic integrals in Lagrange system with nabla derivative

Assume that the configuration of a mechanical system is determined by n generalized coordinates $q_s\,(s=1,2,\ldots,n)$. The Lagrangian is $L=L(t,q_s^\rho,q_s^\nabla)\in C_{\rm ld}$, and the Hamilton principle of the system with nabla derivative is

$$\delta\int_{t_1}^{t_2}L(t,q_s^\rho,q_s^\nabla)\nabla t=0, \tag{9}$$

with commutative relations

$$\delta q_s^{\nabla} = (\delta q_s)^{\nabla}, \quad s = 1,2,\ldots,n, \tag{10}$$

and the boundary conditions

$$\delta q_s \,|_{t=t_1} = \delta q_s \,|_{t=t_2} = 0, \quad s = 1,2,\ldots,n, \tag{11}$$

where $[t_1,t_2] \in \mathbb{T}$, $t \in \mathbb{T}_k$.

Formula (9) can be expressed as

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_s^{\rho}} \delta q_s^{\rho} + \frac{\partial L}{\partial q_s^{\nabla}} \delta q_s^{\nabla} \right) \nabla t = 0. \tag{12}$$

Considering formula (6) and the conditions (11), we have

$$\begin{aligned} \int_{t_1}^{t_2} \frac{\partial L}{\partial q_s^{\rho}} \delta q_s^{\rho} \nabla t &= \left(\int_{t_1}^{t} \frac{\partial L}{\partial q_s^{\rho}}(\tau) \nabla\tau \delta q_s \right)_{t_1}^{t_2} \\ &\quad - \int_{t_1}^{t_2} \int_{t_1}^{t} \frac{\partial L}{\partial q_s^{\rho}}(\tau) \nabla\tau \delta q_s^{\nabla} \nabla t \\ &= - \int_{t_1}^{t_2} \int_{t_1}^{t} \frac{\partial L}{\partial q_s^{\rho}}(\tau) \nabla\tau \delta q_s^{\nabla} \nabla t. \end{aligned} \tag{13}$$

Substituting formula (13) into formula (12), we have

$$\int_{t_1}^{t_2} \left[- \int_{t_1}^{t} \frac{\partial L}{\partial q_s^{\rho}}(\tau) \nabla\tau + \frac{\partial L}{\partial q_s^{\nabla}} \right] \delta q_s^{\nabla} \nabla t = 0. \tag{14}$$

From Lemma 1, we have

$$- \int_{t_1}^{t} \frac{\partial L}{\partial q_s^{\rho}}(\tau) \nabla\tau + \frac{\partial L}{\partial q_s^{\nabla}} = c. \tag{15}$$

Taking nabla derivative of Eq. (15) with respect to t, we have

$$\frac{\nabla}{\nabla t} \frac{\partial L}{\partial q_s^{\nabla}} - \frac{\partial L}{\partial q_s^{\rho}} = 0, \quad s = 1,2,\ldots,n. \tag{16}$$

Equations (16) are called the Lagrange equations of the system with nabla derivative.

If some of coordinates q_s^{ρ} are not contained in Lagrangian L, for instance q_1^{ρ}, i.e.,

$$\frac{\partial L}{\partial q_1^{\rho}} = 0, \tag{17}$$

then q_1^{ρ} is called the cyclic coordinate of the Lagrange system with nabla derivative. Form Eq. (16), we have

$$\frac{\nabla}{\nabla t} \frac{\partial L}{\partial q_1^{\nabla}} = 0. \tag{18}$$

Integrating Eq. (18), we have

$$\frac{\partial L}{\partial q_1^{\nabla}} = c_1 = \text{const}. \tag{19}$$

Equation (19) is the cyclic integral of the Lagrange system with nabla derivative corresponding to the cyclic coordinate q_1^{ρ}.

4. Routh method of reduction for Lagrange system with nabla derivatives

For the Lagrange system with nabla derivatives, the equations of motion for the system are reduced by using the cyclic coordinates. In this section, we establish the Routh method of reduction for Lagrange system with nabla derivative.

Suppose that the k coordinates $q_1^{\rho}, q_2^{\rho}, \ldots, q_k^{\rho}$ $(1 \le k \le n)$ of the Lagrange system with nabla derivative are not explicit, then the system is the k cyclic integrals as

$$\frac{\partial L}{\partial q_j^{\nabla}} = \beta_j, \quad j = 1,2,\ldots,k. \tag{20}$$

From Eq. (20), we have

$$q_j^{\nabla} = f_j(t, \beta_1, \ldots, \beta_k, q_{k+1}^{\rho}, \ldots, q_n^{\rho}, q_{k+1}^{\nabla}, \ldots, q_n^{\nabla}). \tag{21}$$

Define the Routh function with nabla derivative as

$$R = L - \sum_{j=1}^{k} q_j^{\nabla} \frac{\partial L}{\partial q_j^{\nabla}}, \tag{22}$$

where q_j^{∇} are replaced by Eq. (21), i.e.,

$$R = R(t, \beta_1, \ldots, \beta_k, q_{k+1}^{\rho}, \ldots, q_n^{\rho}, q_{k+1}^{\nabla}, \ldots, q_n^{\nabla}). \tag{23}$$

Taking the isochronal variation of Eqs. (22) and (23), we obtain

$$\delta R = \sum_{i=k+1}^{n} \left(\frac{\partial L}{\partial q_i^{\rho}} \delta q_i^{\rho} + \frac{\partial L}{\partial q_i^{\nabla}} \delta q_i^{\nabla} \right) - \sum_{j=1}^{k} q_j^{\nabla} \delta \beta_j, \tag{24}$$

$$\delta R = \sum_{i=k+1}^{n} \left(\frac{\partial R}{\partial q_i^{\rho}} \delta q_i^{\rho} + \frac{\partial R}{\partial q_i^{\nabla}} \delta q_i^{\nabla} \right) + \sum_{j=1}^{k} \frac{\partial R}{\partial \beta_j} \delta \beta_j. \tag{25}$$

Comparing Eq. (24) with Eq. (25), we obtain

$$\frac{\partial R}{\partial q_i^{\rho}} = \frac{\partial L}{\partial q_i^{\rho}}, \quad \frac{\partial R}{\partial q_i^{\nabla}} = \frac{\partial L}{\partial q_i^{\nabla}}, \quad i = k+1, k+2, \ldots, n, \tag{26}$$

$$q_j^{\nabla} = - \frac{\partial R}{\partial \beta_j}, \quad j = 1,2,\ldots,k. \tag{27}$$

Substituting Eq. (26) into Eq. (16), we have

$$\frac{\nabla}{\nabla t} \frac{\partial R}{\partial q_i^{\nabla}} - \frac{\partial R}{\partial q_i^{\rho}} = 0, \quad i = k+1, k+2, \ldots, n. \tag{28}$$

Equations (28) are called Routh equations of Lagrange system with nabla derivative, and hold the form of Lagrange equations (16), but contain only n–k equations. Integrating Eq. (27), we obtain

$$\begin{aligned} q_j^{\rho} &= - \int \frac{\partial R}{\partial \beta_j}(\rho(t)) \nabla t + \int \upsilon^{\nabla} \frac{\partial R}{\partial \beta_j}(\rho(t)) \nabla t, \\ &\quad j = 1,2,\ldots,k. \end{aligned} \tag{29}$$

In fact

$$q_j^{\rho\nabla} = q_j^{\nabla\rho}(1 - \upsilon^{\nabla}).$$

We have

$$q_j^{\nabla\rho} = -\frac{\partial R}{\partial \beta_j}(\rho(t)), \quad j = 1,2,\ldots,k,$$

then we have

$$q_j^{\rho\nabla} = -\left(1-\upsilon^{\nabla}\right)\frac{\partial R}{\partial \beta_j}(\rho(t)), \quad j = 1,2,\ldots,k.$$

Therefore, integrating the formula, we have Eq. (29).

If $\mathbb{T} = \mathbb{R}, \rho(t) = t, \upsilon(t) = 0, \upsilon^{\nabla}(t) = 0$, then equations (28) are reduced to the Routh equations of classical Lagrange system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial \dot{q}_i} - \frac{\partial R}{\partial q_i} = 0, \quad i = k+1, k+2, \ldots, n. \tag{30}$$

If $\mathbb{T} = h\mathbb{Z}$, $\rho(t) = t-h$, $\upsilon(t) = h$, $\upsilon^{\nabla}(t) = 0$, $R = R(t, \beta_j, q_i(t-h), \Delta_t q_j)$ is the discrete Routh function, then equations (28) are reduced to Routh equations of discrete Lagrange system as

$$\begin{aligned}&\Delta_t \frac{\partial R(t-h, \beta_j, q_i(t-2h), \Delta_t q_j(t-h))}{\partial \Delta_t q_i(t-h)}\\ &-\frac{\partial R}{\partial q_i(t-h)}(t+h) = 0,\end{aligned} \tag{31}$$

where the continuous derivatives are substituted by $\Delta_t q_i = \frac{1}{h}(q_i(t) - q_i(t-h))$, and $\Delta_t = \partial_t$. We can refer to Refs. [31] and [32].

Hence, a dynamical problem on time scales with n degrees of freedom, which has k cyclic coordinates, can be reduced to a dynamical problem on time scales which only has n–k degrees of freedom.

5. Generalized energy integral of Lagrange system with nabla derivatives

In order to get the corresponding generalized energy integral of Lagrange system with nabla derivative, we calculate

$$\begin{aligned}\frac{\nabla}{\nabla t}\left(\frac{\partial L}{\partial q_s^{\nabla}}q_s^{\nabla} - L\right) &= \frac{\nabla}{\nabla t}\frac{\partial L}{\partial q_s^{\nabla}}q_s^{\nabla\rho} + \frac{\partial L}{\partial q_s^{\nabla}}\frac{\nabla}{\nabla t}(q_s^{\nabla}) - \frac{\nabla}{\nabla t}L\\ &= \frac{\nabla}{\nabla t}\frac{\partial L}{\partial q_s^{\nabla}}q_s^{\nabla\rho} - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial q_s^{\rho}}\frac{\nabla}{\nabla t}(q_s^{\rho})\\ &= \frac{\nabla}{\nabla t}\frac{\partial L}{\partial q_s^{\nabla}}q_s^{\nabla\rho} - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial q_s^{\rho}}(1-\upsilon^{\nabla})q_s^{\nabla\rho}\\ &= \left(\frac{\nabla}{\nabla t}\frac{\partial L}{\partial q_s^{\nabla}} - \frac{\partial L}{\partial q_s^{\rho}}\right)q_s^{\nabla\rho}\\ &\quad - \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_s^{\rho}}\upsilon^{\nabla}q_s^{\nabla\rho}.\end{aligned} \tag{32}$$

From Eqs. (16) and (32), if the time t is not explicit in function L, and formula (32) satisfies the condition

$$\frac{\partial L}{\partial t} = 0, \quad \frac{\partial L}{\partial q_s^{\rho}}\upsilon^{\nabla}q_s^{\nabla\rho} = 0, \tag{33}$$

then along the dynamical trajectory of Lagrange system with nabla derivative, we have

$$\frac{\nabla}{\nabla t}\left(\frac{\partial L}{\partial q_s^{\nabla}}q_s^{\nabla} - L\right) = 0. \tag{34}$$

Integrating Eq. (34), we have

$$\frac{\partial L}{\partial q_s^{\nabla}}q_s^{\nabla} - L = h = \text{const}. \tag{35}$$

When the conditions (33) are satisfied, formula (35) is the generalized energy integral of Lagrange system with nabla derivative.

6. Whittaker method of reduction for Lagrange system with nabla derivatives

Choose a generalized coordinate to substitute the effect of the time t, such as q_1. Let

$$q_r' = \frac{\nabla q_r}{\nabla q_1}, \quad r = 2,3,\ldots,n, \tag{36}$$

then

$$q_r^{\nabla} = \frac{\nabla q_r}{\nabla q_1}\frac{\nabla q_1}{\nabla t} = q_r' q_1^{\nabla}, \quad r = 2,3,\ldots,n. \tag{37}$$

Suppose that

$$\Omega(q_s^{\rho}, q_1^{\nabla}, q_r') = L(q_s^{\rho}, q_1^{\nabla}, q_r' q_1^{\nabla}). \tag{38}$$

Differentiating formula (38) with respect to q_1^{∇}, q_r', and q_s^{ρ}, we obtain respectively

$$\begin{aligned}\frac{\partial \Omega}{\partial q_1^{\nabla}} &= \sum_{s=1}^{n}\frac{\partial L}{\partial q_s^{\nabla}}\frac{\partial q_s^{\nabla}}{\partial q_1^{\nabla}}\\ &= \frac{\partial L}{\partial q_1^{\nabla}} + \sum_{r=2}^{n}\frac{\partial L}{\partial q_r^{\nabla}}\frac{\partial q_r^{\nabla}}{\partial q_1^{\nabla}}\\ &= \frac{\partial L}{\partial q_1^{\nabla}} + \sum_{r=2}^{n}\frac{\partial L}{\partial q_r^{\nabla}}q_r'\\ &= \frac{\partial L}{\partial q_1^{\nabla}}\frac{q_1^{\nabla}}{q_1^{\nabla}} + \sum_{r=2}^{n}\frac{\partial L}{\partial q_r^{\nabla}}\frac{q_r^{\nabla}}{q_1^{\nabla}}\\ &= \sum_{s=1}^{n}\frac{\partial L}{\partial q_s^{\nabla}}\frac{q_s^{\nabla}}{q_1^{\nabla}},\end{aligned} \tag{39}$$

$$\frac{\partial \Omega}{\partial q_r'} = \frac{\partial L}{\partial q_l^{\nabla}}\frac{\partial q_l^{\nabla}}{\partial q_r'} = \frac{\partial L}{\partial q_r^{\nabla}}q_1^{\nabla}, \quad l = 2,3,\ldots,n, \tag{40}$$

$$\frac{\partial \Omega}{\partial q_s^{\rho}} = \frac{\partial L}{\partial q_s^{\rho}}, \quad s = 1,2,\ldots,n. \tag{41}$$

Using formula (35), we can obtain q_1^{∇} as

$$q_1^{\nabla} = q_1^{\nabla}(q_r', q_r^{\nabla}). \tag{42}$$

From formula (38) and (39), formula (35) can be expressed as

$$\frac{\partial \Omega}{\partial q_1^{\nabla}}q_1^{\nabla} - \Omega = h = \text{const}, \tag{43}$$

where q_1^{∇} is determined by formula (42). Differentiating formula (43) with respect to q_r' and q_r^{ρ}, we have

$$q_1^{\nabla}\frac{\partial^2\Omega}{\partial(q_1^{\nabla})^2}\frac{\partial q_1^{\nabla}}{\partial q_r'}+q_1^{\nabla}\frac{\partial^2\Omega}{\partial q_1^{\nabla}\partial q_r'}-\frac{\partial\Omega}{\partial q_r'}=0,\quad r=2,3,\ldots,n. \tag{44}$$

$$q_1^{\nabla}\frac{\partial^2\Omega}{\partial(q_1^{\nabla})^2}\frac{\partial q_1^{\nabla}}{\partial q_r\rho}+q_1^{\nabla}\frac{\partial^2\Omega}{\partial q_1^{\nabla}\partial q_r^{\rho}}-\frac{\partial\Omega}{\partial q_r^{\rho}}=0,\quad r=2,3,\ldots,n. \tag{45}$$

Construct the Whittaker function with nabla derivative as

$$W=W(q_s^{\rho},q_r')=\frac{\partial\Omega}{\partial q_1^{\nabla}}. \tag{46}$$

Differentiating formula (46) with respect to q_r' and q_r^{ρ}, we obtain

$$\frac{\partial W}{\partial q_r'}=\frac{\partial^2\Omega}{\partial(q_1^{\nabla})^2}\frac{\partial q_1^{\nabla}}{\partial q_r'}+\frac{\partial^2\Omega}{\partial q_1^{\nabla}\partial q_r'},\quad r=2,3,\ldots,n, \tag{47}$$

$$\frac{\partial W}{\partial q_r^{\rho}}=\frac{\partial^2\Omega}{\partial(q_1^{\nabla})^2}\frac{\partial q_1^{\nabla}}{\partial q_r^{\rho}}+\frac{\partial^2\Omega}{\partial q_1^{\nabla}\partial q_r^{\rho}},\quad r=2,3,\ldots,n. \tag{48}$$

Comparing Eq. (44) with Eq. (47), we obtain

$$\frac{\partial W}{\partial q_r'}=\frac{1}{q_1^{\nabla}}\frac{\partial\Omega}{\partial q_r'},\quad r=2,3,\ldots,n. \tag{49}$$

Comparing Eq. (45) with Eq. (48), we obtain

$$\frac{\partial W}{\partial q_r^{\rho}}=\frac{1}{q_1^{\nabla}}\frac{\partial\Omega}{\partial q_r^{\rho}},\quad r=2,3,\ldots,n. \tag{50}$$

According to Eqs. (40) and (41), we obtain

$$\frac{\partial W}{\partial q_r'}=\frac{\partial L}{\partial q_r^{\nabla}},\quad r=2,3,\ldots,n. \tag{51}$$

$$\frac{\partial W}{\partial q_r^{\rho}}=\frac{1}{q_1^{\nabla}}\frac{\partial L}{\partial q_r^{\rho}},\quad r=2,3,\ldots,n, \tag{52}$$

then equations (16) can be written in the form

$$\frac{\nabla}{\nabla t}\frac{\partial W}{\partial q_r'}-\frac{1}{q_1^{\nabla}}\frac{\partial W}{\partial q_r^{\rho}}=0,\quad r=2,\ldots,n, \tag{53}$$

or

$$\frac{\nabla}{\nabla q_1}\frac{\partial W}{\partial q_r'}-\frac{\partial W}{\partial q_r^{\rho}}=0,\quad r=2,\ldots,n. \tag{54}$$

Equations (53) or (54) are called Whittaker equations of Lagrange system with nabla derivative. Equations (53) or (54) hold the form of Lagrange equations (16), but equations (53) or (54) contain only $n-1$ equations.

If $\mathbb{T}=\mathbb{R}$, $\rho(t)=t$, $\upsilon(t)=0$, $\upsilon^{\nabla}(t)=0$, then equations (53) or (54) can be written as the Whittaker equations of classical Lagrange system[33,34]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial W}{\partial q_r'}-\frac{1}{\dot{q}_1}\frac{\partial W}{\partial q_r}=0,\quad r=2,\ldots,n, \tag{55}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}q_1}\frac{\partial W}{\partial q_r'}-\frac{\partial W}{\partial q_r}=0,\quad r=2,\ldots,n. \tag{56}$$

If $\mathbb{T}=h\mathbb{Z}$, $\rho(t)=t-h$, $\upsilon(t)=h$, $\upsilon^{\nabla}(t)=0$, and $W=W(q_s(t-h),\Delta_{q_1}q_r)$ is the discrete Whittaker function, then equations (53) or (54) can be written to Whittaker equations of the discrete Lagrange system as

$$\Delta_t\frac{\partial W(q_s(t-2h),\Delta_{q_1}q_r(t-h))}{\partial\Delta_{q_1}q_r(t-h)}-\Delta_t q_1\frac{\partial W}{\partial q_r(t-h)}(t+h)=0, \tag{57}$$

or

$$\Delta_{q_1}\frac{\partial W(q_s(t-2h),\Delta_{q_1}q_r(t-h))}{\partial\Delta_{q_1}q_r(t-h)}-\frac{\partial W}{\partial q_r(t-h)}(t+h)=0. \tag{58}$$

Hence, the integral of energy enables us to reduce a given dynamical system with nabla derivative with n degrees of freedom to another dynamical system with nabla derivative with $n-1$ degrees of freedom.

7. Examples

Example 1 The Lagrangian of a mechanical system with nabla derivative is

$$L=\frac{2}{3}ml^2\left[(q_1^{\nabla})^2+(q_2^{\nabla})^2\sin^2 q_1^{\rho}\right]-mgl\cos q_1^{\rho}. \tag{59}$$

Clearly, q_2^{ρ} is a cyclic coordinate with nabla derivative and the integral corresponding to the cyclic coordinate q_2^{ρ} of the system is

$$\frac{\partial L}{\partial q_2^{\nabla}}=\frac{4}{3}ml^2q_2^{\nabla}\sin^2 q_1^{\rho}. \tag{60}$$

The Routh function is

$$\begin{aligned}R&=L-q_2^{\nabla}\frac{\partial L}{\partial q_2^{\nabla}}\\&=\frac{2}{3}ml^2\left[(q_1^{\nabla})^2-(q_2^{\nabla})^2\sin^2 q_1^{\rho}\right]-mgl\cos q_1^{\rho}.\end{aligned} \tag{61}$$

From formula (60), we obtain

$$q_2^{\nabla}=\frac{3\beta_2}{4ml^2\sin^2 q_1^{\rho}}. \tag{62}$$

Substituting formula (62) into formula (61), we obtain

$$R=\frac{2}{3}ml^2(q_1^{\nabla})^2-\frac{3\beta_2^2}{8ml^2\sin^2 q_1^{\rho}}-mgl\cos q_1^{\rho}. \tag{63}$$

Substituting Eq. (63) into Eq. (28), we obtain

$$\frac{4}{3}ml^2q_1^{\nabla^2}-\frac{3\beta_2^2\cos q_1^{\rho}}{8ml^2\sin^3 q_1^{\rho}}-mgl\sin q_1^{\rho}=0, \tag{64}$$

then equation (64) is the Routh equation of the system with nabla derivative.

From formula (29), we obtain

$$\begin{aligned}q_2^{\rho}&=-\frac{3\beta_2}{4ml^2}(\cot q_1(t-2h)+\csc q_1(t-2h))\\&\quad+\int\upsilon^{\nabla}\frac{3\beta_2}{4ml^2\sin^2 q_1(t-2h)}\nabla t.\end{aligned} \tag{65}$$

If $\mathbb{T}=\mathbb{R}$, $\rho(t)=t$, $\upsilon(t)=0$, $\upsilon^{\nabla}(t)=0$, formulae (64) and (65) can be written as

$$\frac{4}{3}ml^2\ddot{q}_1-\frac{3\beta_2^2\cos q_1}{8ml^2\sin^3 q_1}-mgl\sin q_1=0, \tag{66}$$

$$q_2=-\frac{3\beta_2}{4ml^2}(\cot q_1+\csc q_1). \tag{67}$$

Equation (66) is the Routh equation of classical Lagrange system.

If $\mathbb{T}=h\mathbb{Z}$, $\rho(t)=t-h$, $\upsilon(t)=h$, $\upsilon^{\nabla}(t)=0$, formulae (64) and (65) are expressed as the Routh equation of discrete Lagrange system

$$\frac{4}{3}ml^2\triangle_t(\triangle_t q_1(t-h))-\frac{3\beta_2^2\cos q_1}{8ml^2\sin^3 q_1}-mgl\sin q_1=0, \tag{68}$$

$$q_2(t-h)=\sum_{t=t+h}^{t_2}\frac{3\beta_2}{4ml^2\sin^2 q_1(t-h)}. \tag{69}$$

Example 2 The Lagrangian of a system with nabla derivative with two degrees of freedom is

$$L=\frac{1}{2}q_2^{\rho}(q_1^{\nabla})^2+\frac{1}{2}(q_2^{\nabla})^2-(q_2^{\rho})^2. \tag{70}$$

From formula (35), we obtain

$$\frac{1}{2}q_2^{\rho}(q_1^{\nabla})^2+\frac{1}{2}(q_2^{\nabla})^2+(q_2^{\rho})^2=h=\text{const}. \tag{71}$$

Satisfy the condition

$$\frac{1}{2}[(q_1^{\nabla})^2-4q_2^{\rho}]\upsilon^{\nabla}q_s^{\nabla\rho}=0. \tag{72}$$

Choose a generalized coordinate q_1 to substitute the effect of the time t. Setting

$$q_2'=\frac{\nabla q_2}{\nabla q_1}, \tag{73}$$

we have

$$q_2^{\nabla}=\frac{\nabla q_2}{\nabla q_1}\frac{\nabla q_1}{\nabla t}=q_2'q_1^{\nabla}. \tag{74}$$

Substituting formula (74) into (71), we have

$$\frac{1}{2}q_2^{\rho}(q_1^{\rho\nabla})^2+\frac{1}{2}(q_2'q_1^{\nabla})^2+(q_2^{\rho})^2=h=\text{const}. \tag{75}$$

Therefore, we have

$$q_1^{\nabla}=\sqrt{\frac{2[h-(q_2^{\rho})^2]}{q_2^{\rho}+(q_2')^2}}. \tag{76}$$

Construct the function Ω as

$$\Omega=\frac{1}{2}q_2^{\rho}(q_1^{\nabla})^2+\frac{1}{2}(q_2'q_1^{\nabla})^2-(q_2^{\rho})^2. \tag{77}$$

The Whittaker function of the system is

$$W=\frac{\partial\Omega}{\partial q_1^{\nabla}}=\sqrt{2[h-(q_2^{\rho})^2][q_2^{\rho}+(q_2')^2]}. \tag{78}$$

Substituting formula (78) into equation (53) or (54), we obtain

$$\frac{\nabla}{\nabla t}\frac{\partial W}{\partial q_2'}-q_1^{\nabla}\frac{\partial W}{\partial q_2^{\rho}}=0, \tag{79}$$

or

$$\frac{\nabla}{\nabla q_1}\frac{\partial W}{\partial q_2'}-\frac{\partial W}{\partial q_2^{\rho}}=0, \tag{80}$$

then equation (79) or (80) is the Whittaker equation of the system with nabla derivative.

Since q_2^{ρ} is not contained in W, we get a new integral of energy

$$\frac{\partial W}{\partial q_2'}-W=\sqrt{\frac{2[h-(q_2^{\rho})^2]}{q_2^{\rho}+(q_2')^2}}(-q_2^{\rho})=K. \tag{81}$$

Therefore, we have

$$q_2'=\sqrt{\frac{2(q_2^{\rho})^2[h-(q_2^{\rho})^2]-K^2q_2^{\rho}}{K^2}}. \tag{82}$$

From formula (9), substituting formula (82) into formula (74), and integrating it, we obtain

$$q_1^{\rho}=K\int\frac{1}{\sqrt{2(q_2^{\rho})^2[h-(q_2^{\rho})^2]-K^2q_2^{\rho}}}\nabla q_2^{\rho}. \tag{83}$$

If $\mathbb{T}=\mathbb{R}$, $\rho(t)=t$, $\upsilon(t)=0$, $\upsilon^{\nabla}(t)=0$, then formulae (75) and (78) can be expressed as

$$\frac{1}{2}q_2\dot{q}_1^2+\frac{1}{2}\dot{q}_1^2+q_2^2=h=\text{const}, \tag{84}$$

$$W=\sqrt{2[h-q_2^2][q_2+(q_2')^2]}. \tag{85}$$

The Whittaker equation (79) or (80) is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial W}{\partial q_2'}-\dot{q}_1\frac{\partial W}{\partial q_2}=0, \tag{86}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}q_1}\frac{\partial W}{\partial q_2'}-\frac{\partial W}{\partial q_2}=0. \tag{87}$$

Formulae (82) and (83) are expressed as

$$q_2'=\sqrt{\frac{2q_2^2[h-q_2^2]-K^2q_2}{K^2}}, \tag{88}$$

$$q_1=\frac{K}{\sqrt{2}}\int\frac{\mathrm{d}q_2}{\sqrt{q_2\{q_2[h-q_2^2]-\frac{K^2}{2}\}}}. \tag{89}$$

If $\mathbb{T}=h\mathbb{Z}$, $\rho(t)=t-h$, $\upsilon(t)=h$, $\upsilon^{\nabla}(t)=0$, then formulae (75) and (78) can be expressed as

$$\frac{1}{2}(\triangle_t q_1)^2[q_2(t-h)+1]+q_2^2(t-h)=h=\text{const}, \tag{90}$$

$$W=\sqrt{2[h-q_2^2(t-h)][q_2(t-h)+(\triangle_{q_1}q_2)^2]}. \tag{91}$$

Equation (80) or (81) is written as

$$\triangle_t\frac{\partial W(q_2(t-2h),\triangle_{q_1}q_2(t-h))}{\partial\triangle_{q_1}q_2(t-h)} -\triangle_t q_1\frac{\partial W}{\partial q_2(t-h)}(t+h)=0, \tag{92}$$

or

$$\triangle_{q_1}\frac{\partial W(q_2(t-2h),\triangle_{q_1}q_2(t-h))}{\partial\triangle_{q_1}q_2(t-h)}-\frac{\partial W}{\partial q_2(t-h)}(t+h)=0. \tag{93}$$

8. Conclusion

The theories of calculus for continuous, discrete, and quantum cases, have been unified and extended by the theories of calculus on time scales. Considering the complexity of dynamical behavior of mechanical systems on time scales, the research of integral theories for mechanical systems on time scales is still in the initial stage of development. In this paper, we study the methods of reduction for Lagrange system with nabla derivative. The Lagrange equations for the system are established, and their cyclic integrals and generalized energy integrals are given. The order of Lagrange equations (16) is reduced, the Routh equations (28) are obtained by using the cyclic integrals, and the Whittaker equations (53) or (54) are derived by using the generalized energy integrals with nabla derivatives. The results show that the reduced Routh equations and Whittaker equations hold the form of Lagrange equations with nabla derivatives. Comparing with the Lagrange equations of the system, the Routh equations (28) have only n–k equations and the Whittaker equations (53) or (54) have only $n-1$ equations, and the cases of continuous and discrete are two special cases with nabla derivative. The results of this paper are of universal significance, the theories of mechanical systems on time scales can be further generalized to the theories of Birkhoffian system, optimal control, and so on.

References

[1] Ahlbrand D, Bohner M and Ridenhour J 2000 *J. Math. Anal. Appl.* **250** 561
[2] Agarwal R, Bohner M and O'Regan D 2004 *J. Comput. Appl. Math.* **141** 1
[3] Bohner M and Guseinov G Sh 2006 *Comput. Math. Appl.* **54** 45
[4] Atici F M, Biles D C and Lebedinsky A 2006 *Math. Comput. Model.* **43** 718
[5] Abdeljawad T, Jarad F and Baleanu D 2009 *Adv. Differ. Eqs.* **30** 840386
[6] Bartosiewicz Z, Martins N and Torres D F M 2011 *Eur. J. Control* **17** 9
[7] Hilger S 1988 *Ein Mkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten* (PhD thesis, Universit at Würzburg)
[8] Hilger S 1990 *Res. Math.* **18** 18
[9] Hilger S 1997 *Nonlinear Anal.: Theor. Methods Appl.* **30** 2683
[10] Bohner M and Peterson A 2001 *Dynamic Equations on Time Scale: An Introduction with Applications* (Birkhäuser Boston)
[11] Bohner M 2004 *Dyn. Sys. Appl.* **13** 339
[12] Martins N and Torres D F M 2009 *Nonlinear Anal.* **71** e763
[13] Torres D F M 2010 *Int. J. Simul. Multi. Des. Optim.* **4** 11
[14] Bartosiewicz Z and Torres D F M 2008 *J. Math. Anal. Appl.* **342** 1220
[15] Martins N and Torres D F M 2010 *Appl. Math. Lett.* **23** 1432
[16] Malinowska A B and Ammi M R S 2014 *Int. J. Differ. Eqs.* **9** 87
[17] Cai P P, Fu J L and Guo Y X 2013 *Sci. China G: Phys. Mech. Astron.* **56** 1017
[18] Song C J and Zhang Y 2015 *J. Math. Phys.* **56** 102701
[19] Herzallah M A E, Muslih S I and Baleanu D 2011 *Nonlinear Dyn.* **66** 549
[20] Cruz A M C B Da, Martins N and Torres D F M 2013 *Appl. Math. Lett.* **26** 264
[21] Bartosiewicz Z, Martins N and Torres D F M 2010 *Eur. J. Control* **17** 9
[22] Malinowska A B and Martins N 2014 *Abstr. Appl. Anal.* **2013** 2013
[23] Routh E J 1877 *A Treatise on the Stability of Motion* (London: Macmillima)
[24] Whittaker E T 1904 *A Treatise on the Analytical Dynamics of Particles any Rigid Bodies* (Cambridge: Cambridge University Press)
[25] Liu D 1988 *Chin. Sci. Bull.* **33** 004
[26] Luo S K 2003 *Chin. Phys.* **12** 140
[27] Luo S K 2002 *Chin. Phys.* **11** 1097
[28] Luo S K, Huang F J and Lu Y B 2004 *Commun. Theor. Phys.* **42** 817
[29] Zhang Y 2008 *Acta Phys. Sin.* **57** 5374 (in Chinese)
[30] Zhang Y 2008 *Chin. Phys. B* **17** 1674
[31] Guo H Y, Li Y Q and Wu K 2002 *Commun. Theor. Phys.* **37** 1
[32] Fu J L, Chen L Q and Chen B Y 2009 *Sci. China Ser. G: Phys. Mech. Astron.* **39** 1320 (in Chinese)
[33] Mei F X 2013 *Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) (in Chinese)
[34] Mei F X and Wu H B 2009 *Dynamics of Constrained Mechanical Systems* (Beijing: Beijjing Institute of Technology Press)