RAPID COMMUNICATION

Energy sharing induced by the nonlinear interaction*

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Strong energy sharing is shown by numerically investigating coupled multi-component Bose–Einstein condensates (BECs) with a harmonic trap to simulate the Fermi–Pasta–Ulam model (FPU). For two-component BECs, the energy exchanging between each part, from regular, quantum beating to complete energy sharing, is explored by simulating their Husimi distributions, the time evolution of energies and the statistical entropy. Meanwhile, in the three-component case, a more complex energy sharing behavior is reported and a strong energy sharing is found.

Keywords: Bose–Einstein condensates, Fermi–Pasta–Ulam model, Husimi distributions, statistical entropy

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1. Introduction

How the energy, initially excited in a particular mode, flows into other nonlinear coupled modes in a chain of classical harmonic oscillators (Fermi–Pasta–Ulam model (FPU)[1]) is still not completely understood.[2] The surprising FPU recurrence of energy, which is opposite to the original expectation,[3] actually gives rise to new phenomena, the integrability of nonlinear equations and the dynamical chaos.[2,4,5] Even so, people are continuing to look for an energy sharing by modifying the FPU model to higher dimensions or coupling forms.[2] Once the nonlinearity is over some threshold, the FPU model shows strong energy sharing or energy equipartition among normal linear modes.[4,5] Linearly dependent coupled frequencies help energy sharing even for a very weakly coupled oscillator system.[6] Further investigation shows that the special initially excited mode under resonance conditions,[6] elastic collision interaction[7] or strong (or weak) chaos conditions,[4,5] leads to strong energy sharing with other coupled normal modes. So far, a statement is reported:[2] energy may flow into other modes with regular behaviors, initially, excited to a higher mode, in some initial period of time (introduction period), and then a stochastic exchange among the modes comes into play, with a practical irreversibility of motion.

Due to the consistency between the Gross–Pitaevskii (GP) equation (governing the basic feature of BECs) and FPU-β model,[8] the investigation of the FPU problem or energy sharing process with BECs has been theoretically suggested almost twenty years ago[9,10] and its time evolution has been experimentally attempted to be observed.[11] Based on the analytically completed eigenstates for a perfect box of the finite size, corresponding to the normal modes in the FPU problem, one criteria for the appearance of the equilibrium state is reported in Ref. [9]. A transition from a regular dynamics to an irregular one coincides with the onset of the Tonks–Girardeau regime.[10] However, it is surprising that no clear evidence was found in Ref. [11], even though a very strong nonlinear interaction (within Tonks–Girardeau regime) was considered.

In Ref. [11], wherein a relatively long-lived oscillation of each BECs trapped in a harmonic trap was reported in the position and momentum spaces, we suggest that each component BECs (with given momentum $\pm 2\hbar k$ in Ref. [11]) trapped in a harmonic trap could be considered as a normal mode of the FPU problem. With this, the nonlinear couplings between the models are realized by the collision interaction between different components of BECs, and then the dynamics of this system is governed by the well-known coupled GP equations. To understand the energy sharing process in this system, the Husimi distribution function (one quantum distribution function),[15,16] the Fourier spectrum of time evolution of energy,[6,17] and the statistical entropy[9,18] are calculated. Initially, for the excited one of the two components of BECs, its energy flows into the initial static mode with Rabi-like oscillation in the case of the weak nonlinear coupling. With the increase of the nonlinear coupling, the time evolution of energy is modified from quantum beating to chaos dynamics after a critical time. When the nonlinear coupling is strong enough, a strong energy sharing is found by checking the time average of energy, Husimi distribution function, and statistic entropy.

In the following, we present our model, the equations, and the nonlinear coupling interaction in Section 2. In Sec-

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tion 3, we analyze the energy exchanging processes of two-component BECs by investigating the Husimi distribution function, the Fourier spectrum of time evolution of energy, and the statistic entropy. Section 4 is devoted ordinarily to the case of three components. A conclusion is given in Section 5.

2. Model and nonlinear coupling force

A general description of n-component condensate mixture with a one-dimensional (1D) harmonic trap $V(x) = x^2/2$ can be given by the following coupled GP equations within the mean-field theory:

$$\mathrm{i}\frac{\partial \Psi_i}{\partial t} = H_0(\Psi_i)\Psi_i + g_{ij}|\Psi_j|^2\Psi_i, \tag{1}$$

with $H_0 = -(1/2)\partial^2/\partial x^2 + V(x) + g_{ii}|\Psi_i|^2$ and $i \neq j$, where, for simplicity, we assume that all components of atoms have the same mass m and the number of atoms of each species is fixed and cannot be converted into a different component. Here, the length and the energy have been rescaled in the units of harmonic trap character length $l = \sqrt{\hbar/m\omega}$ and energy $\hbar\omega$. The atomic interactions are described by g_{ij}, which are proportional to the two-body s-wave scattering length a_{ij}, and describe the intra-nonlinear parameters ($i = j$) and inter-component nonlinear coupling ($i \neq j$), respectively. The dynamics of each component in Eq. (1) performs a harmonic oscillation around its balance position ($x = 0$) due to the harmonic trap $V(x)$ without the inter-component nonlinear coupling ($g_{i\neq j} = 0$). An interesting energy exchanging process is induced by the inter-component nonlinear coupling, which is defined by the scattering lengths between different components and their relative positions $\Delta x = \langle x\rangle_i - \langle x\rangle_j$, where $\langle\cdots\rangle_i \equiv \int \Psi_i^*(x,t)x\Psi_i(x,t)\,\mathrm{d}x$ is the average with the i-th time-dependent component. As shown in Fig. 1, the Gaussian-like interaction energy (defined as $E(\Delta x) = 0.5g_{ij}\int |\Psi_i|^2\left|\Psi_j\right|^2\,\mathrm{d}x$), depends on Δx, accompanied by an antisymmetric force ($F(\Delta x) = -\partial E(\Delta x)/\partial\Delta x$) when $g_{ij} > 0$. The repulsive interaction slows down their approaching process, but speeds up their departing process, which is similar with the elastic scattering interaction except for at very short distance.

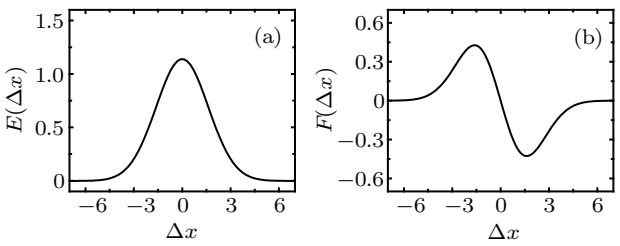


Fig. 1. Interaction energy and force between any two different components with repulsive two-body interaction.

In this paper, we focus on the energy sharing process, which could be a quantum analogy with the well-known FPU problem studied for more than 50 years.[1,2] Compared with the initial FPU model and other existing models in BECs,[9,10] some points should be reminded: i) the same as that in Refs. [1] and [2], the total number of oscillators is an external parameter, but not the same as that in Refs. [9] and [10], wherein it depends on the initial chosen conditions; ii) the intra-nonlinear term is constant, quite different from that in Ref. [1] (frequency-dependence) and Refs. [9] and [10] (dependence on model integral); iii) the modes considered here, are the physically distinguishable components of BECs, but are not the collective motion as before.

3. Rabi, quantum beating and no recurrence energy sharing

Starting with the simplest case governed by Eq. (1), the two components (or species), which may be created by imprinting different momenta on one BECs shown in Ref. [11], are trapped in a harmonic trap $V(x)$, and then all of the intra-nonlinear and inter-coupling parameters in Eq. (1) are the same and are denoted as g. Initially, the two components are exactly the same except for $\Psi_1(x,t)$ with zero momentum, while $\Psi_2(x,t)$ with a nonzero momentum. It is easy to prove that $\Psi_2(x,t)$ performs the oscillated behavior when the inter-component nonlinear coupling ($g_{12} = g_{21} = 0$) is turned off, while $\Psi_1(x,t)$ keeps static at the bottom of $V(x)$. Turning on the inter-component nonlinear coupling ($g_{12} = g_{21} \neq 0$), how initially excited energy of $\Psi_2(x,t)$ flows to or exchanges with that of $\Psi_1(x,t)$ attracts our attention.

Before going into the details, two important quantities are introduced. One is the energy ($E_i(t)$) for the i-th component, which is defined by

$$E_i(t) = \int \Psi_i^*(x,t)H_\mathrm{e}\Psi_i(x,t)\,\mathrm{d}x, \tag{2}$$

where

$$H_\mathrm{e} \equiv -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x) + \frac{1}{2}g_{ii}|\Psi_i(x,t)|^2,$$

and the time-dependent wave function $\Psi_i(x,t)$ can be obtained by numerically solving the coupled GP equations (Eq. (1)) by the second-order splitting-operator method. The other is the Husimi distribution function, which is one kind of quasi-probability distribution function governed by Schrödinger's equation,[15,16] and provides a convenient way to explore the complex dynamics of a quantum system in phase space.[16] Using the characteristic units of a harmonic trap, Husimi distribution function can be written as

$$F_i^\mathrm{H}(q,p,t) = \frac{1}{2\pi\hbar}\left|\int \mathrm{d}x\beta_{qp}^*(x)\,\Psi_i(x,t)\right|^2, \tag{3}$$

where $\beta_{qp}(x)$ is the coherent state in real space,

$$\beta_{qp}(x) = \left(\frac{1}{\sqrt{\pi}}\right)^{1/4} \mathrm{e}^{-(x-q)^2/2}\,\mathrm{e}^{\mathrm{i}px}. \tag{4}$$

The same as in Ref. [16], we have averaged the Husimi distribution over ten-thousand times after the total observation time ($800T_0$ and $T_0 = 6.25$ in this paper), where T_0 is the period of the oscillation of excited nonlinear component at $g_{12} = g_{21} = 0$.

As shown in Fig. 2, with the increase of the value of g, the initially excited energy flows into the nonlinear coupled component as a Rabi-like oscillation, a quantum beating oscillation, and then finally approaches to a strong energy sharing, depending on how strong the nonlinear coupling (g) is. The complex dynamics of energy evolution $E(t)$ shown in Fig. 2 at $g = 9, 10$ is consistent with the general picture described in Ref. [2]: A beating-like oscillation is observed when $t < \tau_c$, but one more complex behavior is found when $t > \tau_c$, where τ_c, which is around $460T_0$ ($4T_0$) for $g = 9$ (10) in Fig. 2, is called the introduction time and the inverse with the strength of inter-component nonlinear coupling g.

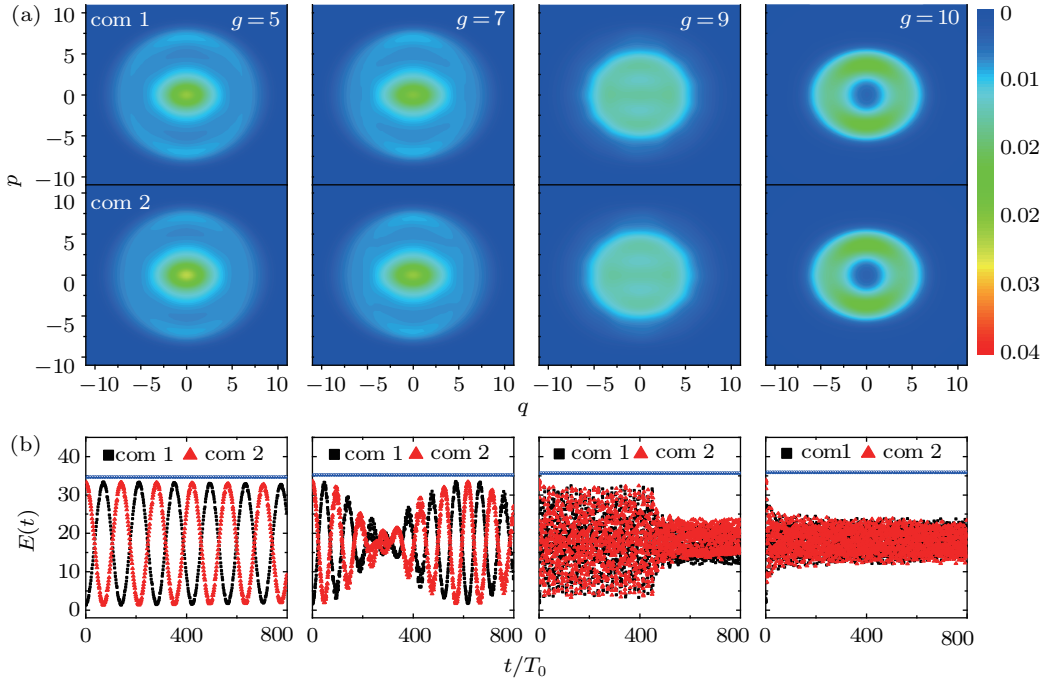


Fig. 2. (color online) The energy exchanging with the increase of the nonlinear coupling strength from $g = 5$ to 10. Panel (a) is corresponding averaged Husimi distributions (Eq. (3)) and panel (b) is the evolution of energy (Eq. (2)).

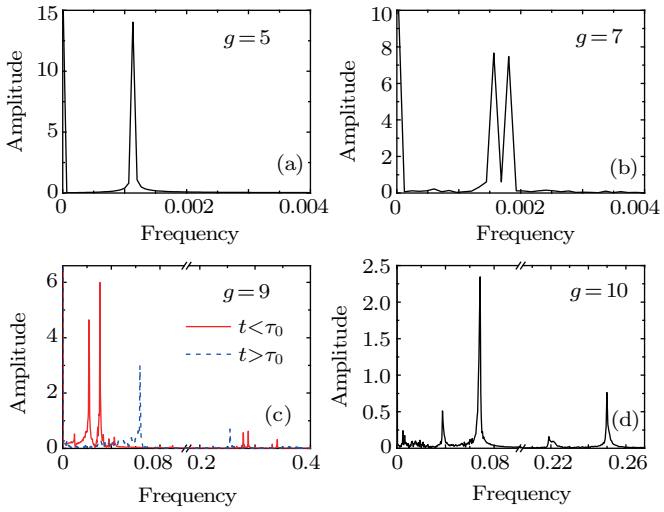


Fig. 3. (color online) The Fourier spectrums of initial excited $E_2(t)$. (a) $g = 5$, (b) $g = 7$, (c) $g = 9$, the red-solid line for $t < \tau_0$ and the blue dashed line for $t > \tau_0$, and (d) $g = 10$.

With the increase of the nonlinear coupling strength from $g = 5$ to 9, the energy exchanging between two components performs from one clear regular (Rabi-like) behavior to a more complex behavior, as shown in Fig. 2. The corresponding average Husimi distribution functions are shown at the top of Fig. 2. The blue-dotted lines of $E(t)$ show that the total energy of this two-component system is conserved during the observation time. In the weak nonlinear coupling (for example, $g < 5$), a Rabi-like energy exchanging can be found, and its frequency is proportional to the nonlinear parameter g. The corresponding Husimi distribution function shows that those two components of BECs share the initial excited energy and develop almost the same region in the phase space. Further increasing g, one more frequency emerges, and the performance of the energy exchanging is the same as a beating-like oscillation, while the behavior of Husimi distribution function does not change any more. A critical time ($\tau_c \sim 460T_0$) is found for $g = 9$, over which a relatively large amplitude collapses to a relatively small one. This critical value can be very short ($\tau_c \sim 4T_0$) when $g = 10$. Meanwhile, the shape of the averaged Husimi distribution function is modified to circles, which may

reveal the completion of the energy sharing process.

To thoroughly understand the dynamic complexity of the time evolution of energy shown in Fig. 2, we make Fourier analysis for $E_2(t)$ and plot the extracted frequencies in Fig. 3. In Ref. [6] and [17], a strong energy sharing (or an equilibrium) could be distinguished by checking its frequency spectrum. The Rabi-like and quantum beating-like oscillations of Fig. 2 are supported by a single or two primary frequencies, shown in Figs. 3(a) and 3(b) for weak nonlinearity. The time evolution of $E_2(t)$ can also be fitted by the following equation:

$$E(t) = A\sum_i \cos\left[2\pi T_0 \omega_i (t+t_0)\right] + B, \tag{5}$$

where i denotes how many primary frequencies ω_i in Fig. 3. The A, t_0, and B are the fitting parameters. For example, the weak nonlinearity $g = 5$, we have only one primary frequency $\omega_1 = 0.001134$, and two frequencies $\omega_1 = 0.001566$ and $\omega_2 = 0.001807$ for $g = 7$. The corresponding fitting behaviors by Eq. (5) can be found in Figs. 4(a) and 4(b). The two different behaviors can also be read from Fourier spectrums. For $g = 9$ and $t < \tau_0 = 460T_0$, we can still fit its time evolution of $E_2(t)$ (see Fig. 4(c)) by Eq. (5) with $\omega_1 = 0.021786$ and $\omega_2 = 0.030501$ (as the red solid curve shown in Fig. 3(c)). It means that the time evolution of $E_2(t)$ can be considered as a quantum beating-like one. Meanwhile, in the case of $t > \tau_c$, we cannot fit $E_2(t)$ by Eq. (5) with only two frequencies (see Fig. 4(d)). Actually, we also see a relatively broad range of frequencies around $\omega \sim 0.06$ in Fig. 3(c), and this implies that a strong energy sharing appears. A similar characteristic in frequency spectrum is also found in Fig. 3(d) for $g = 10$.

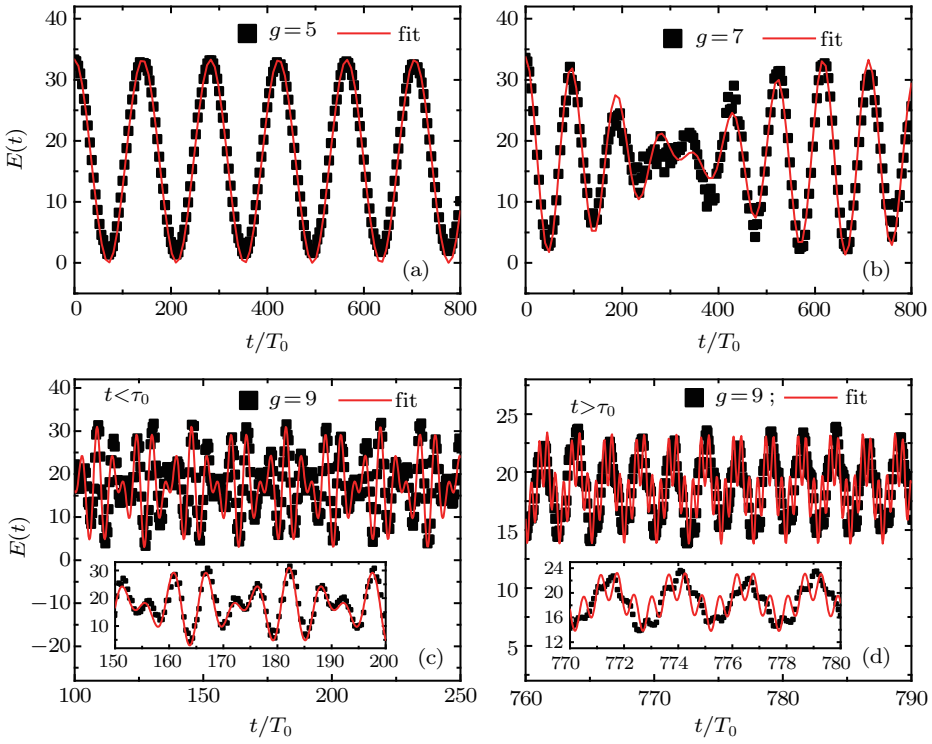


Fig. 4. (color online) The fitting of $E_2(t)$ with Eq. (5). (a) Rabi in $g = 5$; (b) quantum beating in $g = 7$; (c) quantum beating for $t < \tau_0$ in $g = 9$; (d) energy sharing for $t > \tau_0$ in $g = 9$.

Table 1. Probabilities P and statistical entropy S of two components.

	$\eta = 5$	$\eta = 7$	$\eta = 9$	$\eta = 10$
$\overline{E}_1$	17.694	17.759	17.534	17.934
$\overline{E}_2$	16.938	17.406	18.125	17.960
E_{total}	34.632	35.165	35.659	35.894
P_1	0.511	0.505	0.492	0.4996
P_2	0.489	0.495	0.508	0.5004
S	0.692905	0.693097	0.693019	0.693147

The equilibrium or strong energy sharing can also be proved by checking the statistical entropy S,[2,9] which is defined as

$$S = -\sum_i P_i \ln P_i, \tag{6}$$

where P_i is defined as the probability $P_i = \overline{E}_i / E_{\text{total}}$, where $\overline{E}_i$ is the time average of energy of every component and E_{total} is the total energy of all components. The corresponding P_i of process in Fig. 2 and S are calculated and listed in Table 1. It is easy to see that with the increase of the nonlinearity, P_i approaches to 0.5 and the entropy S to $\ln(2) = 0.69314$.

4. Energy sharing of three components

It is interesting to extend the above investigation to more components, for example, three components.

Similar to that in Section 3, the nonlinearity is increased for involving three components. Now, there are at least six

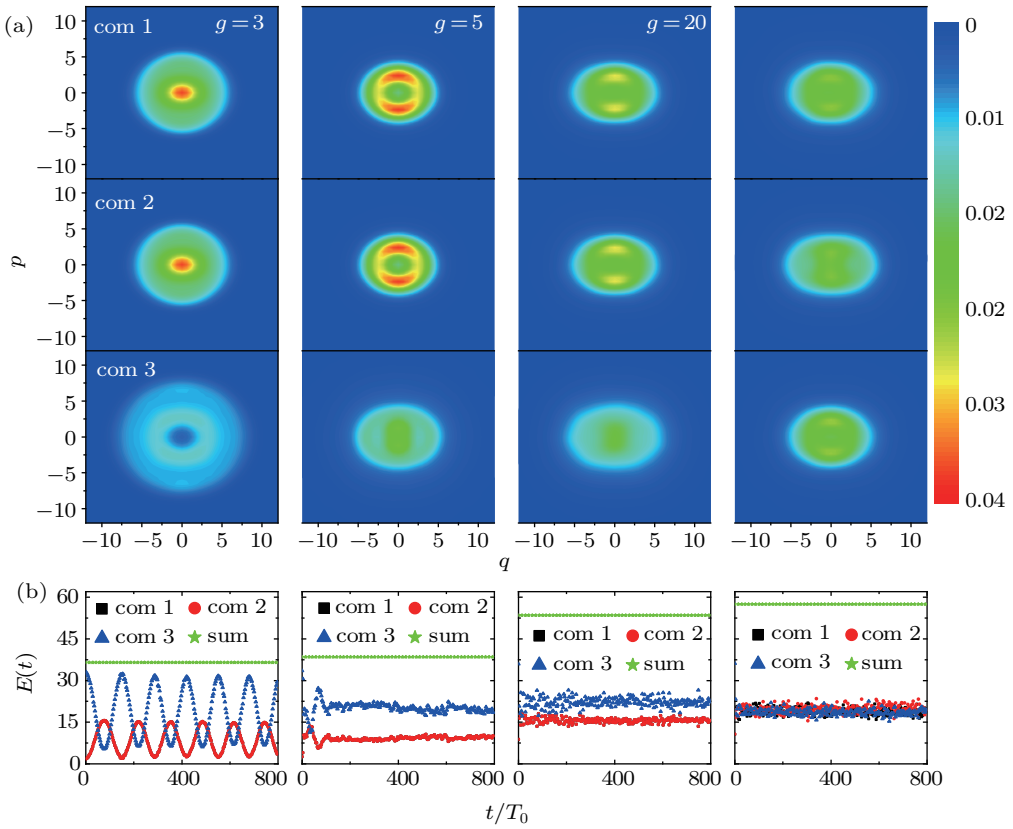


Fig. 5. (color online) The energy exchanging with the increase of the nonlinear coupling strength from $g = 3$ to 20 for the first column, and the last column with $g_{ii} = 20$, $g_{12} \equiv g_{21} = 30$, $g_{13} \equiv g_{31} = 10$, and $g_{23} \equiv g_{32} = 20$. The above is corresponding averaged Husimi distributions (Eq. (3)) and the bottom is the evolution of energy (Eq. (2)).

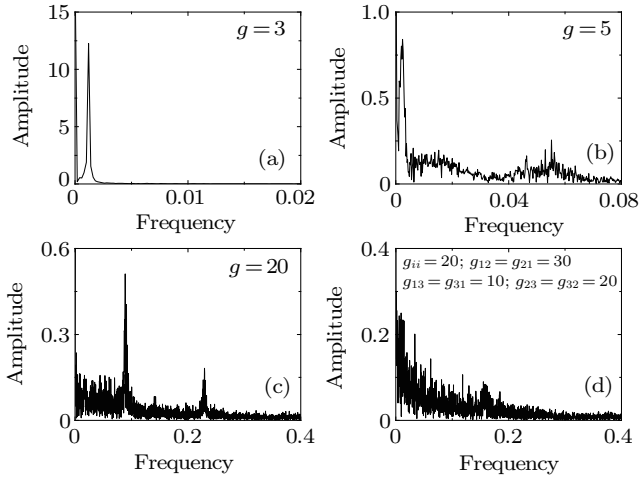


Fig. 6. The Fourier spectrums of initial excited $E_3(t)$. (a) $g = 3$, (b) $g = 5$, (c) $g = 20$, and (d) $g_{ii} = 20$, $g_{12} \equiv g_{21} = 30$, $g_{13} \equiv g_{31} = 10$ and $g_{23} \equiv g_{32} = 20$.

Table 2. Probabilities P and statistical entropy S of three components.

	$\eta = 3$	$\eta = 5$	$\eta = 20$	Different
$\overline{E}_1$	8.8296	9.2426	15.6023	18.9570
$\overline{E}_2$	8.8296	9.2426	15.6023	19.8402
$\overline{E}_3$	18.8329	20.0016	22.2424	18.6392
E_{total}	36.492	38.487	53.447	57.436
P_1	0.242	0.240	0.292	0.330
P_2	0.242	0.240	0.292	0.345
P_3	0.516	0.520	0.416	0.325
S	1.028	1.025	1.084	1.098

nonlinear parameters g_{ij} for $i, j = 1, 2, 3$. With the increase of the nonlinearity for all $g_{ij} = g$ shown in the first three columns of Fig. 5, Rabi-like and strong energy sharing, except for quantum beating, are found. It is easy to note that the Rabi-like energy exchanging can exist through three components, but with different amount. Since we excite initially the third component ($E_3(t)$), but the other two components remain static, we have two equal and small amplifications for $E_{1,2}(t)$ in the case of small nonlinearity. With the increase of the nonlinearity, the amplifications (see probabilities $P_{1,2}$ in Table 2) are increasing. It means that the initial excited energy is going to sharing between the three elements. This can be seen from Fig. 6, where more broad frequency spectrums are displayed with g increasing. To show one equilibrium case, we have to consider one more complex nonlinear coupling, where $g_{ii} = 20$, $g_{12} \equiv g_{21} = 30$, $g_{13} \equiv g_{31} = 10$ and $g_{23} \equiv g_{32} = 20$. The strong energy sharing can be read from statistical entropy S shown in Table 2, where $S \to \ln(3) \sim 1.0986$.

5. Conclusion

In this work, we numerically investigated the energy sharing process in multi-component coupled GP equations with a harmonic trap, with help of the second-order splitting-operator method. Clearly, three different energy sharing behaviors, from regular to chaotic, have been shown for two- and three-

component BECs with a harmonic trap. With the increase of the nonlinear interaction, more frequencies, from single to more broad frequency spectrum, are involved into the time evolution of energy. An introduction time is inverse with the nonlinear interaction for the two-component case. After that, a strong energy sharing is determined by checking the Fourier spectrum of the time evolution of energy and also the statistical entropy. The final value of statistical entropy is close to $\ln(2)$ for two components and $\ln(3)$ for three components, so we would like to say that our system is in an equilibrium state. Considering the situation in experiment,[11] where they initially created two-component BECs with opposite momenta ($\pm 2\hbar k$) and realized the dynamics for many times, this actually already is one energy equilibrium state in the point of view of energy. Therefore, even though they could increase the nonlinear interaction to a very strong value ($\gamma_{\rm d} = 18$) and observe a long oscillation of dynamical interaction, the energy sharing evidence cannot be found. Based on this reason, we suggest a similar experiment with only one-component excitation. It is a crucial factor that the two components of the BECs have different initial energy. Based on the above discussion, the experiment conditions of our project are qualified to observe the energy sharing, and an evidence of energy sharing could be experimentally supposed to be obtained in the near future.

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