Uphill anomalous transport in a deterministic system with speed-dependent friction coefficient*

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We investigate the transport of a deterministic Brownian particle theoretically, which moves in simple one-dimensional, symmetric periodic potentials under the influence of both a time periodic and a static biasing force. The physical system employed contains a friction coefficient that is speed-dependent. Within the tailored parameter regime, the absolute negative mobility, in which a particle can travel in the direction opposite to a constant applied force, is observed. This behavior is robust and can be maximized at two regimes upon variation of the characteristic factor of friction coefficient. Further analysis reveals that this uphill motion is subdiffusion in terms of localization (diffusion coefficient with the form $D(t) \sim t^{-1}$ at long times). We also have observed the non-trivially anomalous subdiffusion which is significantly deviated from the localization; whereas most of the downhill motion evolves chaotically, with the normal diffusion.

Keywords: speed-dependent friction coefficient, anomalous transport, anomalous diffusion

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1. Introduction

Transport of single driven and damped Brownian particle moving in a periodic potential continues to attract considerable interest, due to the possible applications in many different contexts of physics, chemistry, and biology.[1–17] Existing works often model thermal noise effects, and many of significant transport phenomena, such as the current reversals and negative mobility (broadly speaking, a particle traveling in the direction opposite to a dc-bias), have been studied.[4,7,11,14–17] Fewer works[1,2,6,8,10] focus on the deterministic systems, i.e., the noiseless case. The deterministic systems may be, at least experimentally, more controllable and readily accessible than the noisy systems.[18] Recently, the archetypal model of a vibrational motor is proposed,[19] in which an additional driving term yields stochastic-like dynamics. Striking phenomena termed absolute negative mobility (ANM),[19] anomalous diffusion, and enhancement of diffusion[20] are observed in this motor. In a very recent work,[21] the ANM behavior for a deterministic particle with a nonuniform space-dependent damping has been investigated. This behavior is corresponding to the superdiffusive motion of the particle. Even more surprising is that ANM has been observed for a deterministic particle evolving in a phase-modulated (in time) symmetric potential, and this modulated phase assisted by a periodic driving force can lead to the occurrence of ANM.[22]

In order to describe self-propelled motion in biology, appearing on levels ranging from flocks of animals to single cell motility, one class of models, active Brownian particles (ABPs),[23] has been scrutinized in a huge number of works.[24–35] The ABPs can transform the internal energy into energy of motion that drives themselves out of equilibrium. Commonly, it refers to the underdamped Brownian dynamics (requiring the inertial term). Essential thing is not only the mere speed dependence of the friction coefficient, but also that the friction coefficient is negative over a range of velocities. The positive and negative friction coefficients are corresponding to the dissipation and the uptake of energy, respectively.[27,30] There are three popular paradigms for the speed-dependent friction coefficient. In the simplified depot model proposed by Schweitzer, Ebeling and Tilch (SET),[24] the friction coefficient can be expressed as $\gamma(v) = \gamma_0[1-\beta/(1+v^2)]$, where γ_0 and β are two constants.[29] If $\beta > 1$, the friction is negative at speeds between $v = \pm\sqrt{\beta-1}$ and is positive beyond this range. That is, in the range of small velocities, a pumping-like energy occurs as an additional source of energy to accelerate the slow particles. Another typical choice for the friction function is Rayleigh–Helmholtz (RH) friction $\gamma(v) = \gamma_0(v^2 - v_0^2)$ (v_0 constant), which was originally studied by Rayleigh and Helmholtz[36,37] in the propagation of sound and has become a standard model in the study of Brownian dynamics.[29] The third model derived by Schienbein and Gruler (SG)[38] from experiments with a mov-

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ing cell has the form $\gamma(v) = \gamma_0(1 - v_0/v)$ with a discontinuity at $v = 0$. It allows us to describe the active motion of different types, such as granulocytes, monocytes and neural crest cells.[29,38] Understanding the non-equilibrium behavior, particularly transport behavior, of self-propelled objects is one of the major challenges at the interface of physics, biology, and also chemical engineering.[29,39] Consequently, the model investigation has flourished as one of the most active fields in recent times. In the absence of spatially periodic external potentials, for a noisy system with RH friction, the critical force separates parameter regimes of giant diffusion from the regimes with reliable directed transport.[27] The transport of the relevant coupled systems have been reported in Ref. [31]. In the presence of periodic potentials, the diffusive transport of ABPs has also attracted much attention,[28,40] e.g., the direct transport by adjusting the shape of potential.[40] Lately, Ghosh *et al.* propose considering the combined effects of the particle shape and confined geometry, i.e., the dynamics of ABPs of different shapes in deformable channels.[33,34] The giant negative mobility and ANM are observed there.[34] These works on ABPs, to name but a few, are often related to noise effects.

Motivated by the research on the deterministic (noiseless) Brownian motor as mentioned above, it is interesting to exploit the transport for a deterministic ABP in a spatially periodic potential. To the best of our knowledge, the diffusive transport, especially the ANM, for a deterministic ABP has not been reported yet, and thus an inclusion of speed-dependent friction here can provide an alternative scenario for appearance of ANM.

2. System

Our working model describes the motion of a deterministic ABP with mass M, in a symmetric, periodic potential $V(x) = V(x+L)$, written as

$$M\ddot{x} + \gamma(v)\dot{x} = -V'(x) + F(t) + f, \tag{1}$$

where a dot and prime denote differentiation with respect to t and x, respectively. $F(t) = F(t+\tau) = -F(t+\tau/2)$ is a temporally symmetric driving. f is a positive dc-bias. We focus on the simplest example, namely friction coefficient with SET form,[24,28,29] and purely harmonic potential and driving. We adopt dimensionless units of time, length, and mass:

$$\begin{aligned} &M = 1, \quad \gamma(v) = \gamma_0\left(1 - \frac{\beta}{1+v^2}\right), \\ &V(x) = -\sin(x), \quad F(t) = a\cos(\omega t), \end{aligned} \tag{2}$$

where γ_0 is the original friction coefficient and β is the characteristic factor of friction coefficient. The negative and positive friction coefficients are corresponding to the uptake and the dissipation of energy from external environment, respectively.[27,29,30] Note that as $\beta > 1$, the friction coefficient is negative for the speeds $v \in [-\sqrt{\beta-1}, \sqrt{\beta-1}]$ and is positive outside this range. Namely, the slow particle is accelerated while the motion of the fast particle becomes damped, which is just the self-propelled motion mentioned above.

To begin with, the quantity of central interest is the mean velocity of particle (or particle current), defined by[14,15]

$$\langle v \rangle = \frac{\tau}{L}\left\langle \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{d}t'\dot{x}(t') \right\rangle, \tag{3}$$

where $\langle \cdots \rangle$ indicates the ensemble average, $\tau = 2\pi/\omega$, and $L = 2\pi$.

The other quantity employed is the time-dependent diffusion coefficient, given by the standard prescription[41]

$$D(t) = \frac{\langle \Delta x^2(t) \rangle}{2t} = \frac{\langle x(t)^2 \rangle - \langle x(t) \rangle^2}{2t}. \tag{4}$$

Commonly, $D(t)$ at long times has the form $D(t) \sim t^{\lambda}$, where λ is a phenomenological power exponent. $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ are subdiffusion, normal diffusion, and superdiffusion motion, respectively.[42,43] We term the negative mobility as the anomalous transport,[14,15] i.e., $\langle v \rangle < 0$ for $f > 0$, which has a slight difference in the sense of the wide field of anomalous processes exhibiting nonlinear time-dependence of mean particle positions on time.[44,45] Also, we term both the subdiffusion and superdiffusion as the anomalous diffusion as usual.

As the system Eq. (1) is purely deterministic, the resulting asymptotic long time dynamics is not ergodic.[13–15] An additional average over the initial conditions must be performed. We carried out numerical simulations of Eq. (1) with Eq. (2) for $\langle v \rangle$ and $D(t)$ by the second-order Runge–Kutta method with time step $\Delta t = 0.005$. The ensemble average is taken over $N = 1000$ trajectories with initial uniformly distributed position $x(0) \in [-\pi, \pi]$ and vanished velocity $v(0) = 0$.[21] The bifurcation diagram of the system can be constructed in several different and essentially equivalent ways. The relatively standard form used here is that the initial condition is reset whenever β changes.[46] In general, the system may exhibit a rich and varied behavior as a function of its parameters γ_0, a, and ω.[4,15–17] However, one of our main objectives is to explore the effects of nonlinear velocity-dependent friction on the transport and diffusion of particle described by Eqs. (1) and (2). Thus, we will vary the parameters β and f, while fixing the remaining parameters $\gamma_0 = 0.108$, $a = 1.512$, and $\omega = 0.58$ throughout the paper. Specifically, they have been carefully chosen as a set of parameter values for the emergence of ANM in a deterministic system induced by spatial friction inhomogeneity.[21] Parts of our numerical results related to the two key transport quantifiers, $\langle v \rangle$ and $D(t)$, are presented as follows.

3. Transport and diffusion

Figure 1 reveals the dependence of mean velocity on the characteristic factor of the friction coefficient β for different tilted forces f. For $f = 0$, the mean velocity is in the main close to zero. It should be noted that in the range $\beta \in [1.005, 1.482]$, it seems that $\langle v \rangle$ changes its direction many times around the zero, but one may not consider it as current reversals. According to the symmetry analysis,[4,17] a necessary condition for a nonzero particle current is that both the space- and time-reversal symmetries of the system are broken after averaging over ensemble and time for the symmetrically initial distribution. The space-reversal symmetry exists in our case, thus the contributions of the trajectory and its symmetry-related twin will cancel each other. This prohibits the appearance of a directed current and time-shift symmetry. Therefore, the mean velocity of particle should be strictly vanished here. This slight deviation from 0 may be only due to the finite length of time series and the trajectory number in the simulation. With increasing the bias force to $f = 0.1$, one can recognize that two windows with regard to β ($0.72 \leq \beta \leq 1.435$ and $2.353 \leq \beta \leq 3.035$) open to support ANM. In both windows, the particle current is located at the value $\langle v \rangle \approx -3$ and is hardly affected by β, i.e., the robustness of ANM against β. The ANM occurs as both $\beta < 1$ and $\beta > 1$, and thus the self-propulsion nature of the particle as mentioned above is not its prerequisite condition. Outside of the windows, the current is positive and follows the bias. Crucially, most of the β values inside the windows of negative mobility produce an approaching zero current in the absence of bias ($f = 0$). Thus, it can be deduced that the effect of this negative mobility is induced by the static bias force, that is, the tilting of the potential $(V(x) - fx)$ combining a carefully chosen β results in an uphill motion.

According to our extensive simulations, the windows of ANM exist only for $f < f_{\rm cr} \approx 0.2$. There are dual windows with respect to β in the range $0.075 \leq f \leq 0.1375$, beyond which only a single window emerges, for example $f = 0.15$. As $f \to f_{\rm cr}$, the maximal currents with the opposite direction of the bias become smaller and smaller, and eventually vanish. This is due to the fact that for a sufficiently large bias, the periodic portion of the potential of the system becomes unimportant and the particle locates in the running state. An example is for $f = 0.25$ where the current is $\langle v \rangle > 3$.

In order to explore the underlying mechanism of the above observation, the typical bifurcation diagram and the time series are displayed in Figs. 2 and 3. Figure 2 shows that this system exhibits chaotic solutions and periodic solutions in the range of β considered. In Fig. 2(a), there exist periodic solutions as $\beta \in [1.005, 1.482]$, which supports the fluctuation of particle current shown in Fig. 1. Corresponding to the dual-window of ANM with $f = 0.1$ in Fig. 1, just the two windows in Fig. 2(b) support periodic solutions. After a careful inspection of the two curves, one can perceive the connection between the current and bifurcation: the sharp transition from chaotic to periodic (periodic to chaotic) motion related to the transition from downhill to uphill (uphill to downhill) motion, i.e., related closely to the current reversals. The excepted example is the narrow window in the vicinity of $\beta = 2$ for $f = 0.1$, which is only corresponding to a sudden current change (jump current), not to any current reversals. The exact mechanism behind it in general single-particle systems remains open to debate.[2,3,12,21,22] Furthermore, for most non-negative current, the system evolves chaotically.

The representative time series are elucidated in Fig. 3. Constituent parts concerning the uphill motion are given in Fig. 3(c). Initially, the driving (middle panel) becomes

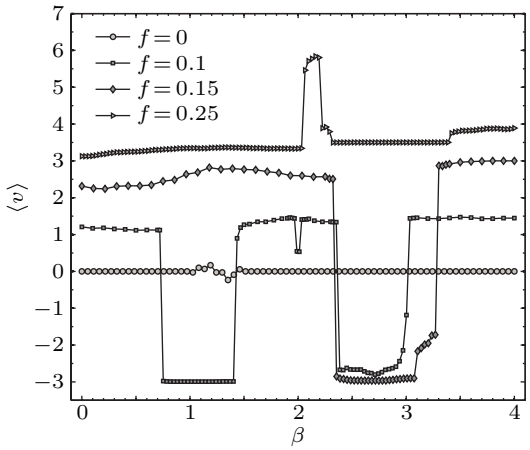


Fig. 1. The mean velocity $\langle v \rangle$ versus β for different dc-bias, obtained by numerical solving Eqs. (1) and (2) with uniformly distributed initial condition and time interval $T_{\rm s} = 2.5 \times 10^6$. The remaining parameters are $\gamma_0 = 0.108$, $a = 1.512$, and $\omega = 0.58$.

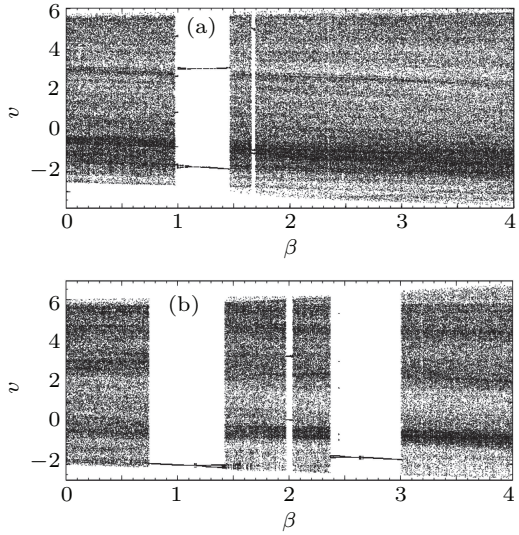


Fig. 2. Bifurcation diagram as a function of β for the two curves in Fig. 1, $f = 0$ (a) and $f = 0.1$ (b), obtained by stroboscopically sampling trajectories after each period of the driving, discarding a considerably long transient. The initial condition (uniformly distributed position $x(0) \in [-\pi, \pi]$ and vanished velocity $v(0) = 0$[21]) is reset whenever β changes.[46]

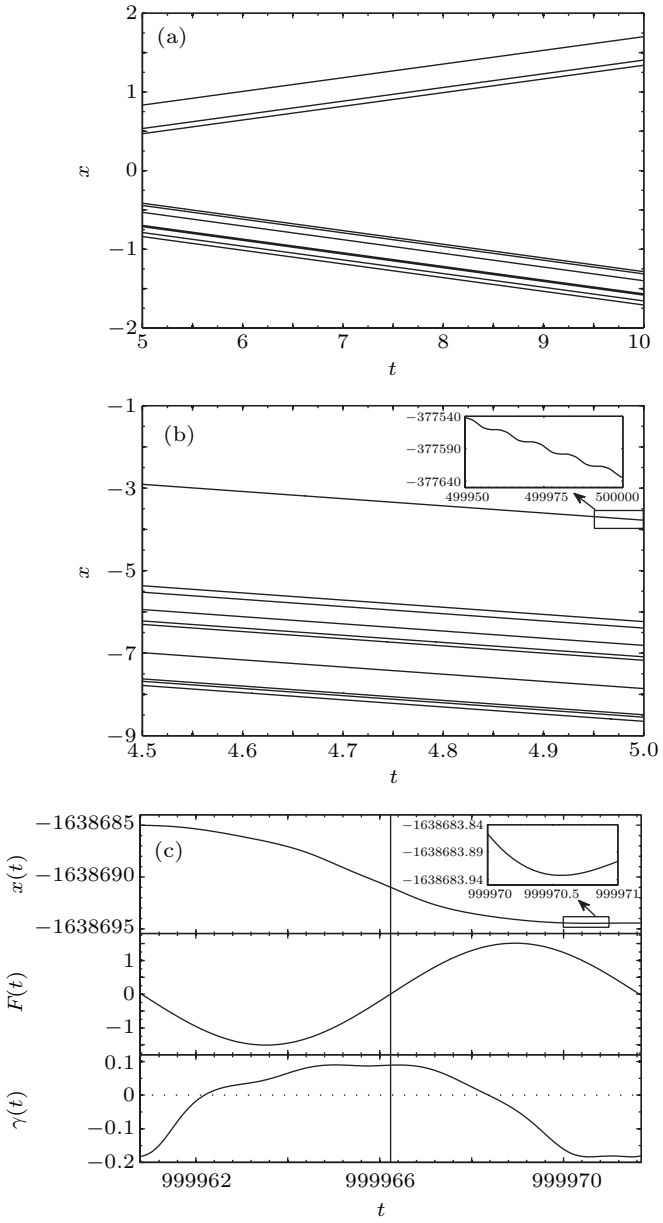


Fig. 3. The evolution of an ensemble with 10 trajectories after a considerably long time from the different initial condition: (a) $f = 0$ and $\beta = 1.2$, the coexisting attractor case, (b) $f = 0.1$ and $\beta = 2.7$, the single attractor case. (c) Constituent parts of a solution exhibiting negative mobility controlled by the single attractor shown in panel (b), evolving for an external driving period τ. The other parameters are the same as that in Fig. 1.

negative, while the damping (bottom panel) is approaching its minimum (≈ -0.183), i.e., this coordination between driving and damping enables the particle to move almost uphill. Subsequently, at $t \approx 999965$ the damping coefficient attains its maximal value (≈ 0.089). However, the particle is driven against the bias as before, owing to the negative-valued driving. After that, the driving becomes positive at $t \approx 999966.24$, indicated by the vertical line, while the damping coefficient is still approaching its maximum. This leads to the slower uphill motion. Lastly, the particle's uphill motion ceases and follows the direction of the bias feebly (the inset). Crucially for a net negative current, this turning point occurs in the final stages $t \approx 9\tau/10$ in the course of a driving period (only short intervals for the downhill motion). This behavior continues in a periodic fashion allowing the particle to travel large distances in the direction opposite to the dc-bias. It should be noted that the occurrence of ANM induced by a time-dependent phase of potential can also be well elucidated by the effective potential of the particle for different snapshot times,[22] while a proper mechanism is still not completely detailed. However, for our purely symmetric potential independent of time in Eq. (1) with Eq. (2), this scheme is not available, and thus not be employed here (see the case of $\lambda = 0$ in Fig. 5 of Ref. [22]).

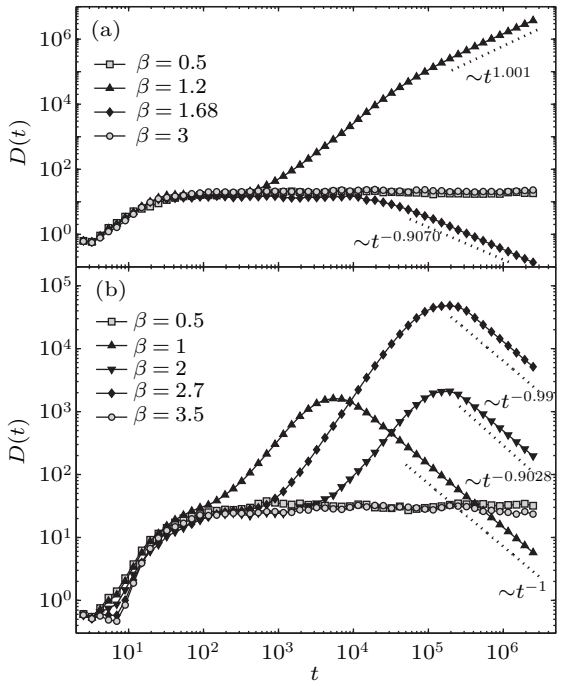


Fig. 4. Log–log plot of the time evolution of the diffusion coefficient $D(t)$ for typical parameter values of β with $f = 0$ (a) and $f = 0.1$ (b). The anomalous diffusive behavior evolving after a considerably long time is indicated by the fitted (dotted) lines.

To further characterize the motion, the time-dependent diffusion coefficient and distribution of particle positions at much longer times are depicted in Figs. 4 and 5. It is shown in Fig. 4 that the early temporal evolution of $D(t)$ is similar, and subsequently at around $t \approx 10^3$ different regimes become apparent for the system with different β values. For $f = 0$ in Fig. 4(a), β substantially affects the diffusive behavior. As $\beta = 0.5$ and 3, $D(t)$ can reach stationary values, which means normal diffusive motion, i.e., $D(t) \sim t^{\lambda}$, $\lambda = 0$. The anomalous diffusion activities in the form of the superdiffusive motion ($D(t) \sim t^{\lambda}$, $\lambda = 1.001 \approx 1$) for $\beta = 1.2$ and the subdiffusive motion ($\lambda = -0.9070$) for $\beta = 1.68$[42] appear. The superdiffusive trajectories have been shown in Fig. 3(a). Here, the coexisting attractors may be responsible for the superdiffusion, due to an ensemble average over them.[21] The superdiffusion ($\lambda \approx 1$) observed above is also termed the ballistic diffusion. For $f = 0.1$ in Fig. 4(b), the normal diffusion for

$\beta = 0.5$ and 3.5 can be identified. Meanwhile, the subdiffusive behavior with different exponents $\lambda = -1, -0.9028, -0.99$, corresponding to $\beta = 1, 2, 2.7$ respectively, can be observed. Notice that a detailed illustration of the trajectories on one kind of subdiffusion, $\lambda \approx -1$, has been provided by Fig. 3(b). That is to say, the trajectories for all initial conditions evolve eventually to the negative-valued attractor. Each trajectory undergoes the same motion resulting in the diffusion of position becoming zero, i.e., coherent transport behavior or dispersionless transport in the long time limit.[11,20,27,31] The other representative subdiffusive behaviors with $\lambda = -0.9070, -0.9028$ are related to the existence of coexisting attractors. This may be explained as follows: the coexisting attractors contribute to the diffusion with different weights,[47] with one attractor being dominant, yielding the subdiffusion. Furthermore, the normal diffusive behaviors in Fig. 4 are associated with the chaotic dynamics (shown in Fig. 2), and related closely to downhill motion (shown in Fig. 1).

Some features of the different dynamical regimes can also be understood by viewing the distribution of the positions of the particle at very long times. In Fig. 5, for the normal diffusion above ($f = 0$ and $\beta = 0.5$), the distribution is clearly tending toward the Gaussian distribution. Compared to this Gaussian shape, we observe the spike-like double peaks of distribution for $f = 0$ and $\beta = 1.2$, indicative of bidirectional motion, corresponding to the ballistic diffusion in Fig. 4. In the subdiffusive regime ($f = 0.1$ and $\beta = 1, 2, 2.7$), the distributions exhibit pronounced asymmetric peaks and are exponential, non-Gaussian. Generally speaking, it is assumed that if the distribution of positions is non-Gaussian, the diffusion is anomalous[48] (for the special case beyond this supposition, see Ref. [49]). These distributions further confirm our results as mentioned.

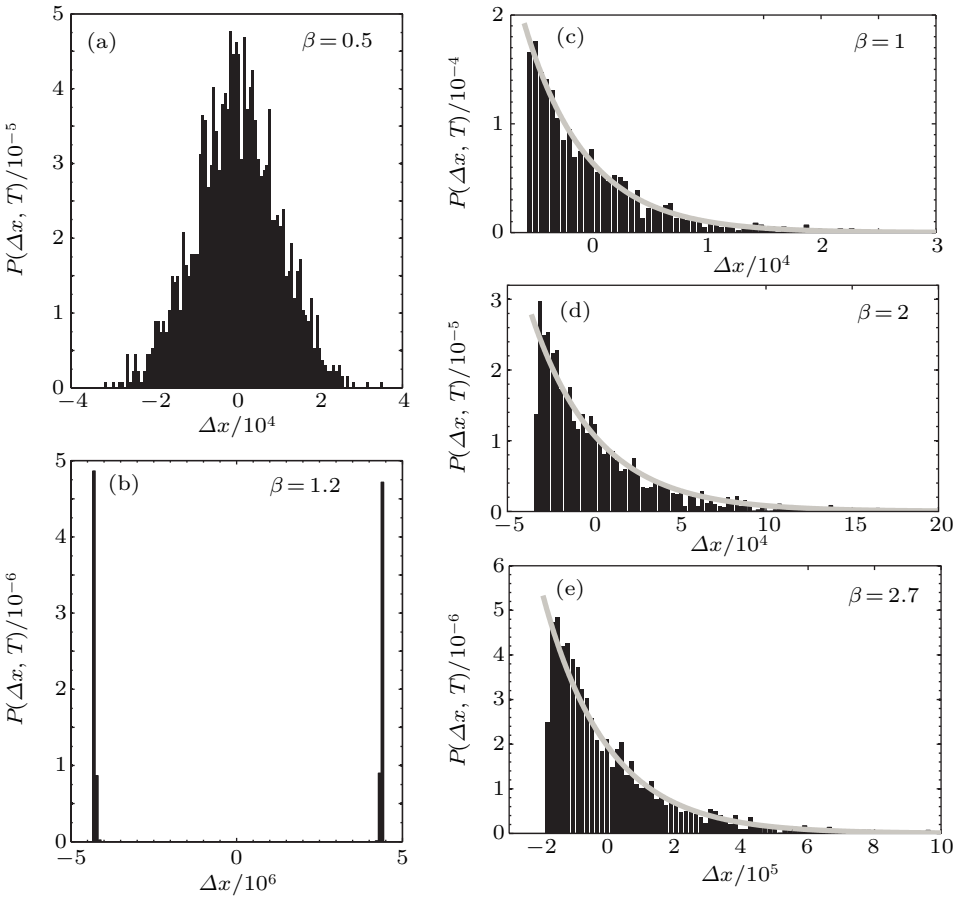


Fig. 5. (color online) Distribution of particle positions $P(\Delta x, T)$, where $\Delta x = x - \langle x(T) \rangle$ at time $T = 2.5 \times 10^6$ with the emblematical parameter values as in Fig. 4, for $f = 0$ (left column) and $f = 0.1$ (right column). In the right column, gray lines are the fitted lines by the exponential function $P(\Delta x) \propto \exp(-\alpha \Delta x)$. The values of α for $\beta = 1, 2, 2.7$ are about 1.0847×10^{-4}, 2.61×10^{-5}, and 5.17×10^{-6}, respectively.

Eventually, we wish to make some comments on the phenomena observed. First, we find that the characteristic factor of friction coefficient β can substantially modulate the transport and diffusion of an ABP in the absence of noise. For the ABP under the action of weak noise, according to our extensive simulations (figures not shown), we can also obtain the analogous characteristics as mentioned and other dynamic properties, such as the negative nonlinear mobil-

ity and the enhancement of diffusion. Moreover, previous works showed that there are only three different exponents for the time-dependent diffusion coefficient in the time-discrete climbing sine map[50,51] or in the vibrational motor,[20] namely $\lambda \approx -1, 0, 1$ in the long time limit. In other words, the subdiffusion only occurs in terms of localization ($\lambda \approx -1$), and superdiffusion in terms of ballistic motion ($\lambda \approx 1$). Here in the deterministic ABP, we find non-trivial anomalous diffusion ($\lambda \approx -0.9070, -0.9028$ as $\beta = 1.68, 2$ for different dc-biases), which may be attributed to the coexisting attractors contributing to the transport and diffusion with different weights. Notice that it occurs only as $\beta > 1$ that the self-propulsion nature of the particle is existent as mentioned above and thus is responsive for this non-trivial behavior. For the absence of self-propulsion ($0 \leq \beta < 1$), the appearance of the above non-trivial anomalous diffusion may be impossible.[20,50,51]

4. Conclusion

We have reported a study on the transport and diffusion of a driven deterministic particle with the speed-dependent friction coefficient, i.e., a deterministic ABP. Within the tailored parameter regime, the uphill anomalous transport, also termed ANM, is optimized at two regimes upon variation of the characteristic factor of friction coefficient, and the robustness can also be observed. A heuristic explanation of its underlying mechanism is the bifurcation from chaotic to periodic orbits, where the current exhibits remarkable jumps that can lead to a situation where the current reverses its direction. Further analysis has revealed that the uphill motion is subdiffusive, whereas most of the downhill motion evolves chaotically with normal diffusion. This is distinct from the ANM phenomenon studied in space-dependent system,[21] where the solutions are superdiffusive. We also have observed the non-trivially anomalous subdiffusion, which is significantly deviated from the localization. It may be attributed to the coexisting attractors contributing to the transport and diffusion with different weights. The self-propulsion nature of the particle is responsible for this non-trivial behavior. An inclusion of speed-dependent friction here can provide an alternative scenario for appearance of ANM in periodic systems under noiseless environment. Moreover, the anomalous diffusion phenomenon has been observed for active motion in the biological cells,[52] plant lice,[53] and humans.[54] The results mentioned above may be helpful for further understanding the nontrivial response of nonlinear dynamics to external bias, and also may have potential applications for activities of biological processes.

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