Robust pre-specified time synchronization of chaotic systems by employing time-varying switching surfaces in the sliding mode control scheme

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In the conventional chaos synchronization methods, the time at which two chaotic systems are synchronized, is usually unknown and depends on initial conditions. In this work based on Lyapunov stability theory a sliding mode controller with time-varying switching surfaces is proposed to achieve chaos synchronization at a pre-specified time for the first time. The proposed controller is able to synchronize chaotic systems precisely at any time when we want. Moreover, by choosing the time-varying switching surfaces in a way that the reaching phase is eliminated, the synchronization becomes robust to uncertainties and exogenous disturbances. Simulation results are presented to show the effectiveness of the proposed method of stabilizing and synchronizing chaotic systems with complete robustness to uncertainty and disturbances exactly at a pre-specified time.

Keywords: chaos synchronization, finite time synchronization, sliding mode controller, time varying switching surfaces

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1. Introduction

In recent decades, chaos synchronization has attracted attention of many researchers. The aim of the synchronization is that a slave system follows completely the master system, despite existence of uncertainties and exogenous disturbances in both systems and difference in their initial conditions. So far various control methods have been utilized for synchronizing chaotic systems, e.g. back stepping method,[1,2] linear feedback control,[3,4] nonlinear feedback control,[5] adaptive control,[6,7] adaptive feedback,[8] fuzzy logic control,[9] LMI-based control,[10] observer-based control,[11,12] active control,[13–16] Lyapunov stability theory,[17–20] and so on. In Refs. [21] and [22], active controls have been used under the assumption of known parameters and Lyapunov stability theory has been used for establishing parameter update laws when parameters are uncertain or fully unknown.

The sliding mode control (SMC) is the other control method that has been also utilized effectively for chaos synchronization. Although robustness to uncertainty and disturbances in sliding phase is one of the most important advantages of SMC, neither uncertainty nor disturbances have been considered in Refs. [23]–[29]. Uncertainty and disturbances have been considered in Refs. [30]–[34] but these studies are not completely robust due to the existence of the reaching phase. However, this problem has been resolved in Refs. [35]–[38] by establishing a reaching law and in Ref. [39] by eliminating the reaching phase. All the papers, which have been so far mentioned, have asymptotical stability. As a result, in these papers, synchronization will be realized at infinity analytically.

Finite-time stability for chaos synchronization, stabilization and control has been achieved in some studies such as Refs. [40]–[48] with control methods other than the sliding mode one. Sun *et al.* in Ref. [49] have proposed an adaptive combination sliding mode controller based on using the nonsingular terminal sliding mode control technique. In Ref. [50], finite-time synchronization has been accomplished by introducing a new nonsingular terminal sliding surface. Similarly, Aghababa and Aghababa in Ref. [51] have managed to synchronize two chaotic systems finite-timely by introducing a novel nonsingular terminal sliding surface. In some papers such as Refs. [52]–[54], finite-time stability is available only for either the reaching phase or sliding phase. Sun *et al.* in Ref. [55] not only have succeeded in finite-time synchronization but also have made synchronization completely robust by eliminating the reaching phase. Nevertheless, in all these papers, only an upper bound of synchronization time has been determined and it depends directly on the initial error. Hence, the greater the differences, the more time the controller must take to synchronize chaotic systems. For reducing this time and subsequently the energy consumption, initial conditions of two chaotic systems must be made nearer to each other at the initial time as far as we can.

Effati *et al.* in Ref. [56] could accomplish the chaos synchronization at a pre-specified time by an optimal control design. In their work, the proposed controller is only applicable to chaotic Chen systems, uncertainty and disturbances must

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not exist and more important than the others is that we have to solve a two-point boundary value problem (TPBVP).

Motivated by the above discussion, the aim of this paper is to resolve the above problems by using the sliding mode technique. To the best of our knowledge, this is the first work in synchronizing and stabilizing a large class of chaotic system in the presence of uncertainty and disturbances precisely at a time when that time has been set in advance. Indeed, what is intended to be carried out is that two chaotic systems with uncertainty and disturbances are synchronized exactly at any time when we want, is needed or asked for free of worrying about how much two chaotic systems are far from each other at the initial time. We will show that it can be achieved remarkably without requiring the solution of a time-consuming TPBVP by employing time-varying switching surfaces. Moreover, these surfaces are determined in such a way that the reaching phase is eliminated such that the synchronization process is completely robust to uncertainty and disturbances.

2. Preliminaries

Theorem 1[57,58] Assume that a continuous, positive-definite function $V(t)$ satisfies the following differential inequalities:

$$\dot{V}(t) \leq -cV^{\eta}(t), \quad \forall t \geq t_0, \quad V(t_0) \geq 0, \tag{1}$$

where $c > 0$ and $0 < \eta < 1$ are constants. Then, for any given t_0, $V(t)$ satisfies the following inequality:

$$V^{1-\eta}(t) \leq V^{1-\eta}(t_0) - c(1-\eta)(t-t_0), \quad t_0 \leq t \leq t_1 \tag{2}$$

and

$$V(t) = 0, \quad \forall t \geq t_1 \tag{3}$$

with t_1 given by

$$t_1 \leq t_0 + \frac{V^{1-\eta}(t_0)}{c(1-\eta)}. \tag{4}$$

Lemma 1[59] Assume that $a_1, a_2, \ldots, a_n$ and $0 < q < 1$ are real numbers, then we have

$$(|a_1| + |a_2| + \cdots + |a_n|)^q \leq |a_1|^q + |a_2|^q + \cdots + |a_n|^q. \tag{5}$$

3. System description and problem statement

It is assumed that master and slave systems are modeled as Eqs. (6) and (7) respectively.

$$\begin{cases} \dot{y}_1 = g_1(Y) + \delta g_1(Y,t) + d_1^m(t), \\ \dot{y}_2 = g_2(Y) + \delta g_2(Y,t) + d_2^m(t), \\ \cdots \\ \dot{y}_n = g_n(Y) + \delta g_n(Y,t) + d_n^m(t), \end{cases} \tag{6}$$

where $Y = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} \in R^n$ is the state vector of the master system, $g_i : R^n \to R$ for $i = 1, \ldots, n$ are given nonlinear functions of Y and t, $\delta g_i(Y,t)$ and $d_i^m(t)$ for $i = 1, \ldots, n$ are the master system uncertainty and disturbances respectively, and

$$\begin{cases} \dot{x}_1 = f_1(X) + \delta f_1(X,t) + d_1^s(t) + u_1(X,Y,t), \\ \dot{x}_2 = f_2(X) + \delta f_2(X,t) + d_2^s(t) + u_2(X,Y,t), \\ \cdots \\ \dot{x}_n = f_n(X) + \delta f_n(X,t) + d_n^s(t) + u_n(X,Y,t), \end{cases} \tag{7}$$

where $X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \in R^n$ is the state vector of the slave system, $f_i : R^n \to R$ for $i = 1, \ldots, n$ are given nonlinear functions of X and t, $\delta f_i(Y,t)$ and $d_i^s(t)$ for $i = 1, \ldots, n$ are the slave system uncertainty and disturbances and finally u_i for $i = 1, \ldots, n$ are control signals.

Synchronization will be realized when the difference between two chaotic systems, which is called the synchronization error henceforth, becomes zero. Accordingly, the synchronization error is defined as follows:

$$e_i(t) = x_i(t) - y_i(t), \quad i = 1, \ldots, n. \tag{8}$$

Taking first-order derivative with respect to time from both sides of the above equation, we have

$$\dot{e}_i = \dot{x}_i - \dot{y}_i, \quad i = 1, \ldots, n. \tag{9}$$

Now if equations (6) and (7) are substituted into Eq. (9), the dynamics of the synchronization error is expressed by

$$\begin{cases} \dot{e}_1 = f_1(X) - g_1(Y) + \delta f_1(X,t) - \delta g_1(Y,t) + d_1^s(t) - d_1^m(t) + u_1(X,Y,t), \\ \dot{e}_2 = f_2(X) - g_2(Y) + \delta f_2(X,t) - \delta g_2(Y,t) + d_2^s(t) - d_2^m(t) + u_2(X,Y,t), \\ \cdots \\ \dot{e}_n = f_n(X) - g_n(Y) + \delta f_n(X,t) - \delta g_n(Y,t) + d_n^s(t) - d_n^m(t) + u_n(X,Y,t). \end{cases} \tag{10}$$

In most cases of practical situations, uncertainty and disturbances are naturally bounded. Therefore, some assumptions are made below.

Assumption 1 It is assumed that the uncertainties existing in Eqs. (6) and (7) are bounded by positive known numbers G_i and F_i respectively, i.e.,

$$\begin{aligned} &|\delta g_i(Y,t)| \leq G_i, \quad G_i \geq 0, \quad i = 1, \ldots, n, \\ &|\delta f_i(X,t)| \leq F_i, \quad F_i \geq 0, \quad i = 1, \ldots, n. \end{aligned} \tag{11}$$

Assumption 2 It is assumed that the disturbances exist-

ing in Eqs. (6) and (7) are bounded by positive known numbers D_i^m and D_i^s respectively, i.e.,

$$|d_i^m(t)| \leq D_i^m, \quad D_i^m \geq 0, \quad i = 1,\ldots,n,$$
$$|d_i^s(t)| \leq D_i^s, \quad D_i^s \geq 0, \quad i = 1,\ldots,n. \tag{12}$$

4. Main result

The sliding mode control procedure is composed of two stages. The first stage is to determine a sliding surface. The next one is to establish a control law, insuring that the state trajectory converges to the sliding surface and eventually reaches it and stays on it forever. Accordingly, in the following subsection, the general form of switching surfaces is described. In the next one, the control law is established by Lyapunov stability theory. Finally, an example of terminal functions whose general properties are specified in the following subsection is presented.

4.1. Switching surfaces definition

The switching surfaces are considered as follows:

$$s_i = e_i - h_i(t), \quad i = 1,\ldots,n, \tag{13}$$

where $h_i(t)$ for $i = 1,\ldots,n$ are called terminal functions.[60] It is assumed that we are able to design a controller enforcing the synchronization error to follow terminal function completely. Thus, complete robustness will be fulfilled provided that

$$h_i(t_0) = e_i(t_0), \quad i = 1,\ldots,n, \tag{14}$$

where t_0 is the initial time. Without loss of generality, the initial time is considered to be zero. According to Eq. (14), the value of the initial error must be available. Synchronization at a pre-specified time will be satisfied if

$$h_i(T) = 0, \quad i = 1,\ldots,n, \tag{15}$$

where T is the synchronization time. For $t > T$, the terminal functions must be zero, otherwise the slave system will deviate from the master system proportional to the terminal functions' value.

As the synchronization error follows the terminal functions completely and it will be shown later that the derivatives of the terminal functions exist in control signal formula, they must be differentiable. In general, two distinct manners can be adopted for determining the terminal functions. We can determine them dependently or independently.

4.2. Controller design

As mentioned earlier, if there is a controller to force the synchronization error to follow the terminal functions completely, then robust synchronization will be realized exactly at a pre-specified time. Mathematically, the switching surfaces (13) must remain at zero. To this end, the controller will be designed on the basis of Lyapunov stability theory in this way that the time derivative of a positive definite function of the switching surfaces is negative definite.

Theorem 2 Synchronization between master and slave systems (6) and (7) can be achieved in the presence of uncertainty and disturbances precisely at a pre-specified time T only if the controller in slave system is chosen as

$$\begin{aligned} u_i(X,Y,t) &= -f_i(X) + g_i(Y) + \dot{h}_i(t) \\ &\quad - \left(\beta_i |s_i|^{\mu} + k_i\right) \operatorname{sign}(s_i), \quad i = 1,\ldots,n, \end{aligned} \tag{16}$$

where β_i and k_i for $i = 1,\ldots,n$ and $\mu < 1$ are positive constant.

Proof Consider the following positive-definite function:

$$V(S) = \sum_{i=1}^{n} |s_i|, \tag{17}$$

and taking the derivative of both sides of Eq. (17) with respect to time, we obtain

$$\dot{V}(S) = \sum_{i=1}^{n} \dot{s}_i \cdot \operatorname{sign}(s_i). \tag{18}$$

Firstly, $\dot{s}_i$ for $i = 1,\ldots,n$ are determined. Thus, according to Eq. (8), $\dot{s}_i$ can be written as

$$\dot{s}_i(t) = \dot{e}_i(t) - \dot{h}_i(t). \tag{19}$$

Replacing $\dot{e}_i(t)$ by Eq. (10), we have

$$\begin{aligned} \dot{s}_i &= f_i(X) - g_i(Y) + \delta f_i(X,t) - \delta g_i(Y,t) \\ &\quad + d_i^s(t) - d_i^m(t) + u_i(X,Y,t) - \dot{h}_i(t). \end{aligned} \tag{20}$$

Control law $u_i(XYt)$ can be replaced by the terms on the right-hand side of Eq. (16). Thus, with some simplifications, equation (20) can be written as

$$\begin{aligned} \dot{s}_i &= \delta f_i(X,t) - \delta g_i(Y,t) + d_i^s(t) - d_i^m(t) \\ &\quad - \left(\beta_i |s_i|^{\mu} + k_i\right) \operatorname{sign}(s_i). \end{aligned} \tag{21}$$

Now if both sides of Eq. (21) are multiplied by sign (s_i), we have

$$\begin{aligned} \dot{s}_i \operatorname{sign}(s_i) &= (\delta f_i(X,t) - \delta g_i(Y,t) + d_i^s(t) \\ &\quad - d_i^m(t)) \operatorname{sign}(s_i) - k_i \cdot \operatorname{sign}(s_i) \operatorname{sign}(s_i) \\ &\quad - \beta_i |s_i|^{\mu} \cdot \operatorname{sign}(s_i) \operatorname{sign}(s_i). \end{aligned} \tag{22}$$

We know that sign (s_i) sign $(s_i) = 1$. Thus, equation (22) can be rewritten as follows:

$$\begin{aligned} \dot{s}_i \operatorname{sign}(s_i) &= (\delta f_i(X,t) - \delta g_i(Y,t) + d_i^s(t) - d_i^m(t)) \operatorname{sign}(s_i) \\ &\quad - k_i - \beta_i |s_i|^{\mu}, \end{aligned} \tag{23}$$

$\dot{s}_i \operatorname{sign}(s_i)$ also satisfies the following inequality

$$\begin{aligned} \dot{s}_i \operatorname{sign}(s_i) &\leq |(\delta f_i(X,t) - \delta g_i(Y,t) + d_i^s(t) - d_i^m(t)) \operatorname{sign}(s_i)| \\ &\quad - k_i - \beta_i |s_i|^{\mu} \end{aligned}$$

$$\leq |\delta f_i(X,t) - \delta g_i(Y,t) + d_i^s(t) - d_i^m(t)||\operatorname{sign}(s_i)| - k_i - \beta_i |s_i|^\mu$$
$$\leq (G_i + F_i + D_i^m + D_i^s) - k_i - \beta_i |s_i|^\mu . \quad (24)$$

If it is assumed that

$$k_i = G_i + F_i + D_i^m + D_i^s + \gamma_i,$$
$$\gamma_i > 0 \text{ for } i = 1,\ldots,n, \quad (25)$$

then

$$\dot{s}_i \operatorname{sign}(s_i) \leq -\gamma_i - \beta_i s_i^\mu \leq -\beta_i |s_i|^\mu . \quad (26)$$

Combining inequality (26) with Eq. (17) yields

$$\dot{V}(S) = \sum_{i=1}^{n} \dot{s}_i \operatorname{sign}(s_i)$$
$$= \dot{s}_1 \operatorname{sign}(s_1) + \dot{s}_2 \operatorname{sign}(s_2) + \cdots + \dot{s}_n \operatorname{sign}(s_n)$$
$$\leq -\beta_1 |s_1|^\mu - \beta_2 |s_2|^\mu - \cdots - \beta_n |s_n|^\mu$$
$$\leq -\beta_{\min} \left(|s_1|^\mu + |s_2|^\mu + \cdots + |s_n|^\mu\right), \quad (27)$$

where $\beta_{\min} = \min(\beta_1, \beta_2, \ldots, \beta_n)$. Using Lemma 1, the following inequality can be obtained:

$$\dot{V}(S) \leq -\beta_{\min} (|s_1| + |s_2| + \cdots + |s_n|)^\mu$$
$$= -\beta_{\min} (V(S))^\mu . \quad (28)$$

According to inequality (28), the time derivative of Eq. (17) is negative-definite. The implication of Eq. (28) is that the controller (16) will return the synchronization error to the sliding surface finite-timely once the smallest possible deviation from the sliding surface is observed. Hence, the error trajectory follows the sliding surface. This guarantees that the slave system (7) is synchronized with its master system (6) precisely at time T.

Remark 1 With an ideal switch, the error trajectory can never ever move apart from the sliding surface even so slightly; however, a practical switch does not have this ability. Therefore, we should anticipate that the error trajectory deviates from the sliding surface. The inequality (28) implies that if the error trajectory is outside the sliding surface at time t_1, it will be returned to the sliding surface at time $t_2 \leq t_1 + V^{1-\mu}(t_1)/\beta_{\min}(1-\mu)$. Since the controller (16) with observing the smallest possible deviation forces the error trajectory to return to the sliding surface, neither the deviation at time t_1 nor $V(t_1)$ is significant at all. Hence, essentially, $(t_2 - t_1)$ is small. Nonetheless, it can be further diminished by making μ as far as possible nearer to zero and by increasing $\beta_{\min}$, but increasing $\beta_{\min}$ results in more energy consumption. Indeed, the controller (16) must consume more energy in order to return the error trajectory to the sliding surface in less time. Therefore, we should make a compromise between two desires. Ultimately, we will ensure that the slave system (7) is synchronized with its master system (6) precisely at time T if the parameters μ and $\beta_{\min}$ are chosen in such a way that $(t_2 - t_1)$ is not comparable to T.

4.3. An example of the terminal functions

It is necessary for the terminal functions to be differentiable and satisfy two conditions (14) and (15). Therefore, we use the polynomial function and for simplicity they are presented for third-order chaotic systems. Consider the polynomial function $Q(t)$ as follows:

$$Q(t) = t^7 + a_6 t^6 + a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0. \quad (29)$$

We define terminal functions dependently for $t \leq T$ as follows:

$$h_1(t) = Q(t), \quad h_2(t) = \dot{Q}(t), \quad h_3(t) = \ddot{Q}(t) \quad (30)$$

and for $t > T$, all of them are equal to zero. Now, two conditions (14) and (15) and differentiability at $t = T$ will be met if there are coefficients $a_0, \ldots, a_6$ satisfying the following algebraic equations:

$$h_1(0) = a_0 = e_1(0), \quad h_2(0) = a_1 = e_2(0),$$
$$h_3(0) = 2a_2 = e_3(0), \quad h_1(T) = 0,$$
$$h_2(T) = 0, \quad h_3(T) = 0, \quad Q^{(3)}(T) = 0. \quad (31)$$

The system of algebraic equations (31) is composed of seven equations and seven unknowns. Assuming that the initial errors are known, a_0, a_1, and a_2 can be directly obtained from the first three equations of Eq. (31):

$$a_0 = e_1(0), \quad a_1 = e_2(0), \quad a_2 = e_3(0)/2. \quad (32)$$

Using Eq. (32), the remaining four equations of Eq. (31) can be rewritten in the matrix form as follows:

$$\begin{bmatrix} T^3 & T^4 & T^5 & T^6 \\ 3T^2 & 4T^3 & 5T^4 & 6T^5 \\ 6T & 12T^2 & 20T^3 & 30T^4 \\ 6 & 24T & 60T^2 & 120T^3 \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$
$$= \begin{bmatrix} -e_1(0) - Te_2(0) - T^2 e_3(0)/2 - T^7 \\ -e_2(0) - Te_3(0) - 7T^6 \\ -e_3(0) - 42T^5 \\ -210T^4 \end{bmatrix}. \quad (33)$$

As the determinant of the coefficient matrix is equal to $12T^{12}$, there will be the terminal functions (30) that satisfy the conditions (14) and (15) and at the same time differentiability at $t = T$. According to the Cramer's rule, the values of the remaining unknowns will be equal to

$$\begin{bmatrix} a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} \frac{20}{T^3} & -\frac{45}{T^4} & \frac{36}{T^5} & -\frac{10}{T^6} \\ -\frac{10}{T^2} & \frac{25}{T^3} & -\frac{21}{T^4} & \frac{6}{T^5} \\ \frac{2}{T} & -\frac{11}{2T^2} & \frac{5}{T^3} & -\frac{3}{2T^2} \\ -\frac{1}{6} & \frac{1}{2T} & -\frac{1}{2T^2} & \frac{1}{6T^3} \end{bmatrix}^{\mathrm{T}}$$

$$\times \begin{bmatrix} -e_1(0) - Te_2(0) - T^2 e_3(0)/2 - T^7 \\ -e_2(0) - Te_3(0) - 7T^6 \\ -e_3(0) - 42T^5 \\ -210T^4 \end{bmatrix}, \quad (34)$$

with knowing the exact values of the coefficients, the control law is determined completely.

5. Simulation results

In this section, effectiveness and applicability of the proposed method is illustrated by some examples. The first-order differential equations are numerically approximated by the Euler method. Numerical simulations are implemented in MATLAB software. For having as accurate simulation results as possible, the step size 0.0001 is taken.

Two non-identical Lu[61] and Liu[62] chaotic systems are considered as master and slave systems and their dynamics are respectively as follows:

$$\begin{cases} \dot{y}_1 = 36(y_2 - y_1), \\ \dot{y}_2 = -y_1 y_3 + 20 y_2, \\ \dot{y}_3 = y_1 y_2 - 3 y_3, \end{cases} \quad (35)$$

and

$$\begin{cases} \dot{x}_1 = 10(x_2 - x_1), \\ \dot{x}_2 = 40 x_1 - x_1 x_3, \\ \dot{x}_3 = -2.5 x_3 + x_1^2. \end{cases} \quad (36)$$

Initial conditions of master and slave systems have been set at $Y_0 = [\,-1.5 \;\; -2 \;\; -2.5\,]$ and $X_0 = [\,2 \;\; 1.5 \;\; 2.2\,]$ respectively. Time solutions of corresponding state variables of master and slave systems are compared in Fig. 1.

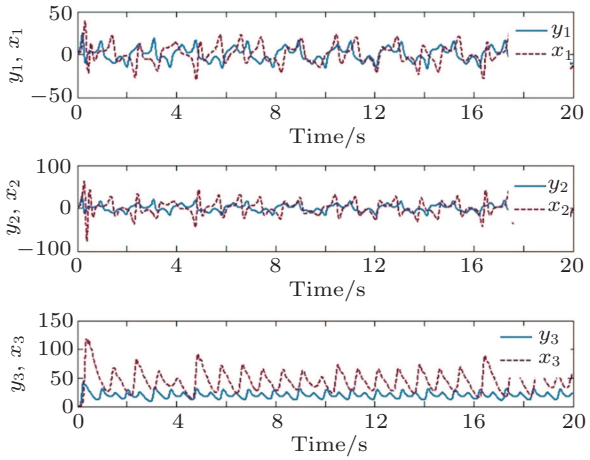


Fig. 1. (color online) Comparison of the time response of the state between Lu system (35) and Liu system (36).

Firstly, simulation is performed without considering uncertainty nor disturbances. Therefore, $k_i = 0$ for $i = 1,2,3$. The synchronization time has been set at $T = 2$ s. The controller is applied to the slave system (36) when the controller parameters have been set as $\beta_i = 1$ for $i = 1,2,3$ and $\mu = 0.5$. Now that the control signals have been applied to the slave system (36), we expect that the slave system is synchronized with the master system exactly at $t = 2$ s. Figure 2 demonstrates that the controller has satisfied our expectation because the error reaches zero exactly at $t = 2$ s. This means that the trajectory of the slave system has reached the trajectory of the master system by the controller just at $t = 2$ s (Fig. 3). The control signals have been depicted in Fig. 4.

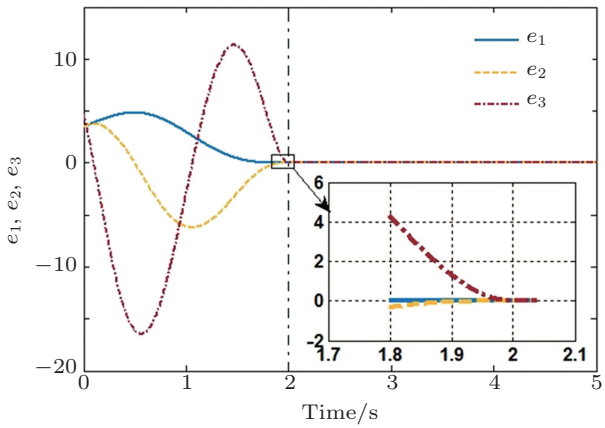


Fig. 2. (color online) Synchronization error between Lu system (35) and Liu system (36) after applying the controller.

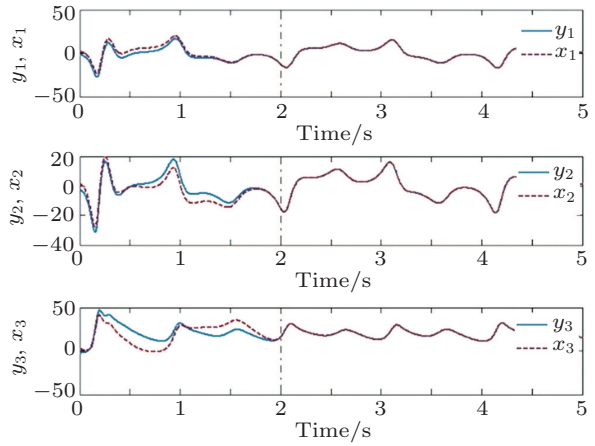


Fig. 3. (color online) The state variable of Lu system (35) and Liu system (36) synchronized at $t = 2$ s with initial conditions $Y_0 = [\,-1.5 \;\; -2 \;\; -2.5\,]$ and $X_0 = [\,2 \;\; 1.5 \;\; 2.2\,]$.

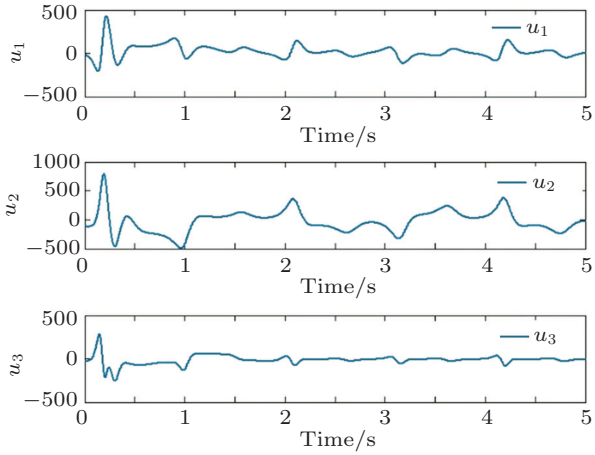


Fig. 4. (color online) Control signals applied to the slave system in order to synchronize Liu system (36) with Lu system (35) at $t = 2$ s with initial conditions $Y_0 = [\,-1.5 \;\; -2 \;\; -2.5\,]$ and $X_0 = [\,2 \;\; 1.5 \;\; 2.2\,]$.

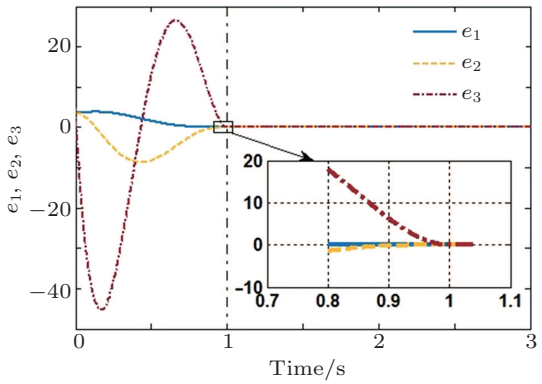


Fig. 5. (color online) Synchronization error between Lu system (35) and Liu system (36) after applying the controller to the slave system to be synchronized with master precisely at $T = 1$ s.

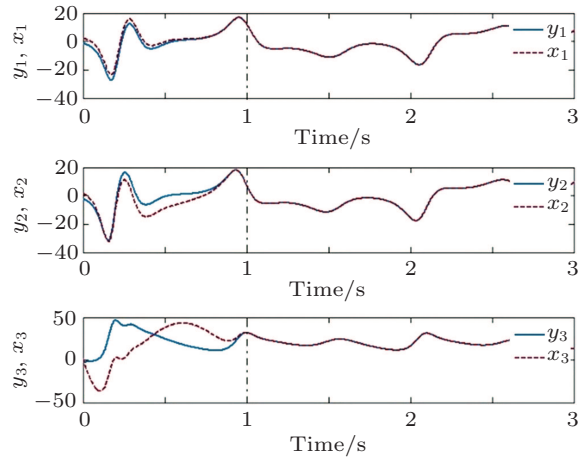


Fig. 6. (color online) State variable of Lu system (35) and Liu system (36) to be synchronized just at $t = 1$ s with initial conditions $Y_0 = [\,-1.5\ \ -2\ \ -2.5\,]$ and $X_0 = [\,2\ \ 1.5\ \ 2.2\,]$.

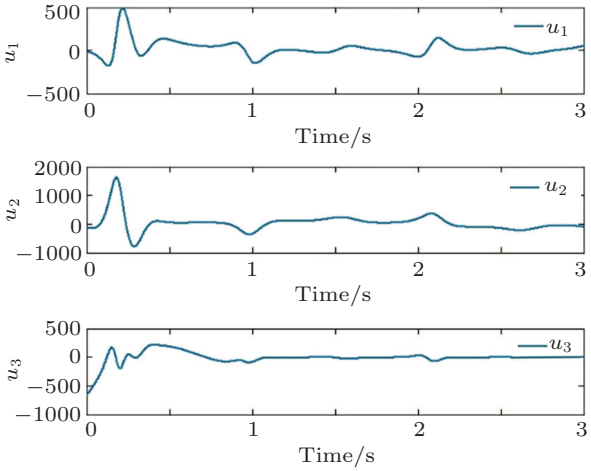


Fig. 7. (color online) Control signals applied to the slave system in order to synchronize Lu system (35) and Liu system (36) at $t = 1$ s with initial conditions $Y_0 = [\,-1.5\ \ -2\ \ -2.5\,]$ and $X_0 = [\,2\ \ 1.5\ \ 2.2\,]$.

Now, to show the ability of the proposed method of synchronizing two chaotic systems at any pre-specified time, the synchronization time is set at $T = 1$ s. Figure 5 shows that synchronization is realized right at $t = 1$ s. Time series of state variables are shown in Fig. 6. Control signals resulting in synchronization exactly at $t = 1$ s are illustrated in Fig. 7. These two simulations testify the ability of the proposed controller in synchronizing two chaotic systems at any pre-specified time free of worrying about how far they are from each other at the initial time.

Here, it is assumed that uncertainty and disturbances are present in both master and slave systems as follows:

$$\begin{cases} \dot{y}_1 = 36\,(y_2 - y_1) + 0.5y_1\cos(3.5t) \\ \qquad + \cos(1.5t) + u(t-1)\,, \\ \dot{y}_2 = -y_1y_3 + 20y_2 + 0.5y_2\cos(5t) \\ \qquad + \cos(-0.5t) + u(t-2)\,, \\ \dot{y}_3 = y_1y_2 - 3y_3 + 0.5y_3\cos(0.75t) \\ \qquad + \cos(1/7t) + u(t-3)\,, \end{cases} \tag{37}$$

and

$$\begin{cases} \dot{x}_1 = 10\,(x_2 - x_1) + 0.5x_1\sin(3t) \\ \qquad + \sin(t) + u(t-4)\,, \\ \dot{x}_2 = 40x_1 - x_1x_3 + 0.5x_2\sin(1.5t) \\ \qquad + \sin(4t) + u(t-5)\,, \\ \dot{x}_3 = -2.5x_3 + x_1^2 + 0.5x_3\sin(0.5t) \\ \qquad + \sin(2t) + u(t-6)\,. \end{cases} \tag{38}$$

The initial conditions are just the same as the first one. Control signals are applied to Eqs. (38) when $\mu = 0.5$, $\beta_i = 7$ for $i = 1,2,3$, $k_1 = k_2 = 12$ and $k_3 = 25$. The controller is intended to synchronize the slave system (38) with the master system (37) exactly at $t = 2$ s in the presence of uncertainty and disturbances. The time series of the synchronization errors is illustrated in Fig. 8. This figure shows that the controller has executed its duty because the synchronization error has arrived at zero exactly at $t = 2$ s despite uncertainty and disturbances in both systems. The time series of the state variable of systems (37) and (38) are shown in Fig. 9. Figure 10 shows the control signals inducing systems (37) and (38) to be synchronized just at $t = 2$ s in the presence of uncertainty and disturbances.

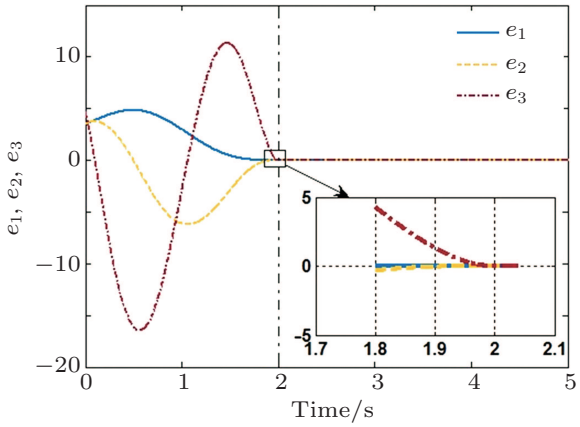


Fig. 8. (color online) Synchronization error between Liu system (36) with Lu system (35) in the presence of uncertainty and disturbances after applying the control signals to the slave system. (36)

A very simple comparison between Fig. 2 and Fig. 8 shows that the existence of uncertainty and the presence of disturbance have no effect on the system behavior. Strictly speaking, the controller forces the synchronization error to follow a pre-determined trajectory whether there exists uncertainties and disturbances or not. This confirms system robustness to uncertainty and disturbance from the beginning of the system motion.

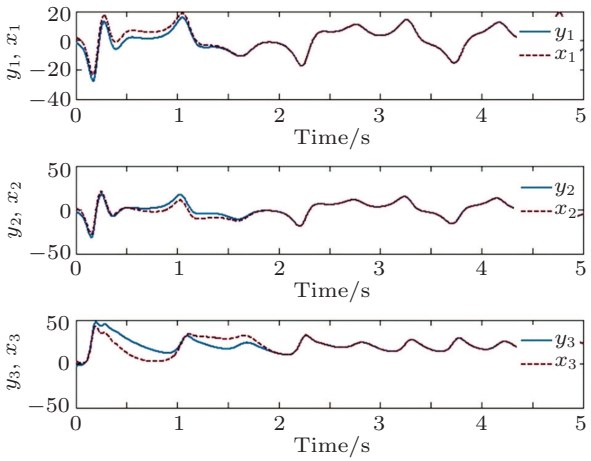


Fig. 9. (color online) State variable of system (37) and system (38) synchronized at $t = 2$ s in the presence of uncertainty and disturbances.

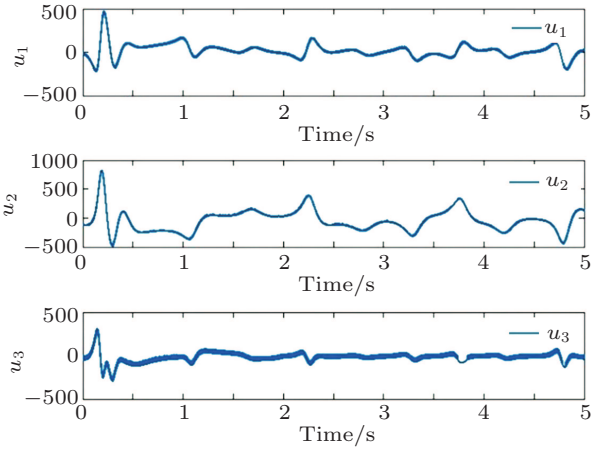


Fig. 10. (color online) Control signals applied to the slave system in order to synchronize system (38) with system (37) at $t = 2$ s in the presence of uncertainty and disturbances.

6. Conclusions

Synchronizing chaotic system exactly at a pre-specified time in the presence of uncertainty and disturbances is achieved for a large class of chaotic systems using time-varying sliding mode control without requiring to solve a TP-BVP for the first time. The complete robustness of the synchronization process to uncertainty and disturbances is also demonstrated. This approach could be applied to other types of synchronization such as projective, function projective or modified function projective synchronization. Central limitation of this approach is the need for exact knowledge of the values of initial conditions. Further work can be carried out about relaxing the assumption of exact knowledge and introducing more suitable terminal functions that do not lower practical applications of this approach.

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