Quantum speed limits of a qubit system interacting with a nonequilibrium environment*

Zhi He(贺志)†, Chun-Mei Yao(姚春梅), Li Li(李莉), and Qiong Wang(王琼)

College of Physics and Electronics, Hunan University of Arts and Science, Changde 415000, *China*

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The speed of evolution of a qubit undergoing a nonequilibrium environment with spectral density of general ohmic form is investigated. First we reveal non-Markovianity of the model, and find that the non-Markovianity quantified by information backflow of Breuer *et al.* [*Phys. Rev. Lett.* **103** 210401 (2009)] displays a nonmonotonic behavior for different values of the ohmicity parameter s in fixed other parameters and the maximal non-Markovianity can be achieved at a specified value s. We also find that the non-Markovianity displays a nonmonotonic behavior with the change of a phase control parameter. Then we further discuss the relationship between quantum speed limit (QSL) time and non-Markovianity of the open-qubit system for any initial states including pure and mixed states. By investigation, we find that the QSL time of a qubit with any initial states can be expressed by a simple factorization law: the QSL time of a qubit with any qubit-initial states are equal to the product of the coherence of the initial state and the QSL time of maximally coherent states, where the QSL time of the maximally coherent states are jointly determined by the non-Markovianity, decoherence factor and a given driving time. Moreover, we also find that the speed of quantum evolution can be obviously accelerated in the wide range of the ohmicity parameter, i.e., from sub-Ohmic to Ohmic and super-Ohmic cases, which is different from the thermal equilibrium environment case.

Keywords: quantum speed limits, non-Markovianity, nonequilibrium environment

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1. Introduction

The speed of evolution of a quantum system is an important question in quantum physics, which is intimately associated with various quantum tasks including quantum communication,[1] quantum computation,[2] quantum state transfer,[3] quantum metrology,[4,5] quantum optimal control theory,[6] as well as nonequilibrium thermodynamics.[7] The maximal rate of evolution may be characterized by the quantum speed limit (QSL) time defined as the minimal time $\tau_{\rm QSL}$ needed to evolve the initial state (pure or mixed) ρ_0 to a target state ρ_τ through a unitary evolution. The shorter time $\tau_{\rm QSL}$ means the faster speed of evolution. For closed systems with unitary evolution, there are two-type famous time bounds i.e., Mandelstam–Tamm (MT) type bound[8] and Margolus–Levitin (ML) type bound.[9] The generalizations of the MT and ML bounds to nonorthogonal states and to driven systems have also been given in Refs. [10]–[16]. For open systems with non-unitary evolution, there appear to be some valuable time bounds for pure states[17–20] and mixed states,[21–24] more recently which has attracted much attention.[25–32] Particularly, in Ref. [19] authors have shown that non-Markovian effects can speed up quantum evolution for the amplitude-damping model in the resonant case, which very recently has been experimentally tested in cavity setups.[20] On the other hand, in Refs. [33] and [34] the author has investigated the decoherence of a qubit interacting with a nonequilibrium environment. More recently, in Refs. [35] and [36] the authors have studied the correction to the geometric phase by non-Markovian effect of nonequilibrium environment with general ohmic form. However, a detailed investigation of the relationship between the QSL time from an initial state ρ_0 to a target state ρ_τ and non-Markovianity of a qubit with a nonequilibrium environment is absent at present.

Motivated by the above considerations, in this paper we will consider the speedup evolution of a qubit system interacting with a nonequilibrium environment. First we discuss non-Markovianity of nonequilibrium environment from the point of view of information backflow.[37] Then we further focus on the relationship between the QSL time and the non-Markovianity of the open system for any initial states. By investigation, we obtain some interesting findings for non-Markovianity of the nonequilibrium environment, which is unrevealed in Refs. [35] and [36]. Concretely speaking, we find that the non-Markovianity displays a nonmonotonic behavior for different values of the ohmicity parameter s in fixed other parameters and the maximal non-Markovianity can

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†Corresponding author. E-mail: hz9209@126.com or zuiguiyixiu@163.com

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be achieved at a specified value s. We also find that the non-Markovianity displays a nonmonotonic behavior with the change of a phase control parameter. More importantly, we derive a simple factorization law for the QSL time of a qubit with any initial states, namely the QSL time of any initial states are equal to the product of the coherence of the initial states and the QSL time of the maximally coherent states, which greatly simplifies the calculation of the QSL time. Finally, we also demonstrate that the speed of quantum evolution can be accelerated in the wide range of the ohmicity parameter, which is different from the thermal equilibrium environment case.

The paper is organized as follows. In Section 2, we introduce a phase damping model with nonequilibrium environment. We give some brief review of non-Markovianity measure and quantum speed limit for a qubit system undergoing a nonequilibrium environment. We give some brief review of non-Markovianity measure and quantum speed limit for a qubit system undergoing a nonequilibrium environment in Section 3. The results and discussions are given in Section 4. Finally, we give the conclusion of our results in Section 5.

2. Model

It is well known that the famous dephasing model consisting of a qubit interacting with a thermal equilibrium environment has been intensely investigated from the different points of view[38–42] in the past few years. In the present paper, we shall consider another dephasing model, namely a qubit interacting with a nonequilibrium environment, which is first put forward by Martens[33] as an analytically solvable model. According to Ref. [33], a distinct characteristic of the model is that the energy gap $E_2(t)-E_1(t)=\hbar\omega(t)$ of the qubit system involved in the model is time-dependent due to the effect of the environment, where $E_j(t)$ $(j=1,2)$ is the instantaneous energy of state j as perturbed by the environment. The environment/bath is represented by a random function of time corresponding to the transition frequency of the qubit system $\omega(t)$, and the Fourier components of the time series denote the different modes of motion of the bath. Hence, the time-dependent frequency $\omega(t)$ can be written as the form $\omega(t)=\omega_0+\delta\omega(t)$, where

$$\delta\omega(t)=\sum_{k=1}^{\infty}c_k\cos(\omega_k t+\theta_k(t)). \tag{1}$$

The Fourier components c_k are positive constants related to the spectral density of the environment and the coupling of the bath modes to the system. It is noted that in this model the randomness stems from the nonstationary distribution of random phases $\theta_k(t)$. Compared with the usual treatment, the statistical properties of this random function are nonstationary, corresponding physically to impulsively excited phonons of the environment with initial phases that are not random, but which have sharply defined values at $t=0$. So, this environment is no longer at thermal equilibrium.

Based on the result of Ref. [33], the off-diagonal or coherence terms of the qubit with time t, $\rho_{01}(t)$ can be given as

$$\begin{aligned}\rho_{01}(t)&=\rho_{01}(0)\left\langle\exp\left[-\mathrm{i}\int_0^t\delta\omega(t')\,\mathrm{d}t'\right]\right\rangle\mathrm{e}^{-\mathrm{i}\omega_0 t}\\&\equiv\rho_{01}(0)\Lambda(t)\,\mathrm{e}^{-\mathrm{i}\omega_0 t},\end{aligned} \tag{2}$$

where $\Lambda(t)=\mathrm{e}^{-\beta(t)}$ is the so-called decoherence factor, and $\beta(t)$ is defined as

$$\begin{aligned}\beta(t)=\frac{1}{4}\int_0^{\infty}I(\omega)[1-\mathrm{e}^{-2Dt}+(\mathrm{e}^{-2Dt}-\mathrm{e}^{-4Dt})\\\times\cos[2(\omega t+\theta(\omega))]].\end{aligned} \tag{3}$$

Here, the continuum limit has already been taken and diffusion coefficients $D(\omega)=D$ independent of ω also have been assumed for simplicity. Similar to the discussion of Refs. [35] and [36], we shall take for the bath spectral density the expression

$$I(\omega)=4\gamma\Omega^{-2}\omega^s\Omega^{1-s}\mathrm{e}^{-\omega/\Omega}, \tag{4}$$

which is a widely used physical spectral density in the theory of spin–boson systems,[38–42] where γ is the dimensionless dissipative constant, Ω is the cutoff frequency. With the change of the s parameter, one goes from sub-Ohmic ($s<1$) reservoirs to Ohmic ($s=1$) and super-Ohmic ($s>1$) reservoirs, respectively.

In the following, for simplicity the initial phase of the bath modes $\theta(\omega)=-\lambda\omega$ has been taken. The initial phase control parameter λ is a key factor for the occurrence of non-Markovian effect, which directly determines the nonmonotonic decay of the coherence of a qubit as pointed out in Ref. [33]. For the spectral density as Eq. (4), by simple evaluation we can obtain the $\beta(t)$ as

$$\begin{aligned}\beta(t)=\gamma\mathrm{e}^{-4Dt}[-1+\mathrm{e}^{2Dt}]\Gamma(1+s)\\\times\{[1+4(t-\lambda)^2\Omega^2]^{-(1+s)/2}\\\times\cos[(1+s)\arctan[2\Omega(t-\lambda)]]+\mathrm{e}^{2Dt}\}.\end{aligned} \tag{5}$$

Recall that the $\beta(t)$ has been obtained in Ref. [36], in which authors discussed the non-Markovian effect on the correction of geometric phase for the same model.

3. The non-Markovianity measure and quantum speed limits

Recently, studying non-Markovianity of an open quantum system has attracted considerable attention in various physical systems.[37,43–49] A popular and faithful non-Markovianity measure is based on information backflow,[37] which is defined as

$$\mathcal{N}=\max_{\rho_{1,2}(0)}\int_{\sigma>0}\mathrm{d}t\sigma(t,\rho_{1,2}(0)), \tag{6}$$

where $\sigma(t,\rho_{1,2}(0))$ is the rate of change of the trace distance defined as

$$\sigma(t,\rho_{1,2}(0)) = \frac{\mathrm{d}}{\mathrm{d}t}D(\rho_1(t),\rho_2(t)), \tag{7}$$

where the trace distance describing the distinguishability between the two states is defined as[50]

$$D(\rho_1,\rho_2) = \frac{1}{2}\mathrm{tr}|\rho_1-\rho_2|, \tag{8}$$

where $|M| = \sqrt{M^\dagger M}$, and $0 \le D \le 1$. If $D = 0$, the two states are the same, and if $D = 1$, the two states are totally distinguishable. The sign of $\sigma(t,\rho_{1,2}(0))$ indicates the Markovian or non-Markovian characteristics of a quantum process. For example, $\sigma(t,\rho_{1,2}(0)) \le 0$ corresponds to all dynamical semigroups and all time-dependent Markovian processes, and a process is non-Markovian if there exists a pair of initial states and at certain time t such that $\sigma(t,\rho_{1,2}(0)) > 0$. Physically, this means that for non-Markovian dynamics the distinguishability of the pair of states increases at certain times. Here, we should remark that for a qubit undergoing single phase damping channel the conditions of non-Markovianity, including being based on information backflow,[37] divisibility,[43] quantum Fisher information flow,[44] and quantum mutual information,[45] are equivalent to each other.[46]

In the light of Eq. (6), the non-Markovianity $\mathcal{N}$ also can be rewritten as the following form

$$\begin{aligned}\mathcal{N} = \max_{\rho_{1,2}(0)} \sum_n [&D(\rho_1(\tau_n^{\max}),\rho_2(\tau_n^{\max})) \\ &- D(\rho_1(\tau_n^{\min}),\rho_2(\tau_n^{\min}))],\end{aligned} \tag{9}$$

where $\tau_n^{\max}$ and $\tau_n^{\min}$ correspond to the time points of the local maximum and minimum of $D(\rho_1(t),\rho_2(t))$, respectively.

Generally speaking, it is very difficult to make the optimization in Eqs. (6) and (9). In our previous work,[47] we have analytically resolved the optimization of Eqs. (6) and (9) for a qubit undergoing a dephasing process, and given the maximal trace distance and corresponding conditions. In Ref. [47] for a dephasing model we have found that the optimal initial states in Eqs. (6) and (9) are the equatorial, antipodal states of the Bloch sphere of a qubit. The optimal trace distance is obtained as $D(\rho_1(t),\rho_2(t)) = |\Lambda(t)|$, where $\Lambda(t) = \mathrm{e}^{-\beta(t)}$ in our considered model. The expression of the non-Markovianity measure can be equally obtained as

$$\mathcal{N} = \sum_n [\,|\Lambda(\tau_n^{\max})| - |\Lambda(\tau_n^{\min})|\,], \tag{10}$$

or

$$\mathcal{N} = \int_{\partial_t|\Lambda(t)|>0} \partial_t|\Lambda(t)|\,\mathrm{d}t, \tag{11}$$

where in the calculation of Eq. (10), the condition $|\Lambda(\tau_n^{\max})| > |\Lambda(\tau_n^{\min})|$ is required.

Here, we would like to give a brief review of a unified lower bound including both MT and ML types for the minimal evolution time between a pure initial state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and its target state ρ_τ for an open quantum system. For a qubit governed by the time-dependent master equation $\dot{\rho}_t = \mathcal{L}_t(\rho_t)$ with the positive generator $\mathcal{L}_t$, the unified lower bound[17–19] can be given by

$$\tau \ge \tau_{\mathrm{QSL}} = \max\left\{\frac{1}{\Lambda_\tau^1}, \frac{1}{\Lambda_\tau^2}, \frac{1}{\Lambda_\tau^\infty}\right\} \sin^2[\mathcal{B}(\rho_0,\rho_\tau)], \tag{12}$$

where

$$\Lambda_\tau^p = \frac{1}{\tau}\int_0^\tau \mathrm{d}t\,\|\mathcal{L}_t(\rho_t)\|_p,$$

the $\|A\|_p = (\sigma_1^p + \cdots + \sigma_n^p)^{1/p}$ denotes the Schatten p norm of operator A, $\sigma_1,\ldots,\sigma_n$ are the singular values of A. When $p = 1$, the $\|A\|_1$ corresponds to the trace norm, which is equal to their sum $\|A\|_{\mathrm{tr}} = \sum_i \sigma_i$. When $p = 2$, the $\|A\|_2$ corresponds to the Hilbert–Schmidt norm, which is defined by $\|A\|_{\mathrm{hs}} = \sqrt{\sum_i \sigma_i^2}$, and when $p = \infty$, the $\|A\|_\infty$ corresponds to the operator norm, which is given by the largest singular value $\|A\|_{\mathrm{op}} = \sigma_{\max}$. Taking into account the relationship[19] $\|A\|_{\mathrm{op}} \le \|A\|_{\mathrm{hs}} \le \|A\|_{\mathrm{tr}}$, thus for the pure initial states the QSL time τ_{QSL} for the operator norm is the tightest bound. $\mathcal{B}(\rho_0,\rho_\tau) = \arccos\sqrt{\langle\psi_0|\rho_\tau|\psi_0\rangle}$ represents the Bures angle[50] between a pure initial state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and its target state ρ_τ.

Further we discuss the QSL time for any mixed initial states case. It should be noted that the time lower bound as shown Eq. (12) is derived from the pure initial states and cannot be directly applied to mixed initial states. In the following, for any mixed initial states we adopt the time lower bound proposed by Zhang *el al.*,[21] which is defined as

$$\tau \ge \tau_{\mathrm{QSL}} = \max\left\{\frac{1}{\overline{\sum_{i=1}^n \sigma_i\rho_i}}, \frac{1}{\overline{\sqrt{\sum_{i=1}^n \sigma_i^2}}}\right\} |f(\tau)-1|\,\mathrm{Tr}(\rho_0^2), \tag{13}$$

where the

$$\bar{X} = \tau^{-1}\int_0^\tau X\,\mathrm{d}t,$$

σ_i and ρ_i are the singular values of $\dot{\rho}_t$ and the mixed initial states ρ_0, respectively. Also, $f(t)$ is the relative purity[51] between the initial state ρ_0 and the target state ρ_t of the quantum system, which is defined as $f(t) = \mathrm{Tr}(\rho_0\rho_t)/\mathrm{Tr}(\rho_0^2)$. As pointed out in Ref. [21], for the pure initial states case equation (13) can reduce to the unified lower bound[17–19] as shown in Eq. (12). Thus for any mixed initial states we will use Eq. (13) as our time bound to discuss the speed of evolution of a qubit system.

4. Results and discussions

We consider the evolution speed of an open-qubit system for any pure initial states $\rho_0 = |\psi(0)\rangle\langle\psi(0)|$ with $|\psi(0)\rangle = \sin\alpha|0\rangle + \cos\alpha e^{i\varphi}|1\rangle$. In principle, we can use the time bounds Eqs. (12) or (13) as our measure of the QSL time. In order to show the equivalence of the time bounds Eqs. (12) and (13) for any pure initial states, we will discuss them one by one. According to Eqs. (2) and (12), by simple algebra the QSL time $\tau_{\rm QSL}$ can be obtained as

$$\tau_{\rm QSL} = \frac{\tau|\sin 2\alpha|[1-{\rm Re}\Lambda(\tau)]}{\int_0^\tau |\partial_t \Lambda(t)|{\rm d}t}. \tag{14}$$

To reveal the relationship between the QSL time $\tau_{\rm QSL}$ in Eq. (13) and non-Markovianity $\mathcal{N}$ in Eq. (11), we rewrite Eq. (11) as[26]

$$\mathcal{N} = \frac{1}{2}\left[\int_0^\tau |\partial_t \Lambda(t)|{\rm d}t + \Lambda(\tau) - 1\right]. \tag{15}$$

As a consequence, the QSL time $\tau_{\rm QSL}$ can be represented as

$$\tau_{\rm QSL} = \frac{\tau|\sin 2\alpha|}{2\mathcal{N}/[1-\Lambda(\tau)]+1}, \tag{16}$$

where the fact that the $\Lambda(t) = e^{-\beta(t)}$ is a real number in our model has been used. Interestingly, the expression of the QSL time $\tau_{\rm QSL}$ for our phase damping model is similar to the expression of the QSL time of amplitude-damping model in Ref. [26], where the QSL time is dependent on both the non-Markovianity $\mathcal{N}$ and the population P_τ in a given driving time and initial states. But in our phase damping model, for a given driving time and initial states the QSL time as shown in Eq. (16) is related to both the non-Markovianity $\mathcal{N}$ and the decoherence factor $\Lambda(\tau)$.

For any mixed initial states of a qubit system, it has been demonstrated in Ref. [21] that $\rho_1\sigma_1 + \rho_2\sigma_2$ is always less than $\sqrt{\sigma_1^2+\sigma_2^2}$. For example, for a generally initial state $\rho_0 = (1/2)(I + n_x\sigma_x + n_z\sigma_x + n_z\sigma_z)$ there are

$$\rho_1\sigma_1 + \rho_2\sigma_2 = \sqrt{n_x^2+n_y^2}|\partial_t\Lambda(t)|/2$$

and

$$\sqrt{\sigma_1^2+\sigma_2^2} = \sqrt{n_x^2+n_y^2}|\partial_t\Lambda(t)|/\sqrt{2}$$

in the dephasing model. According to Eq. (13) and implementing simple calculation, finally we can obtain the explicit expression of the QSL time for any initial states ρ_0 as

$$\tau_{\rm QSL} = \frac{\tau C_{l_1}(\rho_0)}{2\mathcal{N}/[1-\Lambda(\tau)]+1}, \tag{17}$$

where the $C_{l_1}(\rho_0)$ is the amount of coherence for the initially mixed states ρ_0 quantified by the so-called l_1 norm,[52,53] which is defined by

$$C_{l_1}(\rho) = \sum_{i\neq j}|\rho_{ij}|.$$

Obviously, the expression (17) of the QSL time for mixed states is very similar to the expression (16) for pure states except for a factor of dependent-initial states. It is easy to check that for any pure states as $\rho_0 = |\psi(0)\rangle\langle\psi(0)|$ with $|\psi(0)\rangle = \sin\alpha|0\rangle + \cos\alpha e^{i\varphi}|1\rangle$, there is $C_{l_1}(\rho_0) = |\sin 2\alpha|$. Corresponding to the expression (17) for the generally mixed states recovers the expression (16) for the pure states case. Therefore, we derive that the QSL time of a qubit with any initial states not only relates to coherence of the initial state $\hat{\rho_0}$ but also to the non-Markovianity $\mathcal{N}$ and decoherence factor $\Lambda(\tau)$ in a given driving time τ. More interestingly, from Eqs. (16) and (17) we find that the QSL time for an open qubit system can be rewritten as coherence of the initial state $C_{l_1}(\rho_0)$ and the initial states-independent factor

$$\chi = \frac{\tau}{2\mathcal{N}/[1-\Lambda(\tau)]+1}$$

for any initial states. Therefore, for the QSL time of a qubit undergoing a dephasing model we can obtain an interesting factorization law as

$$\tau_{\rm QSL} = \chi C_{l_1}(\rho_0). \tag{18}$$

Further, we consider the extreme case in which the qubits are initially prepared in the maximally coherent states, i.e., $C_{l_1}(\rho_0) = 1$. Then equation (18) reduces to $\tau_{\rm QSL} = \chi$, which means that χ stands for the QSL time of the maximally coherent states. So if we need to calculate the QSL time of a qubit for any initial states, the corresponding QSL time can be simplified as the product of the coherence of the initial state $C_{l_1}(\rho_0)$ and the QSL time χ for the maximally coherent states. This is similar to the factorization law of entanglement as shown in Ref. [54], where it is found that for any pure states, the entanglement under a one-sided noisy channel can be expressed as the products of initial entanglement and the channel action on the maximally entangled state. It is noted that the expression (17) for the dephasing model with a nonequilibrium environment is similar to that in Ref. [21] with a thermal equilibrium environment, where they found that the QSL time for the generally mixed states relates to the coherence of the initial state $C_{l_1}(\rho_0)$ and the quantity $\partial_t\Lambda(t)$. However, here we reveal that the QSL time for the generally mixed states not only relates to the coherence of the initial state $C_{l_1}(\rho_0)$ but also to the non-Markovianity $\mathcal{N}$ and decoherence factor $\Lambda(\tau)$ in a given driving time τ, which deeply excavates the physical mechanism behind the speed of evolution of the open-qubit system. More importantly, in this paper we mainly focus on the QSL time of a qubit interacting with a nonequilibrium environment rather than a thermal equilibrium environment.

Having Eqs. (10), (11), (16), (17), and (18) in mind, we can discuss the non-Markovianity $\mathcal{N}$, and the relationship between the QSL time and the non-Markovianity $\mathcal{N}$ of the open-qubit system for any pure and mixed initial states.

First we plot the non-Markovianity $\mathcal{N}$ as a function of the ohmicity parameter s for other fixed parameters (i.e., $\lambda = 0.3$, $\Omega = 10$, $\gamma = 0.1$, and $D = 2$, which is the same as that of Ref. [36]) in Fig. 1. Recall that for the same model the authors in Ref. [36] have obtained an interesting result that the non-Markovian effect of the nonequilibrium environment can appear in all values of the ohmicity parameter $s > 0$. Further, they have also shown a hierarchy in the coefficients: the bigger the value s, the more negative the diffusion coefficient is, and the more recoherence induced by the environment. In other words, the non-Markovian effect becomes stronger and stronger with the increase of the s.

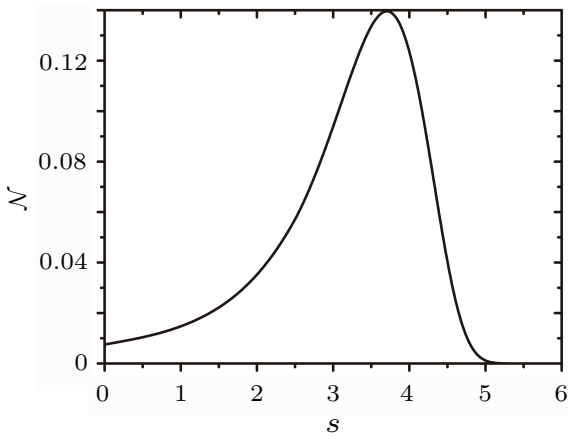


Fig. 1. The non-Markovianity $\mathcal{N}$ as a function of the ohmicity parameter s. Other parameters are $\lambda = 0.3$, $\Omega = 10$, $\gamma = 0.1$, and $D = 2$, which is same to that of the Ref. [36].

From Fig. 1, indeed we can clearly see that the non-Markovianity can occur in all values of the ohmicity parameter $s > 0$. However, it should be mentioned that the $\mathcal{N}$ takes very small values for $s \geq 5$ instead of zero values, which is similar to the assertion of Ref. [55] for the same non-Markovianity measure of the equilibrium environment case. At the same time, from Fig. 1 we also find an unrevealed phenomenon for non-Markovianity $\mathcal{N}$, namely the non-Markovianity $\mathcal{N}$ displays a nonmonotonic behavior for different values s instead of a simply monotonic relationship as shown in Ref. [36]. Specifically, there exists a critical value $s = 3.71$, where the maximal non-Markovianity can be achieved. After the critical value $s = 3.71$, the non-Markovianity $\mathcal{N}$ decreases monotonically with the increase of the s. Here, it should be noted that the critical value s corresponding to the maximal non-Markovianity may change with different λ, Ω, γ, and D, which does not affect our results. Actually, the conclusions in Ref. [36] may be only based on the consideration of the cases for $s = 1$, 2, and 3, where the non-Markovianity $\mathcal{N}$ increases monotonically with the change of ohmicity parameter at $s = 1, 2, 3$. Therefore, the non-Markovianity exhibits a nonmonotonic behavior instead of simply monotonic behavior with the change of the ohmicity parameter s, which is a newly noticed phenomenon.

The above behavior can be understood by a common feature of all non-Markovianity measures, which is based on the nonmonotonic evolution in time of certain coefficients (i.e., here the decoherence factor $\Lambda(t)$) which indicates the information backflow from the environment to the considered system. Conversely, for no memory effect the decoherence factor $\Lambda(t)$ monotonically decreases with time. Actually, the judgment of memory/non-Markovian effect is also equivalent to the fact that the decoherence rate, i.e., $\gamma(t) = -\dot{\Lambda}(t)/(2\Lambda(t)) = \dot{\beta}(t)/2$ becomes temporarily negative. By numerical calculation, it is straightforward to prove that the decoherence rate $\gamma(t)$ becomes temporarily negative, or equally the decoherence factor $\Lambda(t)$ displays a nonmonotonic behavior if and only if $s > 0$. Moreover, in Refs. [33] and [34] it is shown that the decoherence factor displays a nonmonotonic behavior at a "dip" around, and the deeper the dip means the more non-Markovian effect. By simple calculation, we find that the maximal depth of the dip is located at $s = 3.71$ for other fixed parameters. In a word, compared with the thermal equilibrium environment where the appearance of non-Markovian effect only occurs in the super-Ohmic ($s > 2$) at zero temperature,[42] we can find that the nonequilibrium environment shows more abundant non-Markovian effect from sub-Ohmic to Ohmic and super-Ohmic cases.

It is worth noting that recently Haikka *et al.*[42] have proposed a new route to explain the physical mechanism of non-Markovian effect of environment for thermal equilibrium environment. They have shown that the non-monotonic behavior in time of certain coefficients signalling the information backflow from environment back to system is related to the convex to non-convex changes of the spectrum for thermal equilibrium environments. Specifically, they have elucidated that at zero temperature the non-Markovian effect of environment occurs if and only if $s > s_{\text{crit}} = 2$. A question, however, naturally arises: whether or not the critical condition $s > 0$ of non-Markovian effect of environment for the nonequilibrium environment can also be explained by the convexity arguments in Ref. [42]? The answer is positive.

First, from Ref. [42] we learn that the transition from Markovian to non-Markovian effect of environment corresponds to that from convex to nonconvex function $g(\omega, T)$ at s_{crit}. For the thermal equilibrium environment, the function $g(\omega, T)$ is defined as $g(\omega, T) = 2I(\omega)\coth[\hbar\omega/2k_{\text{B}}T]/\omega^2$,[42] where $I(\omega)$ is the bath spectral density and k_{B} is the Boltzmann constant. Obviously, there is $g(\omega, 0) = 2I(\omega)/\omega^2$ at

$T = 0$. If we consider the spectral density as Eq. (4), we will derive that $g(\omega,0) \propto \omega^{s-2} e^{-\omega/\Omega}$. As revealed in Ref. [56], a change of convexity for the $g(\omega,0)$ happens when passing from $s < 2$, for which $g(\omega,0)$ diverges in the origin $\omega = 0$, to $s > 2$, for which $g(\omega,0)$ vanishes in the origin. This behavior is independent of the monotonically decaying cutoff function $e^{-\omega/\Omega}$. As for the nonequilibrium environment case, using the similar idea of Ref. [42] we mainly examine the change of convexity of the function $G(\omega,0) = I(\omega)/4$ in the $\beta(t)$ of Eq. (3). Evidently, for the spectral density as Eq. (4) there is $G(\omega,0) \propto \omega^s e^{-\omega/\Omega}$. Similar to the $g(\omega,0) \propto \omega^{s-2} e^{-\omega/\Omega}$ case, we can obtain that the function $G(\omega,0)$ is always a non-convex function of ω if and only if $s > 0$, for which $G(\omega,0)$ vanishes in the origin $\omega = 0$. This behavior is also independent of the cutoff function $e^{-\omega/\Omega}$. Therefore, the non-Markovian effect of the environment occurs if and only if $s > 0$ for the nonequilibrium environment case.

Note that the authors in Refs. [33] and [34] have pointed out that the decoherence factor displays a nonmonotonic behavior at the dip around $t = \lambda$. In other words, the occurrence of non-Markovian effect is just at the dip around $t = \lambda$ and can be controlled by varying λ. The problem arises: what is the relationship between non-Markovianity and initial phase control parameter λ? In the following we address the problem.

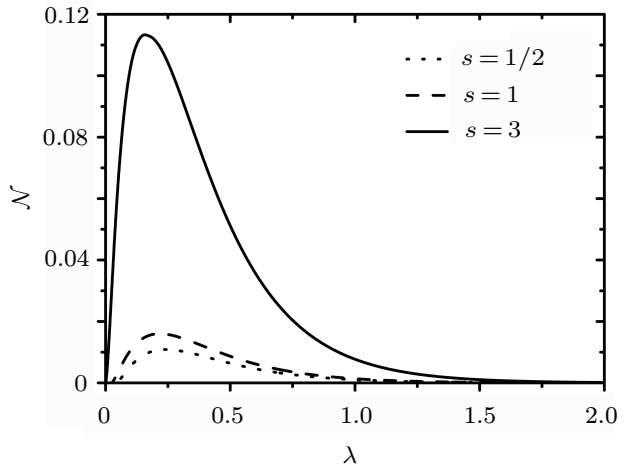


Fig. 2. The non-Markovianity $\mathcal{N}$ as a function of λ for different spectral densities, i.e., $s = 1/2$, $s = 1$, and $s = 3$, respectively. Other parameters are the same as Fig. 1.

We plot the non-Markovianity $\mathcal{N}$ as a function of λ for different spectral densities, i.e., $s = 1/2$, $s = 1$, and $s = 3$, in Fig. 2. Other parameters are the same as those in Fig. 1. From Fig. 2 we can clearly see that the non-Markovianity displays a nonmonotonic behavior with the change of λ for different s, i.e., $s = 1/2$, 1, and 3, respectively. More interestingly, the maximal non-Markovian effect from the environment occurs in an optimal λ value. At first glance, we may think that these optimal λ values are the same for the three spectral densities. However, by careful analysis we find that these optimal λ values for three spectral densities are different from each other. In order to explicitly indicate the relationship between these optimal λ denoted by $\lambda_{\rm opt}$ and different ohmicity parameter s, we plot the $\lambda_{\rm opt}$ point for different values of s in Fig. 3, and other parameters are the same as those in Fig. 1. Obviously, from Fig. 3 we can see that the $\lambda_{\rm opt}$ shows a monotonously decreasing trend with the increase of the s. Therefore, if we get the maximal non-Markovian effect from the environment, it is necessary to carefully consider the choices of λ for different spectral densities.

In the following, we will further focus on the influence of the non-Markovian effect of the environment on the QSL time from any initial state ρ_0 to a target state ρ_τ of the open-qubit system. From Eq. (18), we clearly see that the defined factor χ is independent of initial states. Thus the differences of the QSL time of the open-qubit system for different initial states only are determined by the coherence of the initial states. To this end, we will take the pure initial states as an example to reveal the relationship between the QSL time and non-Markovianity of the open-qubit system. As for the mixed initial states case, the main results are unchangeable. As an example, here we consider a class of important pure states such as the maximally coherent states $\hat{\rho}_0 = |\psi(0)\rangle\langle\psi(0)|$ with $|\psi(0)\rangle = (|0\rangle + e^{i\varphi}|1\rangle)/\sqrt{2}$.

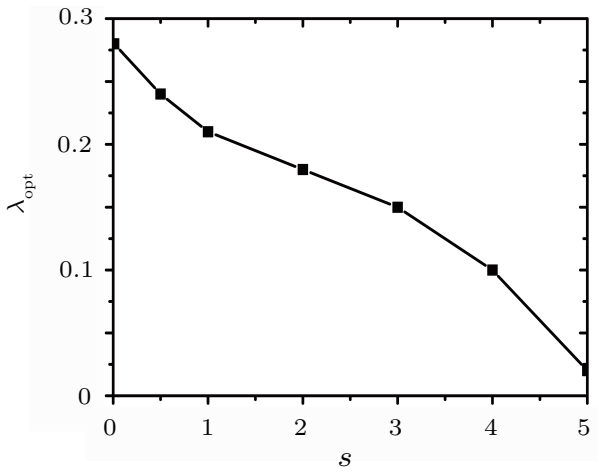


Fig. 3. The $\lambda_{\rm opt}$ points for different values of s. Other parameters are the same as Fig. 1.

To this end, in Fig. 4 we plot the QSL time $\tau_{\rm QSL}$ (a), the non-Markovianity $\mathcal{N}$ (b-solid line) and the quantity $\mathcal{N}/[1 - \Lambda(\tau)]$ (b-dashed line) associated non-Markovianity and decoherence factor as a function of the ohmicity parameter s for the nonequilibrium environment in the actual driving time $\tau = 5$. Other parameters are the same as those in Fig. 1 and the actual driving time $\tau = 5$. In order to compare the following equilibrium environment with the non-Markovian effect in a longer time, here we also choose a longer actual driving time, i.e.,

$\tau = 5$. From Fig. 4(a), it is shown that when $0 < s < 5$, the QSL time $\tau_{\rm QSL}$ is obviously less than the actual driving time τ, namely $\tau_{\rm QSL} < \tau$, and when $s \geq 5$, there is $\tau_{\rm QSL} \approx \tau$. In other words, for the nonequilibrium environment the speedup of quantum evolution can occur from sub-Ohmic ($s < 1$) to Ohmic ($s = 1$) and super-Ohmic ($s > 1$) cases, which is very different from that of the equilibrium environment discussed in the next section. The above phenomenon can be easily explained as follows: from Eq. (16), we can see that the transition point from Markovian to non-Markovian of the environment is just the point from no speedup to speedup of quantum evolution. Since the considered nonequilibrium environment displays non-Markovian effect for all $s > 0$, strictly speaking, from Eq. (16) we know that the QSL time $\tau_{\rm QSL}$ is always less than the actual driving time τ for all $s > 0$. Particularly, for $s \geq 5$ the non-Markovianity $\mathcal{N}$ takes very small (almost negligible) values as shown in Fig. 1, which leads to the QSL time $\tau_{\rm QSL} \approx \tau$.

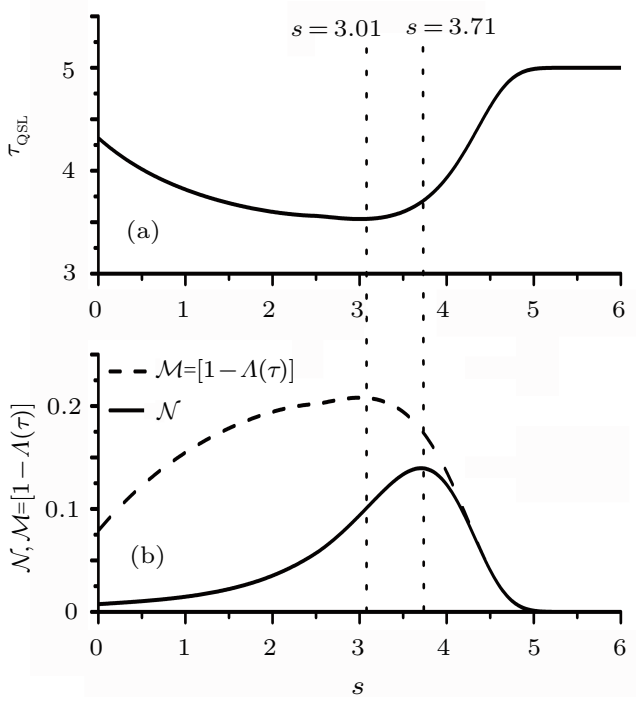


Fig. 4. The QSL time $\tau_{\rm QSL}$ (a), the non-Markovianity $\mathcal{N}$ (b-solid line) and the quantity $\mathcal{N}/[1-\Lambda(\tau)]$ (b-dashed line) associated the non-Markovianity and decoherence factor as a function of the ohmicity parameter s for the nonequilibrium environment. Other parameters are the same as those in Fig. 1 and the actual driving time $\tau = 5$.

Further, comparing Fig. 4(a) and Fig. 4(b), we can find that the stronger quantity $\mathcal{N}/[1-\Lambda(\tau)]$ (dashed line) just corresponds to the smaller QSL time $\tau_{\rm QSL}$. But the stronger non-Markovianity $\mathcal{N}$ (solid line) means the smaller QSL time $\tau_{\rm QSL}$ only at $0 < s \leq 3.01$ and $3.71 \leq s \leq 5$, and the relation is violated at $3.01 < s < 3.71$, which means that the non-Markovian effect of the environment may not monotonically give rise to the speedup of the system evolution. The phenomenon can be easily explained as follows: although the non-Markovian effect is a necessary condition of the speedup of quantum evolution, equation (16) clearly shows that the non-Markovianity $\mathcal{N}$ is not the only factor for the QSL time, and the latter is actually dependent on both the non-Markovianity $\mathcal{N}$ and decoherence factor $\Lambda(\tau)$, namely $\mathcal{N}/[1-\Lambda(\tau)]$. Thus for the nonequilibrium environment we need to carefully consider the monotonous relationship of the QSL time and non-Markovianity, which is determined by the ohmicity parameter s. Recently, it is noted that a similar result about the non-monotonic relation between the QSL time and the non-Markovian effect has been independently found in Ref. [57] for a different environment case. However, they only showed the phenomenon, but did not give any physical explanations. Here except for the different model, more importantly, we also give a reasonable physical explanation for the non-monotonic relation between the QSL time and the non-Markovian effect. In fact, the non-monotonic relation between the QSL time and the non-Markovian effect in dephasing model also exists for equilibrium environment case in the next discussion.

From above analyses we have known that for the nonequilibrium environment the speedup of quantum evolution can occur from sub-Ohmic ($s < 1$) to Ohmic ($s = 1$) and super-Ohmic ($s > 1$) cases. In the following, it is necessary to compare with the speedup of quantum evolution of thermal equilibrium environment. Here we discuss a similar spectral density as Eq. (4) except for a scaled constant, namely[42] $I(\omega) = \omega^s \Omega^{1-s} {\rm e}^{-\omega/\Omega}$ with the cutoff frequency Ω. For simplicity, we only consider at zero temperature case, where the decoherence factor[55] has an explicit expression as $\Lambda(t) = {\rm e}^{-\beta(t)}$ with

$$\beta(t) = 2\frac{\Gamma(s)}{s-1}[1-(1+\Omega^2 t^2)^{-s/2} \times\{\cos[s\arctan(\Omega t)] + \Omega t \sin[s\arctan(\Omega t)]\}],$$

where $\Gamma[x]$ is the Euler gamma function.

In Fig. 5, we plot the QSL time $\tau_{\rm QSL}$ (a), the non-Markovianity $\mathcal{N}$ (b-solid line) and the quantity $\mathcal{N}/[1-\Lambda(\tau)]$ (b-dashed line) combining the non-Markovianity and decoherence factor as a function of the ohmicity parameter s for the thermal equilibrium environment. Other parameters are $\Omega = 1$, $\alpha = \pi/4$ and the actual driving time $\tau = 5$. First comparing Fig. 5 (the equilibrium environment case) with Fig. 4 (the nonequilibrium environment case), we can find that there are some common characteristics for the relationships of the QSL time $\tau_{\rm QSL}$, non-Markovianity $\mathcal{N}$, and the quantity $\mathcal{N}/[1-\Lambda(\tau)]$, namely the stronger quantity $\mathcal{N}/[1-\Lambda(\tau)]$ (Fig. 5 (b-dashed line)) just corresponds to the smaller QSL time $\tau_{\rm QSL}$. But the stronger non-Markovianity $\mathcal{N}$ (Fig. 5 (b-solid line)) means the smaller QSL time $\tau_{\rm QSL}$ only when $2 < s \leq 3.15$ and $s \geq 3.20$, and the relation is violated when

$3.15 < s < 3.20$, which is similar to the nonequilibrium environment case. Of course, the phenomenon can be easily explained by Eq. (16), namely the QSL time is dependent on both the non-Markovianity $\mathcal{N}$ and decoherence factor $\Lambda(\tau)$ rather than the only non-Markovianity $\mathcal{N}$.

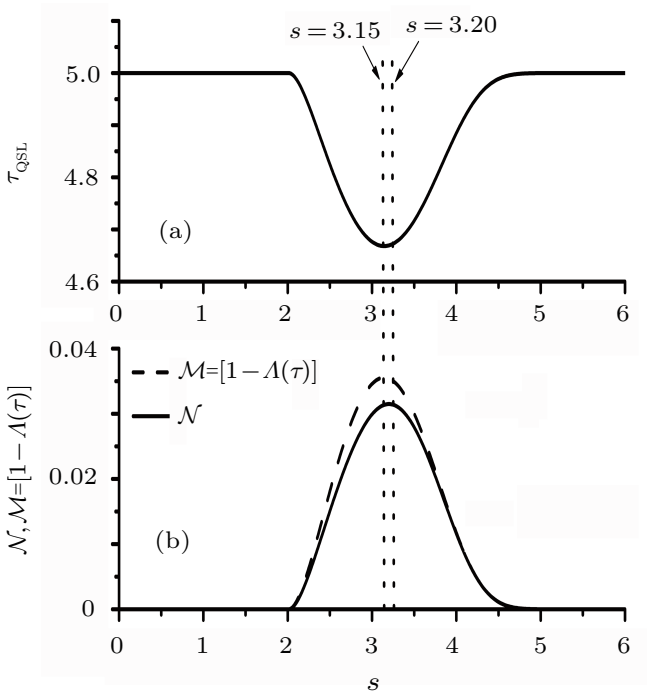


Fig. 5. The QSL time τ_{QSL} (a), the non-Markovianity $\mathcal{N}$ [(b) solid line] and the quantity $\mathcal{N}/[1-\Lambda(\tau)]$ [(b) dashed line] associated the non-Markovianity and decoherence factor as a function of the ohmicity parameter s for the thermal equilibrium environment. Other parameters are $\Omega = 1$, $\alpha = \pi/4$, and the actual driving time $\tau = 5$.

Then we reveal some important differences between the thermal equilibrium environment and nonequilibrium environment cases. From Fig. 5(a), we can clearly see that for the equilibrium environment the speed of quantum evolution can be enhanced only for the super-Ohmic ($s > 2$) cases, but it is a uniform evolution speed with the original speed for the sub-Ohmic and Ohmic cases ($0 < s \leq 2$), which is different from that of the nonequilibrium environment discussed above, where the speedup of quantum evolution can occur from sub-Ohmic ($s < 1$) to Ohmic ($s = 1$) and super-Ohmic ($s > 1$) cases. Further, in Fig. 5(b) (solid line) the non-Markovianity $\mathcal{N}$ shows that the transition from $\mathcal{N} \neq 0$ to $\mathcal{N} = 0$ just corresponds to the QSL time from $\tau_{QSL} < \tau$ to $\tau_{QSL} = \tau$ for the s. The relation between the QSL time τ_{QSL} and the non-Markovianity $\mathcal{N}$ can be easily explained as follows: first recall that in Ref. [42] the authors have indicated that non-Markovian effect occurs (i.e., $\mathcal{N} > 0$) if and only if $s > 2$, i.e., super-Ohmic spectrums, but it is absent (i.e., $\mathcal{N} = 0$) for the $0 < s \leq 2$, i.e., sub-Ohmic and Ohmic spectrums. As a consequence, from Eq. (16) we obtain that there is $\tau_{QSL} = \tau$ for $0 < s \leq 2$, and there is $\tau_{QSL} < \tau$ for $s > 2$. Moreover, it is noted that if we consider the case of the large cutoff frequency $\Omega = 10$, the main results are unchangeable. Summarizing, the characteristics of the QSL time of the nonequilibrium environment with ohmicity parameter s are different from those of the thermal equilibrium environment.

5. Conclusions

We have studied the speed of the evolution of a qubit undergoing a nonequilibrium environment with spectral density of general Ohmic form. First by using a popular measure of non-Markovianity, we analyzed in detail the non-Markovianity of the nonequilibrium environment. Then we further discussed the relationship between the QSL time and the non-Markovianity of the open system for any pure states. By investigations, we have obtained some interesting findings: (i) We find that the non-Markovianity displays a nonmonotonic behavior for different values of the ohmicity parameter s by fixing other parameters, which is an interesting and unrevealed finding in a previous paper;[36] (ii) The non-Markovianity displays a nonmonotonic behavior with the change of a phase control parameter λ so that if we get the maximal non-Markovian effect from the environment, it is necessary to carefully consider the choices of λ for different spectral densities; (iii) The quantum speed limit time of a qubit with any initial states are equal to the product of the coherence of the initial state and the QSL time of the maximally coherent states. Finally, we revealed that the speed of the quantum evolution can be obviously accelerated in the wide range of the ohmicity parameter, which is different from the case of the thermal equilibrium environment. In the present paper, we only considered the relationship between the QSL time and the non-Markovianity for the simple case of initial phases of the bath modes satisfying $\theta(\omega) = -\lambda\omega$ because of having exact solutions of the nonequilibrium environment model as Eq. (5). As for the more complex form of initial phases of the bath modes $\theta(\omega)$, one may show richer non-Markovian effect than our considered case. However, it also holds that the QSL time is related to both the non-Markovianity and decoherence factor for a given driving time and initial states. Note that, during the completion of this work, a new bound[24] from the generation of quantumness point of view has also been proposed to study the QSL time, but it is different from our measure in the paper.

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