New useful special function in quantum optics theory*

Feng Chen(陈锋)[1,†] and Hong-Yi Fan(范洪义)[2]

[1] Department of Mathematics and Physics, Hefei University, Hefei 230601, China
[2] Department of Material Science and Engineering, University of Science and Technology of China, Hefei 230026, China

(Received 16 February 2016; revised manuscript received 6 April 2016; published online 25 June 2016)

By virtue of the operator Hermite polynomial method [Fan H Y and Zhan D H 2014 *Chin. Phys. B* **23** 060301] we find a new special function which is useful in quantum optics theory, whose expansion involves both power-series and Hermite polynomials, i.e.,

$$\sum_{l=0}^{\min(m,n)} \frac{n!m!\,(-1)^l}{l!\,(n-l)!\,(m-l)!}\,\mathrm{H}_{n-l}\,(x)\,y^{m-l} \equiv \mathfrak{G}_{n,m}\,(x,y)\,.$$

By virtue of the operator Hermite polynomial method and the technique of integration within ordered product of operators (IWOP) we derive its generating function. The circumstance in which this new special function appears and is applicable is considered.

Keywords: Hermite polynomial excitation state, IWOP method, new special function, generating function, operator Hermite polynomial method

PACS: 03.65.–w, 42.50.–p **DOI:** 10.1088/1674-1056/25/8/080303

1. Introduction

As a kind of special function, the Hermite polynomial has been widely used in quantum mechanics and mathematical physics.[1–3] The one-variable Hermite polynomials are just eigenfunctions of the Hamiltonian of the quantum harmonic oscillator, while the two-variable Hermite polynomials can represent transition amplitude from the number state $|n\rangle$ to $|m\rangle$ in time evolution of a forced harmonic oscillator.[4] The Hermite–Gaussian modes exist in optical propagation in quadratic Gradient Index (GRIN) lenses.[5] In Ref. [6], the optical field's quadrature excitation state is introduced which turns out to be a Hermite polynomial excited state. Mathematically, the one-variable Hermite polynomial $\mathrm{H}_n\,(x)$ can be introduced by its generating function

$$\mathrm{e}^{-t^2+2tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathrm{H}_n\,(x)\,, \tag{1}$$

or

$$\mathrm{H}_n\,(x) = \frac{\partial^n}{\partial t^n}\,\mathrm{e}^{-t^2+2tx}\Big|_{t=0} = \sum_{l=0}^{[n/2]} \frac{n!\,(-1)^l}{l!\,(n-2l)!}\,(2x)^{n-2l}\,. \tag{2}$$

In Ref. [7] new generating function formulae of even- and odd-Hermite polynomials are obtained, they are

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathrm{H}_{2n}\,(x) = \frac{1}{\sqrt{1+4t}}\exp\left(\frac{4t}{1+4t}x^2\right), \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathrm{H}_{2n+1}\,(x) = \frac{1}{(1+4t)^{3/2}}x\exp\left(\frac{4t}{1+4t}x^2\right), \tag{4}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathrm{H}_{2n+l}\,(x) = \frac{1}{(1+4t)^{(l+1)/2}}\,\mathrm{H}_l\left(\frac{1}{\sqrt{1+4t}}x\right)\exp\left(\frac{4t}{1+4t}x^2\right). \tag{5}$$

On the other hand, associated with the construction of bipartite entangled state representation[8,9]

$$\begin{aligned}|\xi\rangle &= \exp\left(-\frac{|\xi|^2}{2}+a^\dagger\xi+b^\dagger\xi^*-a^\dagger b^\dagger\right)|00\rangle = \exp\left(-\frac{|\xi|^2}{2}\right)\sum_{n,m=0}^{\infty}\frac{a^{\dagger n}b^{\dagger m}}{n!m!}\,\mathrm{H}_{n,m}\,(\xi,\xi)\,|00\rangle \\ &= \exp\left(-\frac{|\xi|^2}{2}\right)\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!m!}}\,\mathrm{H}_{n,m}\,(\xi,\xi)\,|n,m\rangle\,,\end{aligned} \tag{6}$$

we have introduced the generating function of two-variable Hermite polynomials $\mathrm{H}_{n,m}\,(x,y)$

$$\mathrm{e}^{-ts+tx+sy} = \sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\,\mathrm{H}_{n,m}\,(x,y) \tag{7}$$

*Project supported by the Natural Science Fund of Education Department of Anhui Province, China (Grant No. KJ2016A590), the Talent Foundation of Hefei University, China (Grant No. 15RC11), and the National Natural Science Foundation of China (Grant Nos. 11247009 and 11574295).

†Corresponding author. E-mail: chenfeng@hfuu.edu.cn

© 2016 Chinese Physical Society and IOP Publishing Ltd

or

$$\mathrm{H}_{n,m}(x,y)=\frac{\partial^{n+m}}{\partial s^n\partial t^m}\mathrm{e}^{-ts+tx+sy}\Big|_{t=s=0}=\sum_{l=0}^{\min(m,n)}\frac{n!m!(-1)^l}{l!(n-l)!(m-l)!}x^{n-l}y^{m-l}. \tag{8}$$

In this equation when $x^{n-l}y^{m-l}$ is replaced by $\mathrm{H}_{n-l}(\mathrm{i}x)\,\mathrm{H}_{m-l}(\mathrm{i}y)$ we can introduce a kind of special function[10]

$$\mathfrak{F}_{n,m}(x,y)=(2\mathrm{i})^{-n-m}\sum_{l=0}^{\min(m,n)}\frac{4^l n!m!}{l!(n-l)!(m-l)!}\mathrm{H}_{n-l}(\mathrm{i}x)\,\mathrm{H}_{m-l}(\mathrm{i}y) \tag{9}$$

for which we have found its generating function

$$\exp\left(sx+ty-st+\frac{s^2}{4}+\frac{t^2}{4}\right)=\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\mathfrak{F}_{n,m}(x,y) \tag{10}$$

or

$$\mathfrak{F}_{n,m}(x,y)=\frac{\mathrm{d}^n\mathrm{d}^m}{\mathrm{d}s^n\mathrm{d}t^m}\exp\left(sx+ty-st+\frac{s^2}{4}+\frac{t^2}{4}\right)\Big|_{t=s=0}. \tag{11}$$

Such a function $\mathfrak{F}_{n,m}(x,y)$ really appears in some calculations of quantum optics theory, for example, the following anti-normally ordered product of two single-variable Hermite-polynomial operators is equal to

$$\begin{aligned}\mathrm{H}_n(a)\,\mathrm{H}_m(a^\dagger)=&\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathrm{e}^{2ta-t^2}\Big|_{t=0}\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\mathrm{e}^{2\tau a^\dagger-\tau^2}\Big|_{\tau=0}\\=&\frac{\mathrm{d}^n\mathrm{d}^m}{\mathrm{d}t^n\mathrm{d}\tau^m}:\mathrm{e}^{-t^2-\tau^2+2ta+2\tau a^\dagger+4t\tau}:\Big|_{t=\tau=0}\\=&:\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\mathrm{e}^{-t^2+2ta}\mathrm{H}_m(2t+a^\dagger)]:\Big|_{t=0}\\=&:\sum_{l=0}^{n}\binom{n}{l}\frac{\mathrm{d}^{n-l}}{\mathrm{d}t^{n-l}}\mathrm{e}^{-t^2+2ta}\frac{\mathrm{d}^l}{\mathrm{d}t^l}\mathrm{H}_m(2t+a^\dagger)\Big|_{t=0}:\\=&\sum_l\frac{4^l m!n!}{(n-l)!l!(m-l)!}\mathrm{H}_{m-l}(a^\dagger)\,\mathrm{H}_{n-l}(a)\\=&(2\mathrm{i})^{n+m}:\mathfrak{F}_{n,m}(-\mathrm{i}a^\dagger,-ia):\end{aligned} \tag{12}$$

where

$$\frac{\mathrm{d}^l}{\mathrm{d}t^l}\mathrm{H}_m(t)=\frac{2^l m!}{(m-l)!}\mathrm{H}_{m-l}(t)$$

is used. Then the overlap of two Hermite-polynomial excited coherent states is equal to

$$\langle z|\,\mathrm{H}_n(a)\,\mathrm{H}_m(a^\dagger)\,|z\rangle=(2\mathrm{i})^{n+m}\mathfrak{F}_{n,m}(-\mathrm{i}z^*,-\mathrm{i}z). \tag{13}$$

An interesting question thus challenges us: if we replace the power-series function x^{n-l} or y^{m-l} in Eq. (8) by suitable one-variable Hermite polynomial, like: $x^{n-l}\to\mathrm{H}_{n-l}(fx)$ or $y^{m-l}\to\mathrm{H}_{m-l}(y)$, e.g.,

$$\begin{aligned}&\sum_{l=0}^{\min(m,n)}\frac{n!m!(-1)^l}{l!(n-l)!(m-l)!}x^{n-l}y^{m-l}\\\to&\sum_{l=0}^{\min(m,n)}\frac{n!m!(-1)^l}{l!(n-l)!(m-l)!}\mathrm{H}_{n-l}(fx)\,y^{m-l}\\\equiv&\,\mathfrak{G}_{n,m}(fx,y),\end{aligned} \tag{14}$$

then how is such a new function $\mathfrak{G}_{n,m}(x,y)$? Is it a meaningful special function? If it is true, what is its generating function and under what situation does it appear? We shall employ the operator Hermite polynomial method (OHPM)[7] to tackle these problems. In Section 2 we briefly review OHPM. In Section 3 we show how we get the form of $\mathfrak{G}_{n,m}(x,y)$ in quantum optics calculations, noting its expansion involving both power-series and Hermite polynomials. In Section 4 we obtain the generating function of $\mathfrak{G}_{n,m}(x,y)$.

2. Brief review of OHPM

The essential point of OHPM is replacing classical Hermite polynomial $\mathrm{H}_n(x)$ by $\mathrm{H}_n(X)$, where X is the coordinate operator in quantum mechanics

$$X=\frac{a+a^\dagger}{\sqrt{2}},\quad\left[a,a^\dagger\right]=1 \tag{15}$$

and then using the fundamental operator commutative relations among normal, anti-normal or Weyl ordering, to obtain many operator identities about X, finally let X return back to x, we can derive some new relations about special functions. For instance, from Eq. (1) we have

$$\begin{aligned}\sum_{n=0}\frac{t^n}{n!}\mathrm{H}_n(X)=&\,\mathrm{e}^{-t^2+2tX}=\exp\left[-t^2+\sqrt{2}t\left(a+a^\dagger\right)\right]\\=&\exp\left(\sqrt{2}ta^\dagger\right)\exp\left(\sqrt{2}ta\right)=:\mathrm{e}^{2tX}:\\=&\sum_{n=0}\frac{2^n t^n}{n!}:X^n:,\end{aligned} \tag{16}$$

where : : denotes normal ordering, then we have the useful and concise operator identity

$$\mathrm{H}_n(X)=2^n:X^n:. \tag{17}$$

Similarly, for the two-mode case, after introducing the second-mode coordinate operator

$$Y=(b+b^\dagger)/\sqrt{2},\quad\left[b,b^\dagger\right]=1 \tag{18}$$

and noting $\left[a+b^\dagger,b+a^\dagger\right]=0$, from Eq. (7) we have

$$\sum_{n,m=0}\frac{s^n t^m}{n!m!}\mathrm{H}_{n,m}\left(a+b^\dagger,b+a^\dagger\right)=\mathrm{e}^{t\left(a+b^\dagger\right)+s\left(b+a^\dagger\right)-st}=:\mathrm{e}^{t\left(a+b^\dagger\right)+s\left(b+a^\dagger\right)}:=\sum_{n,m=0}\frac{s^n t^m}{n!m!}:\left(a+b^\dagger\right)^n\left(b+a^\dagger\right)^m:, \tag{19}$$

which indicates

$$\mathrm{H}_{n,m}\left(a+b^\dagger,b+a^\dagger\right)=:\left(a+b^\dagger\right)^n\left(b+a^\dagger\right)^m:, \tag{20}$$

this is another concise operator identity. Hence, the OHPM is powerful in deriving new operator identities.

3. The appearance of $\mathfrak{G}_{n,m}(x,y)$

We now demonstrate in what physical circumstances will the form of new special function $\mathfrak{G}_{n,m}(x,y)$ appear? For example, when we calculate photon number distribution of the Hermite-polynomial-excited state $\mathrm{H}_n\left(\sqrt{g}a^\dagger\right)|0\rangle$, i.e., we consider

$$\langle l|\,\mathrm{H}_n\left(\sqrt{g}a^\dagger\right)|0\rangle=\frac{1}{\sqrt{l!}}\langle 0|\,a^l\,\mathrm{H}_n\left(\sqrt{g}a^\dagger\right)|0\rangle, \tag{21}$$

where

$$\langle l|=\frac{1}{\sqrt{l!}}\langle 0|\,a^l$$

is a number state, due to the integration formula

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}\mathrm{H}_n(fx)\,\mathrm{e}^{-(x-y)^2}=\left(1-f^2\right)^{n/2}\mathrm{H}_n\left(fy/\sqrt{1-f^2}\right) \tag{22}$$

and using the integration within normal product of operators, we have

$$\begin{aligned}\mathrm{H}_n(fX)&=\mathrm{H}_n(fX)\int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle\langle x|\\&=\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}\mathrm{H}_n(fx):\mathrm{e}^{-(x-X)^2}:\\&=\left(1-f^2\right)^{n/2}:\mathrm{H}_n\left(fX/\sqrt{1-f^2}\right):,\end{aligned} \tag{23}$$

where $\int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle\langle x|=1$ is the completeness relation of the coordinate eigenstate. When

$$f=\sqrt{\frac{2g}{1+2g}}, \tag{24}$$

then

$$\begin{aligned}\mathrm{H}_n\left(\sqrt{\frac{2g}{1+2g}}X\right)&=\left(\frac{1}{1+2g}\right)^{n/2}:\mathrm{H}_n\left(\sqrt{2g}X\right):\\&=\left(\frac{1}{1+2g}\right)^{n/2}:\mathrm{H}_n\left[\sqrt{g}\left(a+a^\dagger\right)\right]:.\end{aligned} \tag{25}$$

Since

$$:\mathrm{H}_n\left[\sqrt{g}\left(a+a^\dagger\right)\right]:|0\rangle=\mathrm{H}_n\left(\sqrt{g}a^\dagger\right)|0\rangle, \tag{26}$$

so equation (21) becomes

$$\begin{aligned}\langle l|\,\mathrm{H}_n\left(\sqrt{g}a^\dagger\right)|0\rangle&=(1+2g)^{n/2}\frac{1}{\sqrt{l!}}\\&\times\left\langle 0\left|a^l\,\mathrm{H}_n\left(\sqrt{\frac{2g}{1+2g}}X\right)\right|0\right\rangle.\end{aligned} \tag{27}$$

Thus, we see the appearance of the form $a^l\mathrm{H}_n(fX)$, its Hermite conjugate is in the form $\mathrm{H}_n(fX)\,a^{\dagger l}$. Now using

$$\mathrm{e}^{fX}a^\dagger\,\mathrm{e}^{-fX}=a^\dagger+\left[f\frac{a+a^\dagger}{\sqrt{2}},a^\dagger\right]=a^\dagger+\frac{f}{\sqrt{2}}, \tag{28}$$

we consider

$$\begin{aligned}\mathrm{H}_n(fX)\,a^{\dagger l}&=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathrm{e}^{-t^2+2tfX}|_{t=0}a^{\dagger l}=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(a^\dagger+\sqrt{2}tf\right)^l\mathrm{e}^{-t^2+2tfX}|_{t=0}\\&=\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(a^\dagger+\sqrt{2}tf\right)^l\frac{\mathrm{d}^{n-k}}{\mathrm{d}t^{n-k}}\mathrm{e}^{-t^2+2tfX}\Big|_{t=0},\end{aligned} \tag{29}$$

where

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(a^\dagger+\sqrt{2}tf\right)^l|_{t=0}&=\frac{\mathrm{d}^k}{\mathrm{d}t^k}\sum_{j=0}^{l}\binom{l}{j}a^{\dagger l-j}\left(\sqrt{2}tf\right)^j\Big|_{t=0}\\&=\sum_{j=0}^{l}\binom{l}{j}a^{\dagger l-j}\left(\sqrt{2}f\right)^j\delta_{kj}k!=\left(\sqrt{2}f\right)^k k!a^{\dagger l-k}\binom{l}{k},\end{aligned} \tag{30}$$

so the normally ordered expansion of $\mathrm{H}_n(fX)\,a^{\dagger l}$ is

$$\begin{aligned}\mathrm{H}_n(fX)\,a^{\dagger l}&=\sum_{k=0}^{n}\binom{n}{k}\left(\sqrt{2}f\right)^k k!a^{\dagger l-k}\binom{l}{k}\mathrm{H}_{n-k}(fX)\\&=\sum_{k=0}^{n}\frac{n!l!\left(\sqrt{2}f\right)^k}{(l-k)!k!(n-k)!}a^{\dagger l-k}\mathrm{H}_{n-k}(fX)\end{aligned}$$

$$= \left(-\sqrt{2}f\right)^{l}\left(1-f^{2}\right)^{n/2}\sum_{k=0}^{\min(l,n)}\frac{n!l!\,(-1)^{k}}{(l-k)!k!\,(n-k)!}\left(\frac{-1}{\sqrt{2}f}\right)^{l-k}:a^{\dagger l-k}\mathrm{H}_{n-k}\left(fX/\sqrt{1-f^{2}}\right):. \tag{31}$$

Comparing with the definition of $\mathfrak{G}_{n,m}(x,y)$ in Eq. (14), we can see

$$\mathrm{H}_{n}(fX)\,a^{\dagger l}=\left(-\sqrt{2}f\right)^{l}\left(1-f^{2}\right)^{n/2}:\mathfrak{G}_{n,l}\left(\frac{fX}{\sqrt{1-f^{2}}},\frac{-1}{\sqrt{2}f}a^{\dagger}\right):, \tag{32}$$

which indicates that when one calculates photon number distribution of the Hermite-polynomial-excited state, one encounters the new function $\mathfrak{G}_{n,m}(x,y)$.

Similarly, we can derive

$$\begin{aligned}\mathrm{H}_{n}(X)\,a^{l}&=\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\mathrm{e}^{-t^{2}+2tX}\Big|_{t=0}a^{l}=\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\left(a-\sqrt{2}t\right)^{l}\mathrm{e}^{-t^{2}+2tX}\Big|_{t=0}\\&=\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{k}}{\mathrm{d}t^{k}}\left(a-\sqrt{2}t\right)^{l}\frac{\mathrm{d}^{n-k}}{\mathrm{d}t^{n-k}}\mathrm{e}^{-t^{2}+2tX}\Big|_{t=0}=\sum_{k=0}^{n}\frac{n!l!\left(-\sqrt{2}\right)^{k}}{(l-k)!k!\,(n-k)!}a^{l-k}\mathrm{H}_{n-k}(X),\end{aligned} \tag{33}$$

Then using the anti-normally ordered form of $\mathrm{H}_{n}(X)$

$$\mathrm{H}_{n}(X)=2^{n/2}\vdots\mathrm{H}_{n}\left(X/\sqrt{2}\right)\vdots \tag{34}$$

and comparing with Eq. (14) we have

$$\mathrm{H}_{n}(X)\,a^{l}=2^{n/2}\sum_{k=0}^{n}\frac{n!l!\,(-)^{k}}{(l-k)!k!\,(n-k)!}a^{l-k}\vdots\mathrm{H}_{n-k}\left(X/\sqrt{2}\right)\vdots\equiv\vdots\mathfrak{G}_{n,l}\left(a,X/\sqrt{2}\right)\vdots. \tag{35}$$

4. The generating function of $\mathfrak{G}_{n,l}$

Now we explore the generating function of $\mathfrak{G}_{n,l}(x,y)$. According to the idea of OHPM we consider

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^{n}\tau^{l}}{n!l!}\mathrm{H}_{n}(X)\,a^{l}&=\mathrm{e}^{-t^{2}+2tX}\mathrm{e}^{\tau a}=\mathrm{e}^{-t^{2}+\sqrt{2}t\left(a+a^{\dagger}\right)}\mathrm{e}^{\tau a}\\&=\mathrm{e}^{-t^{2}+\sqrt{2}ta}\mathrm{e}^{\sqrt{2}ta^{\dagger}}\mathrm{e}^{\tau a}\mathrm{e}^{-t^{2}}=\mathrm{e}^{-2t^{2}-\sqrt{2t\tau}}\mathrm{e}^{\tau a+\sqrt{2}ta}\mathrm{e}^{\sqrt{2}ta^{\dagger}}.\end{aligned} \tag{36}$$

Combining with Eq. (35) leads to

$$\sum_{n,l}\frac{t^{n}\tau^{l}}{n!l!}\vdots\mathfrak{G}_{n,l}\left(a,X/\sqrt{2}\right)\vdots=\mathrm{e}^{-2t^{2}-\sqrt{2t\tau}}:\mathrm{e}^{(\tau a+2tX)}:. \tag{37}$$

Both sides of Eq. (37) are in anti-normally ordering, so letting $X/\sqrt{2}\to x, a\to y, \sqrt{2t}\to t$, we obtain the generating function of $\mathfrak{G}_{n,l}$

$$\sum_{n,l}\frac{t^{n}\tau^{l}}{n!l!}\mathfrak{G}_{n,l}(x,y)=\mathrm{e}^{-2t^{2}-\sqrt{2t\tau}}\mathrm{e}^{\left(\tau x+2\sqrt{2}ty\right)}. \tag{38}$$

In summary, by virtue of the operator Hermite polynomial method and the technique of integration within the ordered product of operators,[11,12] we have proposed a new kind of special functions $\mathfrak{G}_{n,l}(x,y)$, whose expansion involves both power-series and Hermite polynomials. Its application in calculating physical quantity in quantum optics theory is presented.

References

[1] Mierzejewski J D and Luczka J 2008 *Phys. Rev. A* **77** 042316
[2] Lopez C E, Romero G and Retamal J C 2008 *Phys. Rev. Lett.* **101** 080503
[3] Mazhar Ali 2010 *J. Phys. B: At. Mol. Opt. Phys.* **43** 045504
[4] Fan H Y and Jiang T F 2007 *Mod. Phys. Lett. B* **21** 475
[5] C Yangjian and C Chiyi 2007 *J. Opt. Soc. Am. A* **24** 2394
[6] Fan H Y, He R, Da C and Liang Z F 2013 *Chin. Phys. B* **22** 080301
[7] Fan H Y and Zhan D H 2014 *Chin. Phys. B* **23** 060301
[8] Chen F and Fan H Y 2014 *Chin. Phys. B* **23** 030304
[9] Fan H Y and Klauder J R 1994 *Phys. Rev. A* **49** 704
[10] Fan H Y, Wan Z L and Wu Z 2015 *Chin. Phys. B* **24** 100302
[11] Fan H Y, Lu H L and Fan Y 2006 *Ann. Phys.* **321** 480
[12] de Peixoto F J G 2007 *Eur. Phys. J. D* **42** 153