Non-relativistic scattering amplitude for a new multi-parameter exponential-type potential

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(Received 3 February 2016; revised manuscript received 26 March 2016; published online 25 June 2016)

In this paper, we study the scattering properties of s-wave Schrödinger equation for the multi-parameter potential, which can be reduced into four special cases for different values of potential parameters, i.e., Hulthén, Manning–Rosen, and Eckart potentials. We also obtain and investigate the scattering amplitudes of these special cases. Some numerical results are also obtained and reported.

Keywords: scattering amplitude, Schrödinger equation, multi-parameter potential, phase shifts, cross section

PACS: 03.65.–w, 03.65.Nk **DOI:** 10.1088/1674-1056/25/8/080302

1. Introduction

One of the important tasks of non-relativistic or relativistic quantum mechanics is to study the bound states and scattering properties of a given quantum mechanical system under the action of a potential, which is very useful in studying the physical properties and handling the necessary information about the system. Recently, several authors introduced different potential models to explore the non-relativistic/relativistic bound states and scattering properties.[1–3] Aydoğdu *et al.*[4] derived the scattering and bound states of the Klein–Gordon equation for the Hulthén potential with unequal scalar and vector potentials and considered the position-dependent mass and constant mass, as a special case. They also calculated the transmission and reflection coefficients. In the case of unequal scalar and vector potentials, Wei *et al.*[5] obtained the phase shifts and normalization constant of the Klein–Gordon equation for Hulthén potential by introducing an improved new approximate scheme into the centrifugal term. The authors in Ref. [6] also reported the normalized radial wave functions and obtained the analytical formula of phase shifts within the framework of the partial-wave method, for the Schrödinger equation under the hyperbolic molecular potential. Castilho and Castro[7] investigated the scattering and bound states of fermions in a mixed vector–scalar smooth step potential. In D-dimensional space, Chen *et al.*[8] represented the non-relativistic scattering properties of modified Pöschl–Teller potential. They also provided some numerical results and compared them with others. In view of a spatially one-dimensional (1D) Woods–Saxon potential, the scattering solutions of the Klein–Gordon equation have been obtained in terms of hypergeometric functions by Rojas and Villalba, and they have derived the condition for the existence of transmission resonances.[9] The interested readers can also consult Refs. [10]–[12] for more information.

Recently, Garcia-Martinez[13] proposed the solvable multi-parameter exponential-type potential in the following form:

$$V(r)=\frac{A\,\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}+\frac{B\,\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}+\frac{C\,\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}, \tag{1}$$

where A, B, and C are the potential parameters, and α is the screening parameter. The choices of the A, B, and C parameters lead to specific exponential-type potentials. The Hulthén, Manning–Rosen, and Eckart potentials are special cases of the multi-parameter exponential-type potential. The non-relativistic scattering states of Hulthén and Manning–Rosen potentials can be found in Refs. [14] and [15], respectively.

The main purpose of the present article is to report the scattering amplitude of the s-wave Schrödinger equation for the multi-parameter exponential-type potential and to discuss the special cases of the potential.

2. Scattering amplitude of the multi-parameter exponential-type potential

The radial Schrödinger equation has the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{2\mu}{\hbar^2}V(r)-\frac{l(l+1)}{r^2}+\frac{2\mu E_{n,l}}{\hbar^2}\right]R_{n,l}(r)=0. \tag{2}$$

Substitution of multi-parameter exponential-type potential converts the radial Schrödinger equation into the form (for $l=0$)

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{2\mu}{\hbar^2}\left(\frac{A\,\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}+\frac{B\,\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}+\frac{C\,\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}\right)+\frac{2\mu E_{n,l}}{\hbar^2}\right\}R_{n,l}(r)=0. \tag{3}$$

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By introducing a new variable $z = 1 - e^{-\alpha r}$, equation (3) is transformed into the form

$$z(1-z)\frac{d^2}{dz^2} - z\frac{d}{dz} + \frac{2\mu}{\hbar^2\alpha^2}(-A+C-E_n) - \frac{2\mu}{\hbar^2\alpha^2}(B+C)\frac{1}{z} + \frac{2\mu E_n}{\hbar^2\alpha^2}\frac{1}{1-z}R_n(z) = 0. \tag{4}$$

Substituting $R_n(z) = z^{\xi_1}(1-z)^{\xi_2}G_n(z)$ into Eq. (4) leads to

$$z(1-z)\frac{d^2}{dz^2} + (2\xi_1 - (1+2\xi_1+2\xi_2)z)\frac{d}{dz} - \left(\xi_1^2 + \xi_2^2 + 2\xi_1\xi_2 - \frac{2\mu}{\hbar^2\alpha^2}(C-A-E_n)\right)G_n(z) = 0, \tag{5}$$

where

$$\xi_1 = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8\mu}{\hbar^2\alpha^2}(B+C)}\right), \tag{6}$$

$$\xi_2 = -\frac{ik}{\alpha}, \quad k = \sqrt{\frac{2\mu E_n}{\hbar^2}}. \tag{7}$$

Equation (5) can be written as

$$\left\{z(1-z)\frac{d^2}{dz^2} + (\gamma_3 - (1+\gamma_1+\gamma_2)z)\frac{d}{dz} - \gamma_1\gamma_2\right\}G_n(z) = 0. \tag{8}$$

The solution of Eq. (8) is well known, and it is expressed as hypergeometric function

$$G_n(z) = {}_2F_1(\gamma_1, \gamma_2, \gamma_3; z), \tag{9}$$

where

$$\gamma_1 = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8\mu}{\hbar^2\alpha^2}(B+C)}\right) - \frac{ik}{\alpha} + \sqrt{\frac{2\mu}{\hbar^2\alpha^2}(C-A-E_n)}, \tag{10}$$

$$\gamma_2 = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8\mu}{\hbar^2\alpha^2}(B+C)}\right) - \frac{ik}{\alpha} - \sqrt{\frac{2\mu}{\hbar^2\alpha^2}(C-A-E_n)}, \tag{11}$$

$$\gamma_3 = \left(1 + \sqrt{1 + \frac{8\mu}{\hbar^2\alpha^2}(B+C)}\right). \tag{12}$$

So the radial wave function $R_n(r)$ can be obtained as

$$R_n(r) = N_n e^{ikr}\left(1 - e^{-\alpha r}\right)^{\xi_1} {}_2F_1\left(\gamma_1, \gamma_2, \gamma_3; 1 - e^{-\alpha r}\right), \tag{13}$$

and N_n is the normalization constant. To obtain a finite solution, γ_1 or γ_2 must be a negative integer. This gives the following equations:

$$\frac{1}{2}\left(1 + \sqrt{1 + \frac{8\mu}{\hbar^2\alpha^2}(B+C)}\right) - \frac{ik}{\alpha} \pm \sqrt{\frac{2\mu}{\hbar^2\alpha^2}(C-A-E_n)} = -n, \quad n = 0, 1, 2, \ldots, \tag{14}$$

where, n is the quantum number and the energy eigenvalue equation can be found from the above equation. By using the properties of hypergeometric function

$${}_2F_1(a, b; c; 0) = 1, \tag{15a}$$

$${}_2F_1(a, b; c; s) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a, b; a+b-c+1; 1-s) + (1-s)^{c-a-b} \times \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}{}_2F_1(c-a, c-b; c-a-b+1; 1-s), \tag{15b}$$

for $r \to \infty$, ${}_2F_1\left(\gamma_1, \gamma_2, \gamma_3; 1 - e^{-\alpha r}\right)$ can be written as

$${}_2F_1\left(\gamma_1, \gamma_2, \gamma_3; 1 - e^{-\alpha r}\right) = \Gamma(\gamma_3)\left(\frac{\Gamma(\gamma_3 - \gamma_1 - \gamma_2)}{\Gamma(\gamma_3 - \gamma_1)\Gamma(\gamma_3 - \gamma_2)} + \frac{\Gamma(\gamma_2 + \gamma_1 - \gamma_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}e^{-2ikr}\right). \tag{16}$$

Inserting Eq. (16) into Eq. (13) gives

$$R(r) \xrightarrow{(r\to\infty)} N_n\Gamma(\gamma_3)\frac{(\gamma_2+\gamma_1-\gamma_3)!}{(\gamma_3-\gamma_1-\gamma_2)\Gamma(\gamma_1)\Gamma(\gamma_2)}\left(\frac{(\gamma_3-\gamma_1-\gamma_2)!\Gamma(\gamma_1)\Gamma(\gamma_2)}{(\gamma_2+\gamma_1-\gamma_3)\Gamma(\gamma_3-\gamma_1)\Gamma(\gamma_3-\gamma_2)\Gamma(\gamma_2+\gamma_1-\gamma_3)}e^{ikr} - e^{-ikr}\right). \tag{17}$$

The asymptotic behavior for the wave function is given by[16,17] (for $l = 0$)

$$\lim_{r\to\infty} R_n(r) \simeq (\text{const.})\left[S_{l=0}e^{ikr} - e^{-ikr}\right]. \tag{18}$$

By comparing Eq. (17) with Eq. (18), we find

$$S_{l=0} = \frac{(\gamma_3-\gamma_1-\gamma_2)!\Gamma(\gamma_1)\Gamma(\gamma_2)}{(\gamma_2+\gamma_1-\gamma_3)\Gamma(\gamma_3-\gamma_1)\Gamma(\gamma_3-\gamma_2)\Gamma(\gamma_2+\gamma_1-\gamma_3)}, \tag{19}$$

or equivalently, from Eqs. (10)–(12), we can easily obtain

$$S_{l=0}=\frac{\left(\frac{2ik}{\alpha}\right)!\Gamma\left(\frac{1}{2}\left(1+\sqrt{1+\frac{8\mu}{\hbar^2\alpha^2}(B+C)}\right)-\frac{ik}{\alpha}+\sqrt{\frac{2\mu}{\hbar^2\alpha^2}(C-A-E_n)}\right)}{\left(-\frac{2ik}{\alpha}\right)!\Gamma\left(\frac{1}{2}\left(1+\sqrt{1+\frac{8\mu}{\hbar^2\alpha^2}(B+C)}\right)+\frac{ik}{\alpha}-\sqrt{\frac{2\mu}{\hbar^2\alpha^2}(C-A-E_n)}\right)}$$
$$\times\frac{\Gamma\left(\frac{1}{2}\left(1+\sqrt{1+\frac{8\mu}{\hbar^2\alpha^2}(B+C)}\right)-\frac{ik}{\alpha}-\sqrt{\frac{2\mu}{\hbar^2\alpha^2}(C-A-E_n)}\right)}{\Gamma\left(\frac{1}{2}\left(1+\sqrt{1+\frac{8\mu}{\hbar^2\alpha^2}(B+C)}\right)+\frac{ik}{\alpha}+\sqrt{\frac{2\mu}{\hbar^2\alpha^2}(C-A-E_n)}\right)}. \tag{20}$$

Therefore, the total cross section can be easily obtained from

$$\sigma_{\rm tot}=\frac{4\pi}{k^2}\sum_l(2l+1)\sin^2\delta_l, \tag{21}$$

where

$$S_l=\mathrm{e}^{2i\delta_l},\quad \delta_l=\frac{1}{2}\arg[S_l]. \tag{22}$$

3. Some special cases

In this section, we are going to study four special cases of the multi-parameter exponential-type potential and their scattering properties by choosing approximate parameters in the multi-parameter exponential-type potential.

3.1. Hulthén potential

As already mentioned, for $A=-Z\mathrm{e}^2\lambda$, $\alpha=\lambda$, $B=C=0$, the multi-parameter exponential-type potential reduces to the Hulthén potential

$$V(r)=-Z\mathrm{e}^2\lambda\frac{\mathrm{e}^{-\lambda r}}{1-\mathrm{e}^{-\lambda r}}, \tag{23}$$

where λ is the screening parameter and Z is a constant identified with the atomic number. The Hulthén[18] potential is one of the significant exponential potentials, which behaves like a Coulomb potential for small values of $r$and decreases exponentially for large values without having the problems of the former, and it has many applications in different areas of physics.[19–21] According to Eqs. (6), (7), and (10)–(12) we have

$$\xi_1^{\rm H}=1,\quad \xi_2^{\rm H}=-\frac{ik}{\alpha},$$
$$\gamma_1^{\rm H}=1-\frac{ik}{\alpha}+\sqrt{\frac{2\mu}{\hbar^2\lambda^2}(Z\mathrm{e}^2\lambda-E_n)}, \tag{24}$$
$$\gamma_2^{\rm H}=1-\frac{ik}{\alpha}-\sqrt{\frac{2\mu}{\hbar^2\lambda^2}(Z\mathrm{e}^2\lambda-E_n)},\quad \gamma_3^{\rm H}=2. \tag{25}$$

Therefore, from Eq. (20) we obtain

$$S_{l=0}^{\rm H}=\frac{(\gamma_3^{\rm H}-\gamma_1^{\rm H}-\gamma_2^{\rm H})!\Gamma(\gamma_1^{\rm H})\Gamma(\gamma_2^{\rm H})}{(\gamma_2^{\rm H}+\gamma_1^{\rm H}-\gamma_3^{\rm H})\Gamma(\gamma_3^{\rm H}-\gamma_1^{\rm H})\Gamma(\gamma_3^{\rm H}-\gamma_2^{\rm H})\Gamma(\gamma_2^{\rm H}+\gamma_1^{\rm H}-\gamma_3^{\rm H})}. \tag{26}$$

3.2. Manning–Rosen potential

If we choose $B=0$, $A=-V_0/b^2$, $C=\alpha'(\alpha'-1)/b^2$, and $\alpha=1/b$, the multi-parameter exponential-type potential turns into the Manning–Rosen potential[22,23]

$$V(r)=-\frac{V_0}{b^2}\frac{\mathrm{e}^{-r/b}}{1-\mathrm{e}^{-r/b}}+\frac{\alpha'(\alpha'-1)}{b^2}\frac{\mathrm{e}^{-2r/b}}{\left(1-\mathrm{e}^{-r/b}\right)^2}. \tag{27}$$

The Manning–Rosen potential was first introduced by Manning and Rosen[24] in 1933. It is used for describing the diatomic molecular vibrations. From Eqs. (6), (7), and (10)–(12), one can obtain

$$\xi_1^{\rm MR}=\frac{1}{2}\left(1+\sqrt{1+\frac{8\mu}{\hbar^2}\alpha'(\alpha'-1)}\right),\qquad \xi_2^{\rm MR}=-ikb, \tag{28a}$$
$$\gamma_1^{\rm MR}=\frac{1}{2}\left(1+\sqrt{1+\frac{8\mu}{\hbar^2}\alpha'(\alpha'-1)}\right)-ikb+\sqrt{\frac{2\mu b^2}{\hbar^2}\left(\frac{\alpha'(\alpha'-1)}{b^2}+\frac{V_0}{b^2}-E_n\right)}, \tag{28b}$$
$$\gamma_2^{\rm MR}=\frac{1}{2}\left(1+\sqrt{1+\frac{8\mu}{\hbar^2}\alpha'(\alpha'-1)}\right)-ikb-\sqrt{\frac{2\mu b^2}{\hbar^2}\left(\frac{\alpha'(\alpha'-1)}{b^2}+\frac{V_0}{b^2}-E_n\right)}, \tag{28c}$$
$$\gamma_3^{\rm MR}=\left(1+\sqrt{1+\frac{8\mu}{\hbar^2}\alpha'(\alpha'-1)}\right), \tag{28d}$$

and the scattering amplitude, from Eq. (20), is

$$S_{l=0}^{\mathrm{MR}} = \frac{(\gamma_3^{\mathrm{MR}} - \gamma_1^{\mathrm{MR}} - \gamma_2^{\mathrm{MR}})!\Gamma(\gamma_1^{\mathrm{MR}})\Gamma(\gamma_2^{\mathrm{MR}})}{(\gamma_2^{\mathrm{MR}} + \gamma_1^{\mathrm{MR}} - \gamma_3^{\mathrm{MR}})\Gamma(\gamma_3^{\mathrm{MR}} - \gamma_1^{\mathrm{MR}})\Gamma(\gamma_3^{\mathrm{MR}} - \gamma_2^{\mathrm{MR}})\Gamma(\gamma_2^{\mathrm{MR}} + \gamma_1^{\mathrm{MR}} - \gamma_3^{\mathrm{MR}})}. \tag{29}$$

3.3. Eckart potential

By considering $C = 0$, $A = -\delta$, $B = \beta$, $\alpha = 1/a$, the multi-parameter exponential-type potential changes into the Eckart potential

$$V(r) = -\delta \frac{\mathrm{e}^{-\alpha r}}{1 - \mathrm{e}^{-\alpha r}} + \frac{\beta\, \mathrm{e}^{-\alpha r}}{(1 - \mathrm{e}^{-\alpha r})^2}. \tag{30}$$

The Eckart potential[25] is one of the most important exponential-type potentials in the literature and it has many applications in different areas.[16,26] According to our results, we have

$$\xi_1^{\mathrm{Eck}} = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8\mu\beta a^2}{\hbar^2}}\right), \qquad \xi_2^{\mathrm{Eck}} = -\mathrm{i}ka, \tag{31a}$$

$$\gamma_1^{\mathrm{Eck}} = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8\mu\beta a^2}{\hbar^2}}\right) - \mathrm{i}ka + \sqrt{\frac{2\mu a^2}{\hbar^2}(\delta - E_n)}, \tag{31b}$$

$$\gamma_2^{\mathrm{Eck}} = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8\mu\beta a^2}{\hbar^2}}\right) - \mathrm{i}ka - \sqrt{\frac{2\mu a^2}{\hbar^2}(\delta - E_n)}, \tag{31c}$$

$$\gamma_3^{\mathrm{Eck}} = \left(1 + \sqrt{1 + \frac{8\mu\beta a^2}{\hbar^2}}\right), \tag{31d}$$

thus from Eq. (20) we obtain

$$S_{l=0}^{\mathrm{Eck}} = \frac{(\gamma_3^{\mathrm{Eck}} - \gamma_1^{\mathrm{Eck}} - \gamma_2^{\mathrm{Eck}})!\Gamma(\gamma_1^{\mathrm{Eck}})\Gamma(\gamma_2^{\mathrm{Eck}})}{(\gamma_2^{\mathrm{Eck}} + \gamma_1^{\mathrm{Eck}} - \gamma_3^{\mathrm{Eck}})\Gamma(\gamma_3^{\mathrm{Eck}} - \gamma_1^{\mathrm{Eck}})\Gamma(\gamma_3^{\mathrm{Eck}} - \gamma_2^{\mathrm{Eck}})\Gamma(\gamma_2^{\mathrm{Eck}} + \gamma_1^{\mathrm{Eck}} - \gamma_3^{\mathrm{Eck}})}. \tag{32}$$

Finally, we present some figures which show the scattering cross section (σ) of multi-parameter exponential-type potential and its special cases for different states in Figs. 1–4.

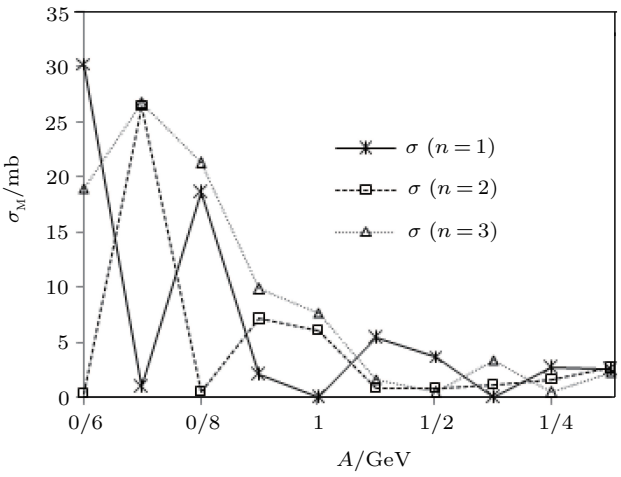


Fig. 1. Variations of scattering cross section with A for different states of multi-parameter exponential-type potential, $C = 0.09$ GeV, $m = 0.938$ GeV, $B = 0.2$ GeV, and $\alpha = 0.01$ GeV. The unit 1 b = 10^{-28} m^2.

It is obvious from these figures that among the special cases of the multi-parameter exponential-type potentials, the Eckart potential has made the most contributions to cross section, and the obtained values of the cross section for this potential are greater than for the Hulthén and the Manning–Rosen potentials. We can also see that in Fig. 4, the cross section of the Eckart potential at $A = 1$ GeV and $A = 1.4$ GeV have minimum values for all the states.

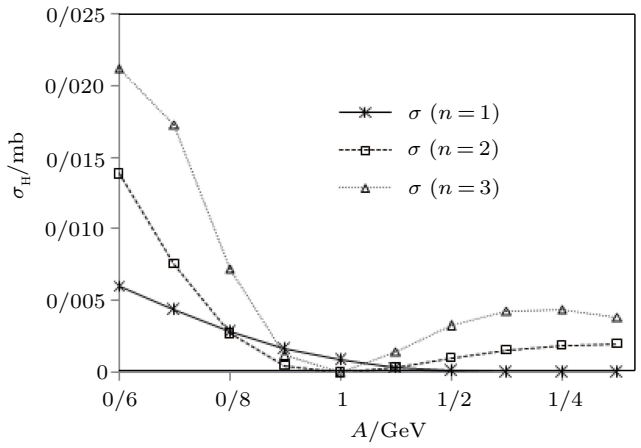


Fig. 2. Variations of scattering cross section with A for different states of Hulthén potential $m = 0.938$ GeV and $\alpha = 0.01$ GeV.

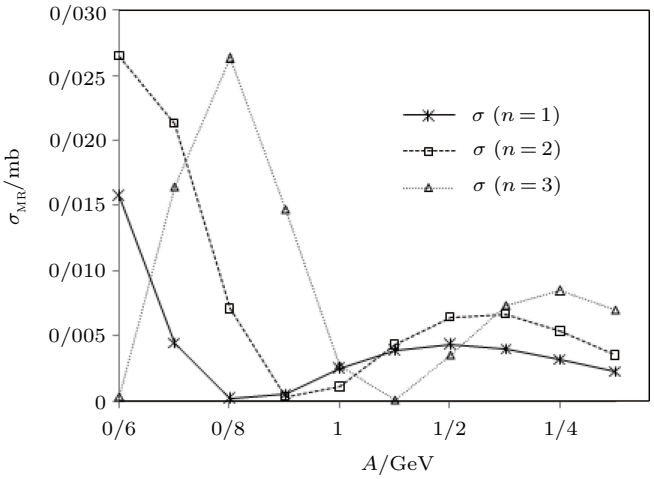


Fig. 3. Variations of scattering cross section with A for different states of Manning–Rosen potential, $C = 0.0002$ GeV, $m = 0.938$ GeV, and $\alpha = 0.01$ GeV.

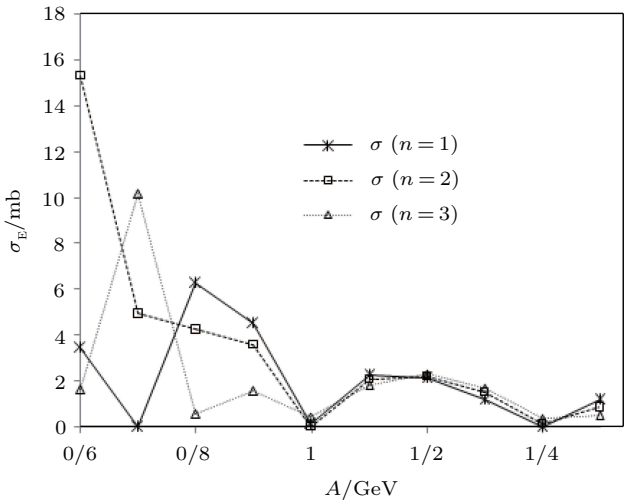


Fig. 4. Variations of scattering cross section with A for different states of Eckart potential, $m = 0.938$ GeV, $B = 0.2$ GeV, and $\alpha = 0.01$ GeV.

4. Conclusions

The multi-parameter exponential-type potential provides an appealing option for theoretical physicists especially for molecular systems. As we have already mentioned, this potential can reduce to special cases of molecular potentials, therefore it can be a suitable choice to study physical systems. Because of these results and the importance of scattering properties, we investigate the scattering amplitude of the Schrödinger equation for multi-parameter exponential-type potential, and report the scattering amplitudes of the special cases. We also report some numerical results. As we see from the figures, of the three potentials (the Eckart potential, the Hulthén potential, and the Manning–Rosen potential), the Eckart potential makes the most contributions to cross section.

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