Gazeau–Klauder coherent states examined from the viewpoint of diagonal ordering operation technique

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In this paper we investigate the Gazeau–Klauder coherent states using a newly introduced diagonal ordering operation technique, in order to examine some of the properties of these coherent states. The results coincide with those obtained from other purely algebraic methods, but the calculations are greatly simplified. We apply the general theory to two cases of Gazeau–Klauder coherent states: pseudoharmonic as well as the Morse oscillators.

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1. Introduction

Let us consider the infinite-dimensional space of ket vectors $|n\rangle (n = 0, 1, 2, \ldots, \infty)$, which are eigenvectors of the Hamiltonian H,

$$H|n\rangle = E_n|n\rangle \equiv \hbar\omega e_n|n\rangle. \tag{1}$$

We will focus on the re-scaled dimensionless Hamiltonian spectrum e_n with the properties: $e_0 = 0$ and $e_n < e_{n+1}$. We will use the following notation:

$$\rho(n) \equiv \prod_{j=1}^{n} e_j. \tag{2}$$

Now, we define two arbitrary partner hermitic conjugate densely defined and dimensionless lowering A_- and raising A_+ operators satisfying the following relations:

$$A_+|n\rangle = \sqrt{e_{n+1}}|n+1\rangle, \quad A_-|n\rangle = \sqrt{e_n}|n-1\rangle, \tag{3}$$

$$A_+A_-|n\rangle = e_n|n\rangle. \tag{4}$$

$$(A_+)^+ = A_-, \quad (A_-)^+ = A_+. \tag{5}$$

Consequently,

$$H = \hbar\omega A_+A_- \tag{6}$$

We have to mention, from the beginning, that the operators A_- and A_+ are simple lowering and raising operators, and they should not be confused with the group generators of quantum groups related to the concrete studied system.

After the repeatable application of the raising operator A_+ on the vacuum state $|0\rangle$, we obtain their counterpart

$$|n\rangle = \frac{1}{\sqrt{\rho(n)}}(A_+)^n|0\rangle, \quad \langle n| = \frac{1}{\sqrt{\rho(n)}}\langle 0|(A_-)^n. \tag{7}$$

In a previous paper,[1] we have introduced a new ordering technique for quantum operators which we have called the diagonal ordering operation technique (DOOT), denoted by the symbols # #. Generally, this technique differs from and may yield different results to those of the normal ordering case in the ordinary sense[2] or the IWOP technique.[3]

We introduce and adopt the following rules for the DOOT calculus.[1]

i) Inside the symbol # #, the order of the operators A_- and A_+ can be permuted like commutable operators, so that finally this results in an operator function depending only on the powers of normally ordered operator product A_+A_-, i.e.,

$$\begin{aligned} &\#A_-A_+\# = \#A_+A_-\# = A_+A_-, \\ &\#(A_-)^n(A_+)^n\# = \#(A_+)^n(A_-)^n\# = (A_+A_-)^n. \end{aligned} \tag{8}$$

ii) A symbol ## inside another symbol ## can be deleted.

iii) If the integration is convergent, a normally ordered product of operators can be integrated or differentiated with respect to c-numbers according to the usual rules. In addition, the c numbers can be taken out from the symbol ##.

iv) The projector $|0\rangle\langle 0|$ of the normalized vacuum state $|0\rangle$, in the frame of DOOT, has the following normal ordered form and this choice is justified as

$$|0\rangle\langle 0| = \#\frac{1}{N(A_+A_-)}\#, \tag{9}$$

where, as we will see later, $N(A_+A_-)$ is the normalization function of the coherent states (CSs), generally being a hypergeometric function depending on the operatorial "variable" A_+A_-, i.e., $[N(A_+A_-)]^2 = {}_pF_q(\{a_i\}_1^p; \{b_j\}_1^q; A_+A_-)$.

Our aim is to apply the DOOT to the Gazeau–Klauder

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coherent states (GK-CSs) introduced in Refs. [4] and [5]

$$|J,\gamma\rangle = \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{\left(\sqrt{J}\right)^n \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\rho(n)}}|n\rangle, \tag{10}$$

where $0 \le J \le R \le \infty$ and $-\infty < \gamma < +\infty$. The convergence radius R can be determined by the limit $R = \lim\limits_{n\to\infty}\sqrt[n]{\rho(n)}$ or $R = \lim\limits_{n\to\infty}\frac{\rho(n+1)}{\rho(n)}\rangle 0$.

Consequently, the normalization function

$$N(J) = \sum_{n=0}^{\infty}\frac{J^n}{\rho(n)} \tag{11}$$

must be an analytical function with respect to the real variable J and the positive constants $\rho(n)$ are assumed to be the moments of a positive definite probability distribution $G(n)$[4] and it must exist for all n

$$\rho(n) \equiv \int_0^R \mathrm{d}u u^n G(n) < +\infty, \quad \rho(0) = 1. \tag{12}$$

Using the property (4) and the DOOT rules, it follows that

$$\#f(A_+A_-)\#|n\rangle = f(e_n)|n\rangle. \tag{13}$$

Particularly, we have

$$\#\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\#|n\rangle = \mathrm{e}^{-\mathrm{i}\gamma e_n}|n\rangle. \tag{14}$$

Consequently, the GK-CSs can be written as

$$|J,\gamma\rangle = \#\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\#|J\rangle, \tag{15}$$

where

$$\begin{aligned}|J\rangle \equiv |J,\gamma=0\rangle &= \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{\left(\sqrt{J}\right)^n}{\sqrt{\rho(n)}}|n\rangle \\ &= \frac{1}{\sqrt{N(J)}}N\left(A_+\sqrt{J}\right)|0\rangle\end{aligned} \tag{16}$$

can be considered as a coherent state depending on the real positive variable J or coherent state defined on the positive real semi axis.

In other words, the GK-CSs introduced in Ref. [4] can be obtained by applying the exponential operator $\#\exp(-\mathrm{i}\gamma A_+A_-)\#$ on the CSs depending on the real positive variable $|J\rangle$ and considering the rules of DOOT.

2. Something about the CSs depending on the real positive variable

Let us examine some properties of the CSs $|J\rangle$.

(i) These states satisfy the continuity condition in the real positive label J

$$\lim_{J'\to J}\left\||J'\rangle - |J\rangle\right\|^2 = \left[2 - \lim_{J'\to J}\left(\langle J|J'\rangle + \langle J'|J\rangle\right)\right] = 0. \tag{17}$$

This property may also be highlighted if we calculate the overlap (scalar product) of two CSs which show that the CSs $|J\rangle$ are normalizable but non-orthogonal

$$\langle J|J'\rangle = \frac{N(\sqrt{JJ'})}{\sqrt{N(J)N(J')}} \neq 0, \tag{18}$$

and then if we apply the above limit $J' \to J$.

(ii) The CSs $|J\rangle$ fulfill the resolution of the unity operator

$$\int \mathrm{d}\mu(J)|J\rangle\langle J| = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1, \tag{19}$$

where we must find the integration measure $\mathrm{d}\mu(J) = \mathrm{d}Jh(J)$ and their positive definite weight function $h(J)$.[4]

Substituting Eq. (16) in the above relation, we obtain

$$\sum_{n,n'=0}^{\infty}\frac{|n\rangle\langle n'|}{\sqrt{\rho(n)\rho(n')}}\int_0^R \mathrm{d}J\frac{h(J)}{N(J)}\left(\sqrt{J}\right)^{n+n'} = 1. \tag{20}$$

In order to have equality, we suppose *a priori* that we must have $n' = n$ in order to accomplish the completeness relation (19). This will be justified below when we demonstrate the completeness relation for the GK-CSs $|J,\gamma\rangle$. Consequently, the reduced weight function $\tilde{h}(J) = h(J)[N(J)]^{-1}$ must satisfy the following moment problem:

$$\int_0^R \mathrm{d}JJ^n\tilde{h}(J) = \rho(n), \tag{21}$$

i.e., exactly Eq. (12) for the positive constants $\rho(n)$.

Depending on the value of convergence radius R, the moment problem is called the Stieltjes moment problem (if $R \to \infty$), or the Hausdorff moment problem (if $R < \infty$).[6]

The solution of such a problem can be expressed through the Meijer's G-functions[7]

$$\tilde{h}(J) = G_{p,q}^{m,n}\left(J\left|\begin{matrix}\{a_i\}_1^n; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q\end{matrix}\right.\right), \tag{22}$$

where, for simplicity, we use the short notation for the parameters (generally, complex numbers): $\{a_i\}_1^n = \{a_1, a_2, \ldots, a_n\}$, and so on.

Finally, the integration measure becomes

$$\mathrm{d}\mu(J) = \mathrm{d}JN(J)G_{p,q}^{m,n}\left(J\left|\begin{matrix}\{a_i\}_1^n; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q\end{matrix}\right.\right), \tag{23}$$

and it provides the resolution of the unity operator.

(iii) Temporal stability means that

$$\mathrm{e}^{-\mathrm{i}Ht/\hbar}|J\rangle = |J(t)\rangle. \tag{24}$$

The proof proceeds the steps below

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}Ht/\hbar}|J\rangle &= \#\mathrm{e}^{-\mathrm{i}\omega tA_+A_-}\#|J\rangle \\ &= \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{\left(\sqrt{J}\right)^n}{\sqrt{\rho(n)}}\mathrm{e}^{-\mathrm{i}\omega te_n}|n\rangle \equiv |J(t)\rangle.\end{aligned} \tag{25}$$

It should be noted that, due to the structure of dimensionless eigenvalues e_n, in the general case the above sum cannot be narrowed. This is possible only for the case of one-dimensional harmonic oscillator (HO-1D) for which we have $e_n = n$ and, consequently, $J(t) = J\exp(-\mathrm{i}2\omega t)$ and the CSs remain stable during the time.

(iv) The last property (condition) is action identity.

Generally, the expectation value of an observable A is

$$\langle J|A|J\rangle = \frac{1}{N(J)}\sum_{n,n'=0}^{\infty}\frac{\left(\sqrt{J}\right)^{n+n'}}{\sqrt{\rho(n)\rho(n')}}\langle n|A|n'\rangle. \tag{26}$$

For $A = H$ we have $\langle n|H|n'\rangle = \hbar\omega e_n\delta_{nn'}$ and, consequently,

$$\langle J|H|J\rangle = \hbar\omega\frac{1}{N(J)}\sum_{n=0}^{\infty}\frac{J^n}{\rho(n)}e_n = \hbar\omega J, \tag{27}$$

where we consider the structure of $\rho(n)$, Eq. (2). So, in unit $\hbar = 1$, it can identify the real variable J as action variable.[5]

In conclusion, the CSs $|J\rangle$ fulfill all requirements imposed to a coherent state,[4] even if they depend on the real variable J.

3. DOOT applied to the GK-CSs

Using Eq. (7) and the DOOT rules, we can rewrite the GK-CSs (10) in an operatorial manner as

$$\begin{aligned}|J,\gamma\rangle &= \#\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\#|J\rangle\\ &= \frac{1}{\sqrt{N(J)}}\#\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\#\sum_{n=0}^{\infty}\frac{\left(\sqrt{J}\right)^n}{\rho(n)}(A_+)^n|0\rangle\\ &= \#\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\#\frac{N\left(A_+\sqrt{J}\right)}{\sqrt{N(J)}}|0\rangle,\end{aligned} \tag{28}$$

as well as their counterpart

$$\langle J,\gamma| = \langle 0|\frac{N\left(A_-\sqrt{J}\right)}{\sqrt{N(J)}}\#\mathrm{e}^{+\mathrm{i}\gamma A_+A_-}\#. \tag{29}$$

Consequently, the projector on the GK-CS $|J,\gamma\rangle$ is

$$\begin{aligned}&|J,\gamma\rangle\langle J,\gamma|\\ &= \#\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\#\frac{N\left(A_+\sqrt{J}\right)}{\sqrt{N(J)}}|0\rangle\langle 0|\frac{N\left(A_-\sqrt{J}\right)}{\sqrt{N(J)}}\#\mathrm{e}^{+\mathrm{i}\gamma A_+A_-}\#\\ &= \frac{1}{N(J)}\#\frac{\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}N\left(A_+\sqrt{J}\right)N\left(A_-\sqrt{J}\right)\mathrm{e}^{+\mathrm{i}\gamma A_+A_-}}{N(A_+A_-)}\#.\end{aligned} \tag{30}$$

If we put $J = 0$ we recover the vacuum projector (9).

Let us demonstrate the unity operator resolution using the DOOT rules. We must have

$$\int \mathrm{d}\mu(J,\gamma)|J,\gamma\rangle\langle J,\gamma| = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1. \tag{31}$$

Now, the integration measure should be structured in the following manner:

$$\begin{aligned}\mathrm{d}\mu(J,\gamma) &= \frac{\mathrm{d}\gamma}{2\pi}\mathrm{d}\mu(J)\\ &= \frac{\mathrm{d}\gamma}{2\pi}\mathrm{d}JN(J)G^{m,n}_{p,q}\left(J\left|\begin{matrix}\{a_i\}_1^n;\ \{a_i\}_{n+1}^p\\ \{b_j\}_1^m;\ \{b_j\}_{m+1}^q\end{matrix}\right.\right).\end{aligned} \tag{32}$$

Substituting Eqs. (32) and (19) into Eq. (31), we obtain

$$\begin{aligned}&\int \mathrm{d}\mu(J,\gamma)|J,\gamma\rangle\langle J,\gamma|\\ &= \#\int_0^{2\pi}\frac{\mathrm{d}\gamma}{2\pi}\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\left[\int \mathrm{d}\mu(J)|J\rangle\langle J|\right]\mathrm{e}^{+\mathrm{i}\gamma A_+A_-}\# = 1,\end{aligned} \tag{33}$$

where we consider that, generally, the angular integral with respect to variable γ is

$$\int_0^{2\pi}\frac{\mathrm{d}\gamma}{2\pi}\mathrm{e}^{-\mathrm{i}(n-n')\gamma} = \delta_{nn'}. \tag{34}$$

This result can be obtained also if we use Eq. (30), by performing the radial and angular integrals and taking into account the DOOT rules which require $n' = n$. We have

$$\begin{aligned}&\#\int \mathrm{d}\mu(J,\gamma)\frac{N\left(A_+\sqrt{J}\right)N\left(A_-\sqrt{J}\right)}{N(J)}\#\\ &= \int_0^{2\pi}\frac{\mathrm{d}\gamma}{2\pi}\sum_{n,n'=0}^{\infty}\frac{\#(A_+)^n(A_-)^{n'}\#}{\rho(n)\rho(n')}\int_0^{\infty}\mathrm{d}J\left(\sqrt{J}\right)^{n+n'}\\ &\quad\times G^{m,n}_{p,q}\left(J\left|\begin{matrix}\{a_i\}_1^n;\ \{a_i\}_{n+1}^p\\ \{b_j\}_1^m;\ \{b_j\}_{m+1}^q\end{matrix}\right.\right) = \sum_{n=0}^{\infty}\frac{\#(A_+A_-)^n\#}{\rho(n)}\\ &= \#N(A_+A_-)\#.\end{aligned} \tag{35}$$

Finally, bearing in mind the DOOT rules, the commutation of the ordered operator's product $\#(A_+A_-)^n\#$, we obtain

$$\begin{aligned}&\int \mathrm{d}\mu(J,\gamma)|J,\gamma\rangle\langle J,\gamma|\\ &= \#\frac{1}{N(A_+A_-)}\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}N(A_+A_-)\mathrm{e}^{+\mathrm{i}\gamma A_+A_-}\# = 1.\end{aligned} \tag{36}$$

The expectation value of a function which depends only on the ordered operator product A_+A_- in the representation of GK-CSs uses Eq. (13)

$$\begin{aligned}&\langle J,\gamma|\#f(A_+A_-)\#|J,\gamma\rangle\\ &= \frac{1}{N(J)}\#\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\left[\sum_{n=0}^{\infty}\frac{J^n}{\rho(n)}f(e_n)\right]\mathrm{e}^{+\mathrm{i}\gamma A_+A_-}\#\\ &= \frac{1}{N(J)}\sum_{n=0}^{\infty}\frac{J^n}{\rho(n)}f(e_n).\end{aligned} \tag{37}$$

Particularly, for $\#f(A_+A_-)\# = H = \hbar\omega\#A_+A_-\#$, we obtain the action identity

$$\begin{aligned}\langle J,\gamma|H|J,\gamma\rangle &= \hbar\omega\langle J,\gamma|\#A_+A_-\#|J,\gamma\rangle\\ &= \hbar\omega\frac{1}{N(J)}\sum_{n=0}^{\infty}\frac{J^n}{\rho(n)}e_n\end{aligned}$$

$$= \hbar\omega \frac{1}{N(J)} J \sum_{n=0}^{\infty} \frac{J^n}{\rho(n)} = \hbar\omega J. \tag{38}$$

Similarly, the temporal stability of the BG-CSs in the frame of DOOT reads

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}Ht/\hbar}|J,\gamma\rangle &= \#\mathrm{e}^{-\mathrm{i}\omega t A_+A_-}\#|J,\gamma\rangle \\ &= \frac{1}{\sqrt{N(J)}} \sum_{n=0}^{\infty} \frac{\left(\sqrt{J}\right)^n \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\rho(n)}} \mathrm{e}^{-\mathrm{i}\omega t e_n}|n\rangle \\ &= \frac{1}{\sqrt{N(J)}} \sum_{n=0}^{\infty} \frac{\left(\sqrt{J}\right)^n \mathrm{e}^{-\mathrm{i}(\gamma+\omega t)e_n}}{\sqrt{\rho(n)}}|n\rangle \\ &\equiv |J,\gamma+\omega t\rangle = |J(t). \end{aligned} \tag{39}$$

4. Mixed thermal states

In order to extend the DOOT approach to the mixed states, let us now consider a quantum system which is in the equilibrium state with a field coupled to a reservoir at temperature $T = (k_{\mathrm{B}}\beta)^{-1}$, where k_{B} is the Boltzmann constant. In this situation, the state of the examined system is a mixed state (thermal state) and the distribution function which describes a thermal state's statistics is similar to the Maxwell–Boltzmann distribution. Then, the corresponding normalized density operator characteristic for this kind of state is (we consider the non-degenerated case)

$$\begin{aligned} \rho = \frac{1}{Z(\beta)}\mathrm{e}^{-\beta H} &= \frac{1}{Z(\beta)}\#\mathrm{e}^{-\beta\hbar\omega A_+A_-}\# \\ &= \frac{1}{Z(\beta)} \sum_{n=0}^{\infty} \mathrm{e}^{-\beta\hbar\omega e_n}|n\rangle\langle n|. \end{aligned} \tag{40}$$

The evaluation of the partition function $Z(\beta)$ can be achieved if the density operator is normalized to unity

$$\mathrm{Tr}\rho = \sum_n \langle n|\rho|n\rangle = \frac{1}{Z(\beta)} \sum_{n=0}^{\infty} \mathrm{e}^{-\beta\hbar\omega e_n} = 1. \tag{41}$$

Using the DOOT rules, the density operator can be written in an operatorial manner as follows:

$$\rho = \frac{1}{Z(\beta)}\#\frac{1}{N(A_+A_-)} \sum_{n=0}^{\infty} \mathrm{e}^{-\beta\hbar\omega e_n} \frac{(A_+A_-)^n}{\rho(n)}\#. \tag{42}$$

For particular cases (quantum systems), the sum can be narrowed or compacted.

The Husimi's Q-function is defined as the diagonal elements of the density operator ρ in the CSs representation.[8] In our case, the GK-CSs $|J,\gamma\rangle$

$$\begin{aligned} Q(J) &\equiv \langle J,\gamma|\rho|J,\gamma\rangle \\ &= \frac{1}{N(J)} \sum_{n,n'=0}^{\infty} \frac{\left(\sqrt{J}\right)^{n+n'} \mathrm{e}^{-\mathrm{i}\gamma(e_n-e_{n'})}}{\sqrt{\rho(n)\rho(n')}}\langle n|\rho|n'\rangle \\ &= \frac{1}{Z(\beta)}\frac{1}{N(J)} \sum_{n=0}^{\infty} \frac{J^n}{\rho(n)} \mathrm{e}^{-\beta\hbar\omega e_n}. \end{aligned} \tag{43}$$

It is straightforward to verify that the Husimi's function is normalized to unity

$$\int \mathrm{d}\mu(J,\gamma)Q(J) = 1, \tag{44}$$

which leads to the correct expression of the partition function $Z(\beta)$

$$Z(\beta) = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta\hbar\omega e_n}. \tag{45}$$

The density operator can be expanded into a superposition of CSs projectors. This expansion is called the diagonal representation of the density operator, or Glauber–Sudarshan representation, or P representation[9,10]

$$\rho = \int \mathrm{d}\mu(J,\gamma)P(J)|J,\gamma\rangle\langle J,\gamma|, \tag{46}$$

where we must find the quasi probability distribution function $P(J)$ which cannot be regarded as a proper probability distribution function.

Substituting Eqs. (32) and (30) into Eq. (44), we have

$$\begin{aligned} \rho &= \int_0^{2\pi} \frac{\mathrm{d}\gamma}{2\pi} \int_0^{\infty} \mathrm{d}J G_{p,q}^{m,n} \left(J \left| \begin{matrix} \{a_i\}_1^n; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q \end{matrix} \right. \right) \\ &\quad \times P(J)\#\frac{\mathrm{e}^{-\mathrm{i}\gamma A_+A_-} N\left(A_+\sqrt{J}\right) N\left(A_-\sqrt{J}\right) \mathrm{e}^{+\mathrm{i}\gamma A_+A_-}}{N(A_+A_-)}\# \\ &= \#\int_0^{2\pi} \frac{\mathrm{d}\gamma}{2\pi} \mathrm{e}^{-\mathrm{i}\gamma A_+A_-} \left[\int_0^{\infty} \mathrm{d}J G_{p,q}^{m,n} \left(J \left| \begin{matrix} \{a_i\}_1^n; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q \end{matrix} \right. \right) \right. \\ &\quad \left. \times P(J) \frac{N\left(A_+\sqrt{J}\right) N\left(A_-\sqrt{J}\right)}{N(A_+A_-)} \right] \mathrm{e}^{+\mathrm{i}\gamma A_+A_-}\#, \end{aligned} \tag{47}$$

$$\begin{aligned} &\left[\int_0^{\infty} \mathrm{d}J \cdots\right] \\ &= \sum_{n,n'=0}^{\infty} \#\frac{\left(A_+\sqrt{J}\right)^n}{\rho(n)} \frac{\left(A_-\sqrt{J}\right)^{n'}}{\rho(n')}\# \\ &\quad \times \int_0^{\infty} \mathrm{d}J G_{p,q}^{m,n} \left(J \left| \begin{matrix} \{a_i\}_1^n; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q \end{matrix} \right. \right) P(J). \end{aligned} \tag{48}$$

In the last double sum, we must have $n' = n$, due to the DOOT rules, in order to have the ordered product $\#(A_+A_-)^n\#$. All terms with $n' \neq n$ will be cancelled. In addition, the result of the angular integral is 1. Consequently, we have

$$\begin{aligned} \rho &= \#\frac{1}{N(A_+A_-)} \sum_{n,n'=0}^{\infty} \frac{(A_+A_-)^n}{[\rho(n)]^2}\# \int_0^{\infty} \mathrm{d}J J^n \\ &\quad \times G_{p,q}^{m,n} \left(J \left| \begin{matrix} \{a_i\}_1^n; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q \end{matrix} \right. \right) P(J). \end{aligned} \tag{49}$$

Comparing with the expression (42) it is evident that we must have

$$\begin{aligned} &\int_0^{\infty} \mathrm{d}J J^n G_{p,q}^{m,n} \left(J \left| \begin{matrix} \{a_i\}_1^n; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q \end{matrix} \right. \right) P(J) \\ &= \frac{1}{Z(\beta)} \mathrm{e}^{-\beta\hbar\omega e_n}\rho(n). \end{aligned} \tag{50}$$

If we perform the function change

$$\tilde{P}(J) \equiv G_{p,q}^{m,n}\left(J\left|\begin{matrix}\{a_i\}_1^n;\ \{a_i\}_{n+1}^p\\ \{b_j\}_1^m;\ \{b_j\}_{m+1}^q\end{matrix}\right.\right)P(J), \tag{51}$$

we obtain the integral

$$\int_0^\infty \mathrm{d}J J^n \tilde{P}(J) = \frac{1}{Z(\beta)}\,\mathrm{e}^{-\beta\hbar\omega e_n}\rho(n). \tag{52}$$

This integral cannot be solved in a general case. The solutions are dependent on the concrete expressions of e_n and $\rho(n)$. Ultimately, all calculations are reduced to a moment problem (Stieltjes or Hausdorff), as we will see in the last section dedicated to applications.

5. Applications

Let us apply the above-considered DOOT rules to two cases: pseudoharmonic oscillator and Morse oscillator. We have chosen these cases because in the first case it is a matter of an oscillator with an infinite number of bound states (and a linear energy spectrum with respect to the main quantum number n) and the second is an anharmonic oscillator with a finite number of bound states and with an energy spectrum expressed as a polynomial of the second degree in n.

5.1. Pseudoharmonic oscillator (PHO)

Let us consider a pseudoharmonic oscillator (PHO) with mass m and angular frequency $\omega = 2\omega_0$ (the corresponding one-dimensional harmonic oscillator has an angular frequency ω_0) and equilibrium distance r_0, with the potential (for simplicity, we consider that the PHO is in fundamental rotational state, with rotational quantum number $J_{\mathrm{rot}} = 0$)[11,12]

$$V(r) = \frac{m\omega^2}{8}r_0^2\left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2. \tag{53}$$

Their energy eigenvalues are

$$E_n = 2\hbar\omega_0(n+k) - m\omega_0^2 r_0^2 \equiv \hbar\omega_0 e_n, \tag{54}$$

where k is the Bargmann index labeling the UIR of the SU(1,1) group characteristic for PHO. Consequently, we have

$$e_n = 2n + 2k - \frac{m\omega_0}{\hbar}r_0^2. \tag{55}$$

Then, the GK-CSs are[12]

$$\begin{aligned}|J,\gamma\rangle_{\mathrm{PHO}} &= \#\,\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\#|J\rangle_{\mathrm{PHO}}\\ &= \frac{1}{\sqrt{N_{\mathrm{PHO}}(J)}}\sum_{n=0}^\infty \frac{\left(\sqrt{J}\right)^n \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\rho_{\mathrm{PHO}}(n)}}|n,k\rangle\\ &= \mathrm{e}^{-\mathrm{i}\gamma\left(2k-\frac{m\omega_0}{\hbar}r_0^2\right)}\frac{1}{\sqrt{N_{\mathrm{PHO}}(J)}}\\ &\quad\times\sum_{n=0}^\infty \frac{\left(\sqrt{J}\,\mathrm{e}^{-2\mathrm{i}\gamma}\right)^n}{\sqrt{\rho_{\mathrm{PHO}}(n)}}|n,k\rangle,\end{aligned} \tag{56}$$

with

$$\rho_{\mathrm{PHO}}(n) = \prod_{j=1}^n e_j = 2^n\frac{\Gamma(n+k+1)}{\Gamma(k+1)} = 2^n(k+1)_n, \tag{57}$$

where the CSs depending on the real positive variable J are

$$|J\rangle_{\mathrm{PHO}} = \frac{1}{\sqrt{N_{\mathrm{PHO}}(J)}}\sum_{n=0}^\infty \frac{\left(\sqrt{J}\right)^n}{\sqrt{\rho_{\mathrm{PHO}}(n)}}|n,k\rangle. \tag{58}$$

Here $\Gamma(x)$ is Euler's gamma function, while $(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)}$ is the Pochhammer symbol[13] and also the normalization function, expressed as the hypergeometric function

$$N_{\mathrm{PHO}}(J) = \sum_{n=0}^\infty \frac{(1)_n}{(k+1)_n}\frac{(J/2)^n}{n!} = {}_1\mathrm{F}_1\left(1;k+1;\frac{J}{2}\right). \tag{59}$$

Using Eqs. (3) and (4), the two arbitrary partner hermitic conjugate lowering A_- and raising A_+ operators satisfy the following relations:

$$A_+|n,k\rangle = \sqrt{2(n+1+k)}|n+1,k\rangle;$$
$$A_-|n,k\rangle = \sqrt{2(n+k)}|n-1,k\rangle, \tag{60}$$
$$A_+A_-|n,k\rangle = 2(n+k)|n,k\rangle. \tag{61}$$

Consequently, the PHO Hamiltonian is

$$H_{\mathrm{PHO}} = \hbar\omega_0 A_+A_- - m\omega_0^2 r_0^2. \tag{62}$$

The following relations are valid, due to the recurrence procedure and hermitic conjugation:

$$\begin{aligned}|n,k\rangle &= \frac{1}{\sqrt{2^n\frac{\Gamma(n+k+1)}{\Gamma(k+1)}}}(A_+)^n|0,k\rangle\\ &= \frac{1}{\sqrt{\rho_{\mathrm{PHO}}(n)}}(A_+)^n|0,k\rangle,\end{aligned} \tag{63}$$

$$\begin{aligned}\langle n,k| &= \frac{1}{\sqrt{2^n\frac{\Gamma(n+k+1)}{\Gamma(k+1)}}}\langle 0,k|(A_-)^n\\ &= \frac{1}{\sqrt{\rho_{\mathrm{PHO}}(n)}}\langle 0,k|(A_-)^n.\end{aligned} \tag{64}$$

Substituting the last two equations in the completeness relation for Fock vectors and using the DOOT rules, we obtain

$$\begin{aligned}&\sum_{n=0}^\infty |n,k\rangle\langle n,k|\\ &= \#\sum_{n=0}^\infty \frac{(A_+)^n|0,k\rangle\langle 0,k|(A_-)^n}{\rho_{\mathrm{PHO}}(n)}\#\\ &= \#|0,k\rangle\langle 0,k|\sum_{n=0}^\infty \frac{(1)_n}{(k+1)_n}\frac{\left(\frac{1}{2}A_+A_-\right)^n}{n!}\#\\ &= \#|0,k\rangle\langle 0,k|{}_1\mathrm{F}_1\left(1;k+1;\frac{1}{2}A_+A_-\right)\#=1,\end{aligned} \tag{65}$$

where we obtain the DOOT ordered vacuum projector

$$|0,k\rangle\langle 0,k| = \#\frac{1}{{}_1\mathrm{F}_1\left(1;k+1;\frac{1}{2}A_+A_-\right)}\#, \tag{66}$$

as expected, according to Eq. (9).

Taking into account the above relations, the GK-CS $|J,\gamma\rangle_{\rm PHO}$ can be written in an operatorial manner as

$$|J,\gamma\rangle_{\rm PHO} = {\rm e}^{-{\rm i}\gamma\left(2k-\frac{m\omega_0}{\hbar}r_0^2\right)}\frac{1}{\sqrt{{}_1{\rm F}_1\left(1;k+1;\dfrac{J}{2}\right)}} \times\sum_{n=0}^{\infty}\frac{(1)_n}{(k+1)_n}\frac{\left(\dfrac{1}{2}A_+\sqrt{J}\,{\rm e}^{-{\rm i}\gamma}\right)^n}{n!}|0,k\rangle$$

$$= {\rm e}^{-{\rm i}\gamma\left(2k-\frac{m\omega_0}{\hbar}r_0^2\right)}\frac{1}{\sqrt{{}_1{\rm F}_1\left(1;k+1;\dfrac{J}{2}\right)}} \times{}_1{\rm F}_1\left(1;k+1;\frac{1}{2}A_+\sqrt{J}\,{\rm e}^{-{\rm i}\gamma}\right)|0,k\rangle. \quad (67)$$

Similarly, their conjugate counterpart ${}_{\rm PHO}\langle J,\gamma|$.

Consequently, the projector on the GK-CS $|J,\gamma\rangle_{\rm PHO}$ can be written, according to Eq. (30), as

$$|J,\gamma\rangle_{\rm PHO\,PHO}\langle J,\gamma| = \frac{1}{{}_1{\rm F}_1\left(1;k+1;\dfrac{J}{2}\right)}\#\frac{{\rm e}^{-{\rm i}\gamma A_+A_-}\,{}_1{\rm F}_1\left(1;k+1;\dfrac{1}{2}A_+\sqrt{J}\,{\rm e}^{-{\rm i}\gamma}\right){}_1{\rm F}_1\left(1;k+1;\dfrac{1}{2}A_-\sqrt{J}\,{\rm e}^{+{\rm i}\gamma}\right){\rm e}^{+{\rm i}\gamma A_+A_-}}{{}_1{\rm F}_1\left(1;k+1;\dfrac{1}{2}A_+A_-\right)}\#. \quad (68)$$

In order to find the integration measure ${\rm d}\mu_{\rm PHO}(J,\gamma) = \frac{{\rm d}\gamma}{2\pi}{\rm d}Jh_{\rm PHO}(J)$, we appeal to Eqs. (21), (22), and (57) and we have

$$\int_0^{\infty}{\rm d}JJ^n\tilde{h}_{\rm PHO}(J) = \rho_{\rm PHO}(n) = 2^n\frac{\Gamma(n+k+1)}{\Gamma(k+1)}, \quad (69)$$

where

$$\tilde{h}_{\rm PHO}(J) \equiv h_{\rm PHO}(J)\left[{}_1{\rm F}_1\left(1;k+1;\frac{J}{2}\right)\right]^{-1}.$$

This is a Stieltjes moment problem and the function $\tilde{h}_{\rm PHO}(J)$ may be found using the substitution $n = s-1$,[5]

$$\int_0^{\infty}{\rm d}JJ^{s-1}\tilde{h}_{\rm PHO}(J) = \frac{1}{2}\frac{1}{\Gamma(k+1)}\frac{1}{(1/2)^s}\Gamma(s+k), \quad (70)$$

which leads to the solution

$$\tilde{h}_{\rm PHO}(J) = \frac{1}{2}\frac{1}{\Gamma(k+1)}G_{0,1}^{1,0}\left(\frac{J}{2}\middle|k\right) = \frac{1}{2}\frac{1}{\Gamma(k+1)}{\rm e}^{-\frac{J}{2}}\left(\frac{J}{2}\right)^k. \quad (71)$$

Finally, the integration measure is

$${\rm d}\mu_{\rm PHO}(J,\gamma) = \frac{1}{\Gamma(k+1)}\frac{{\rm d}\gamma}{2\pi}{\rm d}\,(J/2) \times{\rm e}^{-J/2}\left(\frac{J}{2}\right)^k{}_1{\rm F}_1\left(1;k+1;\frac{J}{2}\right). \quad (72)$$

Using this last expression, as well as the projector (68), and taking into account the DOOT rules and the value of angular integration after straightforward calculations, we prove the validity of the unity operator resolution identity

$$\int{\rm d}\mu_{\rm PHO}(J,\gamma)|J,\gamma\rangle_{\rm PHO\,PHO}\langle J,\gamma| = \sum_{n=0}^{\infty}|n,k\rangle\langle n,k| = 1. \quad (73)$$

In Ref. [12] we have also demonstrated the validity of other Klauder's requirements imposed to a coherent state.[4]

With regard to the thermal states, the corresponding normalized density operator can be obtained using Eq. (42)

$$\rho_{\rm PHO} = \frac{1}{Z_{\rm PHO}(\beta)}\#\frac{1}{{}_1{\rm F}_1\left(1;k+1;\dfrac{1}{2}A_+A_-\right)} \times\sum_{n=0}^{\infty}{\rm e}^{-\beta\hbar\omega_0\epsilon_n}\frac{(A_+A_-)^n}{\rho_{\rm PHO}(n)}\# = \frac{1}{\bar{n}+1}\#\frac{{}_1{\rm F}_1\left(1;k+1;\dfrac{1}{2}A_+A_-\dfrac{\bar{n}}{\bar{n}+1}\right)}{{}_1{\rm F}_1\left(1;k+1;\dfrac{1}{2}A_+A_-\right)}\#, \quad (74)$$

where we have used the expression of the Bose–Einstein population number (i.e., the average number of quanta)

$$\bar{n} = \frac{1}{{\rm e}^{2\beta\hbar\omega_0}-1}, \quad (75)$$

as well as the expression of the partition function[12]

$$Z_{\rm PHO}(\beta) = \sum_{n=0}^{\infty}{\rm e}^{-\beta E_n} = {\rm e}^{\beta m\omega_0^2r_0^2}\left(\frac{\bar{n}}{\bar{n}+1}\right)^k(\bar{n}+1). \quad (76)$$

The Husimi's Q function is obtained using Eq. (43)

$$Q_{\rm PHO}(J) \equiv {}_{\rm PHO}\langle J,\gamma|\rho_{\rm PHO}|J,\gamma\rangle_{\rm PHO} = \frac{1}{\bar{n}+1}\frac{{}_1{\rm F}_1\left(1;k+1;\dfrac{J}{2}\dfrac{\bar{n}}{\bar{n}+1}\right)}{{}_1{\rm F}_1\left(1;k+1;\dfrac{J}{2}\right)}. \quad (77)$$

The $P_{\rm PHO}$, quasi distribution function of the diagonal expansion of PHO normalized density operator $\rho_{\rm PHO}$, i.e.,

$$\rho_{\rm PHO} = \int{\rm d}\mu_{\rm PHO}(J,\gamma)P_{\rm PHO}(J)|J,\gamma\rangle_{\rm PHO\,PHO}\langle J,\gamma| \quad (78)$$

may be obtained by inserting Eqs. (68) and (72).

The angular integral is easy to calculate, if we consider the definition of hypergeometric function ${}_1\mathrm{F}_1\,(a;b;x)$, as well as Eq. (72) and the DOOT rules,

$$\begin{aligned}&\int_0^{2\pi}\frac{\mathrm{d}\gamma}{2\pi}\#\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}{}_1\mathrm{F}_1\left(1;k+1;\frac{1}{2}A_+\sqrt{J}\mathrm{e}^{-\mathrm{i}\gamma}\right)\\&\times{}_1\mathrm{F}_1\left(1;k+1;\frac{1}{2}A_-\sqrt{J}\mathrm{e}^{+\mathrm{i}\gamma}\right)\mathrm{e}^{+\mathrm{i}\gamma A_+A_-}\#\\&=\sum_{n=0}^{\infty}\left[\frac{\Gamma(k+1)}{\Gamma(n+k+1)}\right]^2\#\left(\frac{1}{2}A_+A_-\right)^n\left(\frac{J}{2}\right)^n\#.\end{aligned}\tag{79}$$

Also, the appearing radial integral

$$\begin{aligned}&\int_0^{\infty}\mathrm{d}\left(\frac{J}{2}\right)\mathrm{e}^{-J/2}\left(\frac{J}{2}\right)^{n+k}P_{\mathrm{PHO}}(J)\\&=\frac{1}{\bar{n}+1}\frac{1}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^n}\Gamma(n+k+1)\end{aligned}\tag{80}$$

can be brought to a Stieltjes moment problem by a suitable function, respectively exponent changes $n+k=s-1$, similar to those of Eq. (68).

The final expression of P_{PHO} becomes

$$P_{\mathrm{PHO}}(J)=\frac{1}{\bar{n}+1}\left(\frac{\bar{n}+1}{\bar{n}}\right)^k\mathrm{e}^{-\frac{1}{\bar{n}}\frac{J}{2}}.\tag{81}$$

The diagonal expansion of PHO normalized density operator ρ_{PHO} is

$$\begin{aligned}\rho_{\mathrm{PHO}}&=\frac{1}{\bar{n}+1}\left(\frac{\bar{n}+1}{\bar{n}}\right)^k\\&\times\int\mathrm{d}\mu_{\mathrm{PHO}}(J,\gamma)\,\mathrm{e}^{-\frac{1}{\bar{n}}\frac{J}{2}}|J,\gamma\rangle_{\mathrm{PHO\,PHO}}\langle J,\gamma|.\end{aligned}\tag{82}$$

This result was obtained earlier, by driving along an algebraic route (see Eq. (81) of Ref. [12]).

5.2. Morse oscillator (MO)

The one-dimensional non-rotational Morse Hamiltonian has the form of[14]

$$\tilde{H}_{\mathrm{MO}}(r)=-\frac{\hbar^2}{2m_{\mathrm{red}}}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+D_{\mathrm{e}}\left[1-\mathrm{e}^{-\alpha(r-r_{\mathrm{e}})}\right]^2,\tag{83}$$

where m_{red} is the reduced mass, D_{e} the dissociation energy, r_{e} the inter nuclear distance, and α the exponential parameter connected by the fundamental vibrational frequency ω through the relation $\alpha=\omega\sqrt{m_{\mathrm{red}}/2D_{\mathrm{e}}}$.

The Schrödinger equation for the Morse oscillator (MO) $\tilde{H}_{\mathrm{MO}}|n,N\rangle=E_n|n,N\rangle$ can be solved exactly, with the vibrational energy eigenvalues being

$$E_n=\varepsilon\left(\frac{1}{2}+n-\frac{1}{N}n^2\right),\tag{84}$$

where $\varepsilon\equiv\hbar\omega N/(N+1)$, $N+1=2\sqrt{2m_{\mathrm{red}}D_{\mathrm{e}}}/\alpha\hbar$, and $n=0,1,2,\ldots,[N/2]$ is a main quantum number. Note that the constant N is not an integer. Thus, the Morse potential has maximum $[N/2]$ bound states.

Previously, using the Gazeau–Klauder formalism, Roy and Roy[15] as well as Popov[16] have constructed CSs corresponding to the Morse potential and examined some of their properties. Next we examine the same problem, but in the frame of DOOT. For this purpose, we use the shifted Hamiltonian H_{MO}, as well as the shifted dimensionless eigenvalues e_n

$$H_{\mathrm{MO}}=\tilde{H}_{\mathrm{MO}}-E_0,\quad e_n=\frac{1}{N+1}n(N-n).\tag{85}$$

In order to find the two arbitrary partner hermitic conjugate lowering A_- and raising A_+, we use Eqs. (3) and (4). These operators (which are not the group generators) satisfy the following relations:

$$A_+|n,N\rangle=\sqrt{\frac{1}{N+1}(n+1)(N-n-1)}|n+1,N\rangle,\tag{86}$$

$$A_-|n,N\rangle=\sqrt{\frac{1}{N+1}n(N-n)}|n-1,N\rangle,\tag{87}$$

$$A_+A_-|n,N\rangle=\frac{1}{N+1}n(N-n)|n,N\rangle=e_n|n,N\rangle.\tag{88}$$

In this manner, the MO shifted Hamiltonian can be written as

$$H_{\mathrm{MO}}=\hbar\omega A_+A_-.\tag{89}$$

Applying the recurrence procedure and hermitic conjugation we find that

$$\begin{aligned}|n,N\rangle&=\frac{1}{\sqrt{\Gamma(N)\dfrac{\Gamma(n+1)}{(N+1)^n\Gamma(N-n)}}}(A_+)^n|0,N\rangle\\&=\frac{1}{\sqrt{\rho_{\mathrm{MO}}(n)}}(A_+)^n|0,N\rangle,\end{aligned}\tag{90}$$

$$\begin{aligned}\langle n,N|&=\frac{1}{\sqrt{\Gamma(N)\dfrac{\Gamma(n+1)}{(N+1)^n\Gamma(N-n)}}}\langle 0,N|(A_-)^n\\&=\frac{1}{\sqrt{\rho_{\mathrm{MO}}(n)}}\langle 0,N|(A_-)^n.\end{aligned}\tag{91}$$

Using the property of the Euler gamma functions as well as the Pochhammer symbols[17]

$$\Gamma(z-n)=(-1)^n\frac{\Gamma(z)}{(1-z)_n},\tag{92}$$

the positive constants $\rho_{\mathrm{MO}}(n)$ can be transformed as follows:

$$\begin{aligned}\rho_{\mathrm{MO}}(n)&=\Gamma(N)\frac{\Gamma(n+1)}{(N+1)^n\Gamma(N-n)}\\&=\frac{(-1)^n}{(N+1)^n}(1-N)_n n!.\end{aligned}\tag{93}$$

Consequently, the completeness relation for Fock vectors and the DOOT rules lead to the expression

$$\sum_{n=0}^{[N/2]}|n,N\rangle\langle n,N|$$

$$
\begin{aligned}
&= \#\sum_{n=0}^{[N/2]} \frac{(A_+)^n |0,N\rangle\langle 0,N| (A_-)^n}{\rho_{\rm MO}(n)}\# \\
&= \#|0,N\rangle\langle 0,N| \sum_{n=0}^{[N/2]} \frac{1}{(1-N)_n} \frac{(-(N+1)A_+A_-)^n}{n!}\# \\
&= \#|0,N\rangle\langle 0,N|{}_0{\rm F}_1\left(;1-N;-(N+1)A_+A_-\right)\# \\
&= I_{[N/2]+1}, \qquad (94)
\end{aligned}
$$

where we have the DOOT ordered vacuum projector for MO

$$
|0,N\rangle\langle 0,N| = \# \frac{1}{{}_0{\rm F}_1\left(;1-N;-(N+1)A_+A_-\right)}\#. \qquad (95)
$$

The function ${}_0{\rm F}_1\left(;1-N;-(N+1)A_+A_-\right)$ is the truncated confluent hypergeometric function, in fact a confluent hypergeometric polynomial with the degree $[N/2]$. The result (95) is as expected, according to Eq. (9).

The corresponding expansion of the GK-CSs for the MO in terms of the energy eigenvectors $|n,N\rangle$ are[16]

$$
|J,\gamma\rangle_{\rm MO} = \frac{1}{\sqrt{N_{\rm MO}(J)}} \sum_{n=0}^{[N/2]} \frac{\left(\sqrt{J}\right)^n {\rm e}^{-{\rm i}\gamma e_n}}{\sqrt{\rho_{\rm MO}(n)}} |n,N\rangle, \qquad (96)
$$

where the normalization function is

$$
N_{\rm MO}(J) = {}_0{\rm F}_1\left(;1-N;-(N+1)J\right)
$$

$$
= \sum_{n=0}^{[N/2]} \frac{1}{(1-N)_n} \frac{(-(N+1)J)^n}{n!}. \qquad (97)
$$

Using Eqs. (15), (16), and (90), the MO GK-CSs can be written as

$$
|J,\gamma\rangle_{\rm MO} = \#{\rm e}^{-{\rm i}\gamma A_+A_-}\#|J\rangle_{\rm MO}, \qquad (98)
$$

where

$$
\begin{aligned}
|J\rangle_{\rm MO} &\equiv |J,\gamma=0\rangle_{\rm MO} = \frac{1}{\sqrt{N_{\rm MO}(J)}} \\
&\times \sum_{n=0}^{[N/2]} \frac{\left(\sqrt{J}\right)^n}{\sqrt{\rho_{\rm MO}(n)}} |n,N\rangle \\
&= \frac{1}{\sqrt{{}_0{\rm F}_1\left(;1-N;-(N+1)J\right)}} \\
&\times {}_0{\rm F}_1\left(;1-N;-(N+1)A_+\sqrt{J}\right)|0,N\rangle, \qquad (99)
\end{aligned}
$$

and similar for their counterpart.

Using the DOOT rules, the projector is

$$
|J,\gamma\rangle_{\rm MO\,MO}\langle J,\gamma| = \#{\rm e}^{-{\rm i}\gamma A_+A_-} \frac{{}_0{\rm F}_1\left(;1-N;-(N+1)A_+\sqrt{J}\right){}_0{\rm F}_1\left(;1-N;-(N+1)A_-\sqrt{J}\right)}{{}_0{\rm F}_1\left(;1-N;-(N+1)J\right){}_0{\rm F}_1\left(;1-N;-(N+1)A_+A_-\right)} {\rm e}^{+{\rm i}\gamma A_+A_-}\#, \qquad (100)
$$

according to Eq. (30). By putting $J=0$ we recover the vacuum projector (95).

In order to find the integration measure ${\rm d}\mu_{\rm MO}(J,\gamma) = \frac{{\rm d}\gamma}{2\pi}{\rm d}J h_{\rm MO}(J)$ from the diagonal expansion

$$
\begin{aligned}
&\int {\rm d}\mu_{\rm MO}(J,\gamma)|J,\gamma\rangle_{\rm MO\,MO}\langle J,\gamma| \\
&= \int_0^{2\pi} \frac{{\rm d}\gamma}{2\pi}\#{\rm e}^{-{\rm i}\gamma A_+A_-} \left[\int_0^\infty {\rm d}J h_{\rm MO}(J)|J\rangle_{\rm MO\,MO}\langle J|\right] {\rm e}^{+{\rm i}\gamma A_+A_-}\# \\
&= I_{[N/2]+1}, \qquad (101)
\end{aligned}
$$

according to the DOOT rules, we can take out the radial integral $[\ldots]$ (it does not contain operators) from the symbol # #.

$$
\begin{aligned}
&\sum_{n,n'=0}^{[N/2]} \frac{|n,N\rangle\langle n',N|}{\sqrt{\rho_{\rm MO}(n)}\sqrt{\rho_{\rm MO}(n')}} \\
&\times \int_0^\infty {\rm d}J \left(\sqrt{J}\right)^{n+n'} h_{\rm MO}(J) \frac{1}{{}_0{\rm F}_1\left(;1-N;-(N+1)J\right)} \\
&\times \int_0^{2\pi} \frac{{\rm d}\gamma}{2\pi}\#{\rm e}^{-{\rm i}\gamma A_+A_-}\,{\rm e}^{+{\rm i}\gamma A_+A_-}\# = I_{[N/2]+1}. \qquad (102)
\end{aligned}
$$

The result of the angular integration is unity, and in order to obtain the completeness relation (94), we must have $n'=n$. We appeal to Eqs. (21) and (93)

$$
\int_0^\infty {\rm d}J J^n \tilde{h}_{\rm MO}(J) = \rho_{\rm MO}(n) = \Gamma(N)\frac{\Gamma(n+1)}{(N+1)^n\Gamma(N-n)}, \qquad (103)
$$

where $\tilde{h}_{\rm MO}(J) = h_{\rm MO}(J)\left[{}_0{\rm F}_1\left(;1-N;-(N+1)J\right)\right]^{-1}$.

After the standard procedure (the exponent change $n = s-1$), we obtain after solving the Stieltjes moment problem

$$
\begin{aligned}
\tilde{h}_{\rm MO}(J) &= \Gamma(N)(N+1)G^{1,0}_{0,2}\left(\,(N+1)J\,|\,0;-N\right) \\
&= (N+1)^{-\frac{N}{2}+1}\Gamma(N)J^{-\frac{N}{2}}J_N\left(2\sqrt{(N+1)J}\right), \qquad (104)
\end{aligned}
$$

and finally the integration measure

$$
\begin{aligned}
&{\rm d}\mu_{\rm MO}(J,\gamma) \\
&= (N+1)^{-\frac{N}{2}+1}\Gamma(N)\frac{{\rm d}\gamma}{2\pi}{\rm d}J J^{-\frac{N}{2}} \\
&\times {}_0{\rm F}_1\left(;1-N;-(N+1)J\right)J_N\left(2\sqrt{(N+1)J}\right), \qquad (105)
\end{aligned}
$$

which is the same as that obtained in our previous paper (Eq. (32) in Ref. [16]).

The normalized density operator for a mixed (thermal) state of MO can be obtained using Eq. (42)

$$
\rho_{\rm MO} = \frac{1}{Z_{\rm MO}(\beta)}\#\frac{1}{N_{\rm MO}(A_+A_-)}\sum_{n=0}^{[N/2]} {\rm e}^{-\beta\hbar\omega e_n}\frac{(A_+A_-)^n}{\rho_{\rm MO}(n)}\#. \qquad (106)
$$

The exponential from this equation can be written as

$$
\begin{aligned}
\exp(-\beta\hbar\omega e_n) &= \exp\left(-\beta\hbar\omega\frac{N}{N+1}n + \beta\hbar\omega\frac{1}{N+1}n^2\right) \\
&\equiv \exp\left(-an+bn^2\right). \qquad (107)
\end{aligned}
$$

Due to the fact that the Morse vibrational energy E_n is a polynomial of degree two in main quantum number n, in order to evaluate the density operator in a closed manner, we must apply another method. Taking into account that in the most cases (e.g., for diatomic molecules)

$$b \equiv \beta\hbar\omega\frac{1}{N+1} \ll a \equiv \beta\hbar\omega\frac{N}{N+1},$$

we can apply our previously introduced ansatz[16] and expand the exponential $\exp(-\beta\hbar\omega e_n)$ in the following manner:

$$\begin{aligned}\mathrm{e}^{-\beta\hbar\omega e_n} &\equiv \mathrm{e}^{-an+bn^2} = \mathrm{e}^{-an}\sum_{k=0}^{\infty}\frac{b^k}{k!}n^{2k} = \sum_{k=0}^{\infty}\frac{b^k}{k!}n^{2k}\left(\mathrm{e}^{-an}\right)\\ &= \sum_{k=0}^{\infty}\frac{b^k}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^{2k}\left(\mathrm{e}^{-an}\right) = \mathrm{e}^{b\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^2}\left(\mathrm{e}^{-an}\right). \quad (108)\end{aligned}$$

Firstly, we apply this ansatz in order to calculate the partition function $Z_{\mathrm{MO}}(\beta)$

$$\begin{aligned}Z_{\mathrm{MO}}(\beta) &= \sum_{n=0}^{[N/2]}\mathrm{e}^{-\beta\hbar\omega e_n} \equiv \sum_{n=0}^{[N/2]}\mathrm{e}^{-an+bn^2}\\ &= \mathrm{e}^{b\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^2}\sum_{n=0}^{[N/2]}\left(\mathrm{e}^{-a}\right)^n\\ &= \mathrm{e}^{b\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^2}\left[\frac{1-\mathrm{e}^{-a([N/2]+1)}}{1-\mathrm{e}^{-a}}\right], \quad (109)\end{aligned}$$

which is suitable for the numerical calculations.

In a similar manner we can transform also the density operator (106) into

$$\begin{aligned}\rho_{\mathrm{MO}} &= \frac{1}{Z_{\mathrm{MO}}(\beta)}\#\frac{1}{N_{\mathrm{MO}}(A_+A_-)}\mathrm{e}^{b\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^2}\sum_{n=0}^{[N/2]}\frac{\left(\mathrm{e}^{-a}A_+A_-\right)^n}{\rho_{\mathrm{MO}}(n)}\#\\ &= \frac{1}{Z_{\mathrm{MO}}(\beta)}\#\frac{1}{{}_0\mathrm{F}_1\left(;1-N;-(N+1)A_+A_-\right)}\mathrm{e}^{b\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^2}\\ &\quad\times{}_0\mathrm{F}_1\left(;1-N;-(N+1)\mathrm{e}^{-a}A_+A_-\right)\#. \quad (110)\end{aligned}$$

Starting from this expression, we will find the P-quasi distribution function $P_{\mathrm{MO}}(J)$ from the diagonal expansion of density operator

$$\rho_{\mathrm{MO}} = \int \mathrm{d}\mu_{\mathrm{MO}}(J,\gamma)P_{\mathrm{MO}}(J)|J,\gamma\rangle_{\mathrm{MO}\,\mathrm{MO}}\langle J,\gamma| \quad (111)$$

by equalizing the two manners of expressing the density operator, i.e., Eqs. (110) and (111).

After transforming Eq. (111) by substituting the expressions for the integration measure and GK-CSs projector, and equalizing the right-hand sides of Eqs. (110) and (111), it is obvious that we must have the following equality:

$$\begin{aligned}&\frac{1}{Z_{\mathrm{MO}}(\beta)}\#\frac{1}{{}_0\mathrm{F}_1\left(;1-N;-(N+1)A_+A_-\right)}\mathrm{e}^{b\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^2}\\ &\times\sum_{n=0}^{[N/2]}\frac{\left(\mathrm{e}^{-a}A_+A_-\right)^n}{\rho_{\mathrm{MO}}(n)}\#\\ &= (N+1)^{-\frac{N}{2}+1}\Gamma(N)\\ &\quad\times\int_0^{2\pi}\frac{\mathrm{d}\gamma}{2\pi}\#\mathrm{e}^{-\mathrm{i}\gamma A_+A_-}\left[\mathrm{Int}_J\right]\mathrm{e}^{+\mathrm{i}\gamma A_+A_-}\#, \quad (112)\end{aligned}$$

where $[\mathrm{Int}_J]$ is the radial integral

$$\begin{aligned}\left[\mathrm{Int}_J\right] \equiv{}& \int_0^{\infty}\mathrm{d}J J^{-\frac{N}{2}}J_N\left(2\sqrt{(N+1)J}\right)P_{\mathrm{MO}}(J)\\ &\times\#{}_0\mathrm{F}_1\left(;1-N;-(N+1)A_+\sqrt{J}\right)\\ &\times{}_0\mathrm{F}_1\left(;1-N;-(N+1)A_-\sqrt{J}\right)\#. \quad (113)\end{aligned}$$

After developing the hypergeometric polynomials and using the DOOT rules, we see that in the products of raising and lowering operators we must have $\#(A_+)^n(A_-)^{n'}\# = \#(A_+A_-)^n\#\delta_{nn'}$, and the terms with $n' \neq n$ are cancelled.

By inspecting the right-hand side of Eq. (112) it is evident that the P-quasi distribution function $P_{\mathrm{MO}}(J)$ must have the following structure:

$$P_{\mathrm{MO}}(J) = \frac{1}{Z_{\mathrm{MO}}(\beta)}\mathrm{e}^{b\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^2}P_{\mathrm{MO}}(J;a). \quad (114)$$

Consequently, we have to solve the following Stieltjes moment problem:

$$\begin{aligned}&\int_0^{\infty}\mathrm{d}J J^{-\frac{N}{2}+n}\tilde{P}_{\mathrm{MO}}(J;a)\\ &= \frac{1}{(N+1)^{-\frac{N}{2}+1}}\frac{1}{\left[(N+1)\mathrm{e}^a\right]^n}\frac{\Gamma(n+1)}{\Gamma(N-n)}, \quad (115)\end{aligned}$$

where $\tilde{P}_{\mathrm{MO}}(J;a) \equiv P_{\mathrm{MO}}(J;a)\left[J_N\left(2\sqrt{(N+1)J}\right)\right]^{-1}$. After the substitution $-\frac{N}{2}+n = s-1$ we get the following result:

$$\begin{aligned}\tilde{P}_{\mathrm{MO}}(J;a) &= \left(\mathrm{e}^a\right)^{-\frac{N}{2}+1}G^{1,0}_{0,2}\left((N+1)\mathrm{e}^aJ\middle|\,\tfrac{N}{2};\,-\tfrac{N}{2}\right)\\ &= \left(\mathrm{e}^a\right)^{-\frac{N}{2}+1}J_N\left(2\sqrt{(N+1)\mathrm{e}^aJ}\right). \quad (116)\end{aligned}$$

Finally, the P-quasi distribution function $P_{\mathrm{MO}}(J)$ becomes

$$\begin{aligned}&P_{\mathrm{MO}}(J)\\ &= \frac{1}{Z_{\mathrm{MO}}(\beta)}\mathrm{e}^{b\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^2}\left[\left(\mathrm{e}^a\right)^{-\frac{N}{2}+1}\frac{J_N\left(2\sqrt{(N+1)\mathrm{e}^aJ}\right)}{J_N\left(2\sqrt{(N+1)J}\right)}\right], \quad (117)\end{aligned}$$

which is identical to Eq. (60) in Ref. [16].

With these elements the properties of the GK-CSs of MO can be completely examined.

6. Concluding remarks

In the paper we have revisited the properties of the Gazeau–Klauder coherent states (GK-CSs) using a new operator ordering approach, the diagonal ordering operation technique (DOOT), introduced in our previous paper. In essence, this approach consists of applying the ordered product $\#A_+A_-\#$ (or the operatorial functions of this ordered product) to different entities connected with the coherent states (in

the present case the GK-CSs) in order to simplify respective algebraic calculations. The obtained results are the same as those obtained by using pure algebraic methods, which confirms the validity of the DOOT method.

We have defined two arbitrary partner hermitic conjugates densely defined and dimensionless lowering A_- and raising A_+operators (which, generally, are not identical to the group generators of the corresponding quantum group), the only restriction being that they are connected with the Hamiltonian of the examined system by the relation $H = \hbar\omega A_+ A_-$.

Regarding the GK-CSs introduced in Ref. [4], they can be obtained by applying the DOOT ordered exponential operator $\#\exp(-\mathrm{i}\gamma A_+ A_-)\#$ on the CSs which depend on the real positive variable $|J\rangle$ and by considering the rules of DOOT.

In our opinion, the main conclusions of the present paper are as follows.

i) By using the new approach DOOT, we implicitly prove once again the validity and consistency of the elegant Gazeau and Klauder construction for coherent states of some quantum systems with a known discrete dimensionless spectrum e_n.

ii) The applicability of the DOOT was extended also to the GK-CSs, as well as to the other (Barut–Girardello) kind of coherent states for many kinds of simple physical systems (one-dimensional quantum oscillator (HO-1D), pseudo-harmonic oscillator (PHO), Morse oscillator, infinite quantum well).[18,19]

iii) DOOT facilitates multiple calculations compared to purely algebraic methods.

iv) The present results, which are identical to those obtained by using other purely algebraic methods, are demonstrated that the DOOT may be applied to the systems with infinite as well as finite number of bound energy levels.

v) Moreover, the DOOT calculations are a suitable generalization of the integration within an ordered product (IWOP) technique for Bose operators, referring to the CSs of the HO-1D, elaborated by Hong-Yi Fan and coauthors,[3] in order to be applied to other kinds of operators, respectively coherent states. In this manner, the validity of Fan's pioneering ideas are demonstrated once again.

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