Degree distribution of random birth-and-death network with network size decline*

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In this paper, we provide a general method to obtain the exact solutions of the degree distributions for random birth-and-death network (RBDN) with network size decline. First, by stochastic process rules, the steady state transformation equations and steady state degree distribution equations are given in the case of $m \geq 3$ and $0 < p < 1/2$, then the average degree of network with n nodes is introduced to calculate the degree distributions. Specifically, taking $m = 3$ for example, we explain the detailed solving process, in which computer simulation is used to verify our degree distribution solutions. In addition, the tail characteristics of the degree distribution are discussed. Our findings suggest that the degree distributions will exhibit Poisson tail property for the declining RBDN.

Keywords: random birth-and-death network (RBDN), Markov chain, generating function, degree distribution

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1. Introduction

In the real world, there exist many birth-and-death networks such as the World Wide Web,[1–3] the communication networks,[4–6] the friend relationship networks,[7–10] and the food chain networks,[11–13] in which nodes may enter or exit at any time. For these kinds of evolving networks, the degree distribution is always one of the most important statistical properties.[14–31] Several methods have been proposed to calculate their degree distributions like the first-order partial-differential equation method,[14] the mean-field approach,[15,16] the rate equation approach,[17–20,30] and the master-equation approach.[31] While these approaches aim at the networks whose sizes keep growing or remain unchanged at each time step, Zhang *et al.* put forward the stochastic-process-rules (SPR)-based Markov chain method to solve the degree distributions of evolving networks in which the network size may increase or decrease at each time step.[21]

Furthermore, a random birth-and-death network model (RBDN) is considered,[27] in which at each time step, a new node is added into the network with probability p $(0 < p < 1)$ and connected with m old nodes uniformly, or an existing node is deleted from the network with the probability $q = 1 - p$. For $0 < p < 1/2$, since m is a critical parameter for the degree distributions, solution methods may vary for different values of m. Zhang *et al*. only considered special cases, i.e., $m = 1$ and 2.[27] In this paper, a general approach is proposed for solving the degree distributions of RBDN with network size decline in the case of $m \geq 3$. Taking $m = 3$ for example, we provide the exact solutions of the degree distributions for RBDN. Our findings indicate that the tail of the degree distribution for the declining RBDN is subject to Poisson tail.

The rest of this paper is structured as follows: Section 2 gives the RBDN model with $0 < p < 1/2$ and its steady-state equations; Section 3 provides the method of solving degree distribution of RBDN with $0 < p < 1/2$; Section 4 further discusses the tail characteristics of the degree distribution; Section 5 concludes the paper.

2. Steady state equations of RBDN with network size decline

2.1. RBDN model

Consider the RBDN model:[27]

(i) The initial network is a complete graph with $m + 1$ nodes, where m is a positive integer and $m \geq 3$;

(ii) At each time step t $(t > 0)$, for the network size n, there exist three cases. For each case, its corresponding evolving role is as follows:

a) In the case of $n \geq m$: at time $t + 1$, we add a new node to the network with probability p $(0 < p < 1/2)$ and connect it with m old nodes uniformly, or randomly delete a node from the network with probability $q = 1 - p$;

b) In the case of $1 < n < m$: at time $t + 1$, we add a new node to the network with probability p $(0 < p < 1/2)$ and connect it with all old nodes, or randomly delete a node from the network with probability $q = 1 - p$;

c) In the case of $n = 1$: at time $t + 1$, we add a new node to the network with probability p $(0 < p < 1/2)$ and connect it to the old node, or keep the network unchanged with probability $q = 1 - p$.

By the above evolving roles, the network size of the RBDN is always fluctuating in the evolving process. In par-

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ticular, in the limit $t \to +\infty$, the distribution of the network size follows geometric distributions.[27]

2.2. Steady state transformation equations

Using SPR,[21] we use (n,k) to describe the state of node v, where n is the number of nodes in the network that contains v, and k is the degree of node v. Let $NK(t)$ denote the state of node v at time t, the stochastic process $\{NK(t), t \geq 0\}$ is an ergodic aperiodic homogeneous Markov chain with the state space $E = \{(n,k), n \geq 1, 0 \leq k \leq n-1\}$.

Let $\tilde{P}(t)$ be the probability distribution of $NK(t)$, i.e.,

$$\tilde{P}_{(n,k)}(t) = P\{NK(t) = (n,k)\}. \tag{1}$$

The state transformation equation is as follows (see Appendix A for the details):

$$\tilde{P}(t+1) = \tilde{P}(t) \cdot \boldsymbol{P}, \tag{2}$$

where $\boldsymbol{P}$ is the one-step transition probability matrix. Let

$$\Pi_{(i,k)} = \lim_{t \to +\infty} P\{NK(t) = (i,k)\}. \tag{3}$$

Taking the limit of Eq. (2) as $t \to +\infty$, the steady state transformation equations can be obtained

$$\begin{cases} \Pi_{(1,0)} = q\Pi_{(1,0)} + q\Pi_{(2,0)} + q\Pi_{(2,1)}, \\ 2\Pi_{(2,0)} = 2q\Pi_{(3,0)} + q\Pi_{(3,1)}, \\ \quad \vdots \\ (m+1)\Pi_{(m+1,0)} = (m+1)q\Pi_{(m+2,0)} + q\Pi_{(m+2,1)}, \\ (m+2)\Pi_{(m+2,0)} = (m+2)q\Pi_{(m+3,0)} \\ \qquad + q\Pi_{(m+3,1)} + p\Pi_{(m+1,0)}, \\ \quad \vdots \\ n\Pi_{(n,0)} = nq\Pi_{(n+1,0)} + q\Pi_{(n+1,1)} + (n-m-1)p\Pi_{(n-1,0)}, \\ \quad \vdots \end{cases} \tag{4}$$

$$\begin{cases} 2\Pi_{(2,1)} = q\Pi_{(3,1)} + 2q\Pi_{(3,2)} + p\Pi_{(1,0)} + p\Pi_{(1,0)}, \\ 3\Pi_{(3,1)} = 2q\Pi_{(4,1)} + 2q\Pi_{(4,2)} + 2p\Pi_{(2,0)}, \\ \quad \vdots \\ (m+1)\Pi_{(m+1,1)} = mq\Pi_{(m+2,1)} + 2q\Pi_{(m+2,2)} + mp\Pi_{(m,0)}, \\ (m+2)\Pi_{(m+2,1)} = (m+1)q\Pi_{(m+3,1)} + 2q\Pi_{(m+3,2)} \\ \qquad + mp\Pi_{(m+1,0)} + p\Pi_{(m+1,1)}, \\ \quad \vdots \\ n\tilde{\Pi}_{(n,1)} = (n-1)q\Pi_{(n+1,1)} + 2q\Pi_{(n+1,2)} mp\Pi_{(n-1,0)} \\ \qquad + (n-m-1)p\Pi_{(n-1,1)}, \\ \quad \vdots . \end{cases} \tag{5}$$

$$\begin{cases} m\Pi_{(m,m-1)} = q\Pi_{(m+1,m-1)} + mq\Pi_{(m+1,m)} + (m-1)p\Pi_{(m-1,m-2)} + p\left[\sum_{i=0}^{m-2} \Pi_{(m-1,i)}\right], \\ (m+1)\Pi_{(m+1,m-1)} = 2q\Pi_{(m+2,m-1)} + mq\Pi_{(m+2,m)} + mp\Pi_{(m,m-2)}, \\ (m+2)\Pi_{(m+2,m-1)} = 3q\Pi_{(m+3,m-1)} + mq\Pi_{(m+3,m)} + mp\Pi_{(m+1,m-2)} + (m+2-m-1)p\Pi_{(m+1,m-1)}, \\ \quad \vdots \\ n\Pi_{(n,m-1)} = (n-m+1)q\Pi_{(n+1,m-1)} + mq\Pi_{(n+1,m)} + mp\Pi_{(n-1,m-2)} + (n-m-1)p\Pi_{(n-1,m-1)}, \\ \quad \vdots \end{cases} \tag{6}$$

$$\begin{cases} (m+1)\Pi_{(m+1,m)} = q\Pi_{(m+2,m)} + (m+1)q\Pi_{(m+2,m+1)} + mp\Pi_{(m,m-1)} + p\left[\sum_{i=0}^{m-1} \Pi_{(m,i)}\right], \\ (m+2)\Pi_{(m+2,m)} = 2q\Pi_{(m+3,m)} + (m+1)q\Pi_{(m+3,m+1)} + mp\Pi_{(m+1,m-1)} + p\Pi_{(m+1,m)} + p\left[\sum_{i=0}^{m} \Pi_{(m+1,i)}\right], \\ \quad \vdots \\ n\Pi_{(n,m)} = (n-m)q\Pi_{(n+1,m)} + (m+1)q\Pi_{(n+1,m+1)} + mp\Pi_{(n-1,m-1)} + (n-m-1)p\Pi_{(n-1,m)} + p\left[\sum_{i=0}^{n-2} \Pi_{(n-1,i)}\right], \\ \quad \vdots . \end{cases} \tag{7}$$

For $r \geq m+1$, we have

$$\begin{cases} (r+1)\Pi_{(r+1,r)} = q\Pi_{(r+2,r)} + (r+1)q\Pi_{(r+2,r+1)} + mp\Pi_{(r,r-1)}, \\ (r+2)\Pi_{(r+2,r)} = 2q\Pi_{(r+3,r)} + (r+1)q\Pi_{(r+3,r+1)} + mp\Pi_{(r+1,r-1)} + (r+2-m-1)p\Pi_{(r+1,r)}, \\ \quad \vdots \\ n\Pi_{(n,r)} = (n-r)q\Pi_{(n+1,r)} + (r+1)q\Pi_{(n+1,r+1)} + mp\Pi_{(n-1,r-1)} + (n-m-1)p\Pi_{(n-1,r)}, \\ \quad \vdots . \end{cases} \tag{8}$$

2.3. Steady state degree distribution equations

Let K be the steady state degree distribution,[15,30,31] and $\Pi(k)$ be the probability distribution of K, then we will have

$$\Pi(k) = P\{K = k\} = \lim_{t \to +\infty} \sum_{i \geq k+1} \tilde{P}_{(i,k)}(t) = \sum_{i \geq k+1} \Pi_{(i,k)}. \tag{9}$$

Combining Eq. (9) and steady state transformation equations (4)–(8), we can obtain the steady state degree distribution equations as follows:

$$\begin{cases} (q+mp)\,\Pi(0) = q\Pi(1) + q\Pi_{(1,0)} + \sum_{i=1}^{m-1}(m-i)\,p\Pi_{(i,0)}, \\ (2q+mp)\,\Pi(1) = 2q\Pi(2) + mp\Pi(0) + p\Pi_{(1,0)} - \sum_{i=1}^{m-1}(m-i)\,p\Pi_{(i,0)} + \sum_{i=2}^{m-1}(m-i)\,p\Pi_{(i,1)}, \\ \qquad\vdots \\ (mq+mp)\,\Pi(m-1) = mq\Pi(m) + mp\Pi(m-2) + p\sum_{i=0}^{m-3}\Pi_{(m-1,i)}, \\ [(m+1)\,q+mp]\,\Pi(m) = (m+1)\,q\Pi(m+1) + mp\Pi(m-1) + p - p\sum_{i=1}^{m-1}\Pi_N(i), \\ [(m+2)\,q+mp]\,\Pi(m+1) = (m+2)\,q\Pi(m+2) + mp\Pi(m), \\ \qquad\vdots \\ [(r+1)\,q+mp]\,\Pi(r) = (r+1)\,q\Pi(r+1) + mp\Pi(r-1), \\ \qquad\vdots . \end{cases} \tag{10}$$

3. Degree distribution of RBDN with network size decline

Let $N(t)$ be the number of nodes in RBDN at time t, $N(0) = m+1$, N the steady state network size, and $\Pi_N(n)$ the probability distribution of N, then we will have

$$\begin{aligned} \Pi_N(n) &= P\{N=n\} = \lim_{t\to+\infty} P\{N(t)=n\} \\ &= \sum_{k=0}^{n-1}\Pi_{(n,k)}, \quad n \geq 1. \end{aligned} \tag{11}$$

Considering the fact that $\{N(t), t \geq 0\}$ is a one-dimensional (1D) random walk with a left bound 1, we have

$$\Pi_N(n) = \frac{q-p}{q}\left(\frac{p}{q}\right)^{n-1}, \quad n \geq 1. \tag{12}$$

As shown in Eq. (10), we may employ the probability generating function method to obtain the exact solution of the degree distribution,[27] in which, $\Pi_{(i,k)}$ is a critical parameter. In the case of m=1, 2, as a special case, $\Pi_{(1,0)}$ can be obtained directly as follows:

$$\Pi_{(1,0)} = \Pi_N(1) = \frac{q-p}{q}. \tag{13}$$

However, in the case of $m \geq 3$, as a general case, $\Pi_{(i,k)}$ is much more difficult to obtain. Thus in the following subsection, we focus on the calculation of $\Pi_{(i,k)}$.

3.1. Calculation of $\Pi_{(i,k)}$

To obtain $\Pi_{(i,k)}$, $\tilde{e}_n$ is introduced, i.e.

$$\tilde{e}_n = \sum_{k=0}^{n-1} k \cdot \Pi_{(n,k)}. \tag{14}$$

Since

$$E[K] = \sum_k k\Pi(k) = \sum_k \sum_{i\geq k+1} k\Pi_{(i,k)} = \sum_{i=1}^{+\infty}\sum_{k=0}^{i-1} k\Pi_{(i,k)} = \sum_{i=1}^{+\infty}\tilde{e}_i, \tag{15}$$

$\tilde{e}_n$ represents the contribution of the network with n nodes to the average degree $E[K]$. From Eq. (14) and the steady state transformation equations (4)–(8), we can obtain the equations of $\tilde{e}_i$ $(i = 1, 2, \ldots)$ as follows:

$$\begin{cases} n\tilde{e}_n = (n-1)\,p\tilde{e}_{n-1} + (n-1)\,q\tilde{e}_{n+1} \\ \qquad\qquad + 2\,(n-1)\,p\Pi_N(n-1), \quad 2 \leq n \leq m, \\ n\tilde{e}_n = (n-1)\,p\tilde{e}_{n-1} + (n-1)\,q\tilde{e}_{n+1} \\ \qquad\qquad + 2mp\Pi_N(n-1), \quad n \geq m+1, \end{cases} \tag{16}$$

where

$$\tilde{e}_1 = 0. \tag{17}$$

To sum up the first n $(n > m)$ items in Eq. (16), we can obtain

$$\begin{aligned} 2q\sum_{i=2}^{n+1}\tilde{e}_i &= -(n+1)\,p\tilde{e}_{n+1} + nq\tilde{e}_{n+2} + 2p\sum_{i=1}^{m-1} i\Pi_N(i) \\ &\quad + 2mp\sum_{i=m}^{n}\Pi_N(i). \end{aligned} \tag{18}$$

Note

$$\tilde{e}_i \geq 0 \quad \text{and} \quad \sum_{i=2}^{+\infty}\tilde{e}_i < 2m, \tag{19}$$

then we will have

$$\lim_{n\to+\infty} n\tilde{e}_n = 0. \tag{20}$$

From Eq. (18), we have

$$\begin{aligned} \sum_{i=2}^{+\infty}\tilde{e}_i &= \frac{p}{q}\left[\sum_{i=1}^{m-1} i\Pi_N(i) + m\sum_{i=m}^{+\infty}\Pi_N(i)\right] \\ &= \frac{p}{q}\left[m - \frac{q-p}{q}\sum_{i=1}^{m-1}(m-i)\left(\frac{p}{q}\right)^{i-1}\right]. \end{aligned} \tag{21}$$

Then the generation function for Eq. (16) can be rewritten as

$$T(x)=\sum_{i=0}^{+\infty}\tilde{e}_{2+i}x^i, \tag{22}$$

satisfying

$$\begin{cases} T'(x)=\dfrac{2(1-px)}{(1-x)(q-px)}T(x) \\ \qquad -\dfrac{2p(q-p)}{q^2}\dfrac{\left(q^2+pqx+p^2x^2\right)}{(1-x)(q-px)^2}, \\ T(1)=\displaystyle\sum_{i=2}^{+\infty}\tilde{e}_i=\frac{p}{q}\left[m-\frac{q-p}{q}\sum_{i=1}^{m-1}(m-i)\left(\frac{p}{q}\right)^{i-1}\right]. \end{cases} \tag{23}$$

Solving the differential equation (23), $\tilde{e}_i$ $(i\geq 2)$ can be obtained. Combining Eqs. (14) and (4)–(8), we have

$$\begin{cases} \Pi_{(2,1)}=\tilde{e}_2, \\ \Pi_{(3,1)}+2\Pi_{(3,2)}=\tilde{e}_3, \\ \qquad\vdots \\ \Pi_{(2,0)}+\Pi_{(2,1)}=\Pi_N(2), \\ \Pi_{(3,0)}+\Pi_{(3,1)}+\Pi_{(3,2)}=\Pi_N(3), \\ \qquad\vdots \\ 2q\Pi_{(3,0)}+q\Pi_{(3,1)}=2\Pi_{(2,0)}, \\ q\Pi_{(3,1)}+2q\Pi_{(3,2)}=2\Pi_{(2,1)}-p\Pi_{(1,0)}-p\Pi_{(1,0)}, \\ \qquad\vdots. \end{cases} \tag{24}$$

Then $\Pi_{(i,k)}$ can be solved by Eq. (24).

3.2. Exact solutions of the degree distributions for $m=3$, $0<p<1/2$

Once $\Pi_{(i,k)}$ is obtained, the probability generating function approach can be employed for Eq. (10) to obtain the steady state degree distribution $\Pi(k)$. In this section, taking $m=3$ for example, we explain how this approach is used. From Eq. (10), we can obtain the degree distribution equations of RBDN with $0<p<1/2$, $m=3$ as follows:

$$\begin{cases} (q+3p)\Pi(0)=q\Pi(1)+(2p+q)\Pi_{(1,0)}+p\Pi_{(2,0)}, \\ (2q+3p)\Pi(1)=2q\Pi(2)+3p\Pi(0)-p\Pi_{(1,0)} \\ \qquad -p\Pi_{(2,0)}+p\Pi_{(2,1)}, \\ (3q+3p)\Pi(2)=3q\Pi(3)+3p\Pi(1)+p\Pi_{(2,0)}, \\ (4q+3p)\Pi(3)=4q\Pi(4)+3p\Pi(2)+p-p\Pi_{\rm N}(1) \\ \qquad -p\Pi_{\rm N}(2), \\ (5q+3p)\Pi(4)=5q\Pi(5)+3p\Pi(3), \\ \qquad\vdots \\ [(r+1)q+3p]\Pi(r)=(r+1)q\Pi(r+1)+3p\Pi(r-1), \\ \qquad\vdots. \end{cases} \tag{25}$$

From Eq. (25) we may find that $\Pi_{(2,0)}$ and $\Pi_{(2,1)}$ are needed before we solve the degree distribution. In the case of $m=3$, equation (16) can be rewritten as

$$\begin{cases} n\tilde{e}_n=(n-1)p\tilde{e}_{n-1}+(n-1)q\tilde{e}_{n+1} \\ \qquad +2(n-1)p\Pi_N(n-1), \quad 2\leq n\leq 3, \\ n\tilde{e}_n=(n-1)p\tilde{e}_{n-1}+(n-1)q\tilde{e}_{n+1} \\ \qquad +6p\Pi_N(n-1), \quad n\geq 4. \end{cases} \tag{26}$$

Introducing the following generating function:

$$T(x)=\sum_{i=2}^{+\infty}\tilde{e}_i x^{i-2} \tag{27}$$

and combining Eq. (26), we have

$$\begin{cases} T'(x)=\dfrac{2(1-px)}{(1-x)(q-px)}T(x) \\ \qquad -\dfrac{2p(q-p)}{q^2}\dfrac{\left(q^2+pqx+p^2x^2\right)}{(1-x)(q-px)^2}, \\ T(1)=\dfrac{p(1-pq)}{q^3}. \end{cases} \tag{28}$$

Solving the differential equation (28), we have

$$\begin{aligned} T(x) &= \frac{2p(q-p)}{q^2}\frac{(q-px)^{2p/(q-p)}}{(1-x)^{2q/(q-p)}} \\ &\quad\times\int_x^1\frac{\left(q^2+pqt+p^2t^2\right)(1-t)^{1/(q-p)}}{(q-pt)^{2q/(q-p)}}\mathrm{d}t. \end{aligned} \tag{29}$$

Thus

$$\begin{aligned} \tilde{e}_2 &= T(0)=\frac{2p(q-p)}{q^2}q^{2p/(q-p)} \\ &\quad\times\int_0^1\frac{\left(q^2+pqt+p^2t^2\right)(1-t)^{1/(q-p)}}{(q-pt)^{2q/(q-p)}}\mathrm{d}t. \end{aligned} \tag{30}$$

Let

$$\frac{1-t}{q-pt}=y, \tag{31}$$

then

$$t=\frac{1-qy}{1-py}, \quad \mathrm{d}t=-\frac{q-p}{(1-py)^2}\mathrm{d}y. \tag{32}$$

So, we have

$$\begin{aligned} \tilde{e}_2 &= \frac{2p(q-p)^2}{q^2}q^{2p/(q-p)}\left[\int_0^{1/q}\frac{p^2(q-p)y^{2q/(q-p)+1}}{(1-py)^3}\mathrm{d}y\right. \\ &\quad-\int_0^{1/q}\frac{p(1+p)y^{2q/(q-p)}}{(1-py)^2}\mathrm{d}y \\ &\quad\left.+\int_0^{1/q}\frac{1-pq}{q-p}\frac{y^{1/(q-p)}}{1-py}\mathrm{d}y\right] \\ &= \frac{p(q-p)}{q^2}q^{2p/(q-p)}\left[2\left(1+2q+3q^2\right)\right. \\ &\quad\left.\times\int_0^{1/q}\frac{y^{1/(q-p)}}{1-py}\mathrm{d}y-(2+3q)q^{-1/(q-p)}\right]. \end{aligned} \tag{33}$$

For

$$\begin{cases} \Pi_{(2,1)}=\tilde{e}_2, \\ \Pi_{(2,0)}+\Pi_{(2,1)}=\Pi_N(2), \end{cases} \tag{34}$$

Rewriting Eq. (34), we have

$$\begin{cases} \Pi_{(2,1)}=\tilde{e}_2, \\ \Pi_{(2,0)}=\Pi_N(2)-\tilde{e}_2. \end{cases} \tag{35}$$

To solve $\Pi(k)$, let the probability generating function be

$$G(x)=\sum_{i=0}^{+\infty}\Pi(i)\cdot x^i,\quad G(1)=\sum_{i=0}^{+\infty}\Pi(i)=1. \tag{36}$$

According to Eqs. (25) and (35), we can obtain

$$\begin{aligned}G'(x)=&\frac{3px-q-3p}{q(x-1)}G(x)+\frac{p(1-\Pi_N(1)-\Pi_N(2))}{q(x-1)}x^3\\&+\frac{p\Pi_{(2,0)}}{q(x-1)}x^2+\frac{p\left(\Pi_{(2,1)}-\Pi_{(2,0)}-\Pi_N(1)\right)}{q(x-1)}x\\&+\frac{\Pi_N(1)+p\Pi_N(1)+p\Pi_{(2,0)}}{q(x-1)}.\end{aligned} \tag{37}$$

Solving Eq. (37), we can obtain

$$\begin{aligned}G(x)=&\frac{e^{3px/q}}{1-x}\int_x^1\left[\frac{p(1-\Pi_N(1)-\Pi_N(2))}{q}t^3+\frac{p\Pi_{(2,0)}}{q}t^2\right.\\&+\frac{p\left(\Pi_{(2,1)}-\Pi_{(2,0)}-\Pi_N(1)\right)}{q}t\\&\left.+\frac{(1+p)\Pi_N(1)+p\Pi_{(2,0)}}{q}\right]e^{-3ptq}\,dt\\=&\left(\frac{6q^2-2q+6}{27pq}-\frac{2\tilde{e}_2q^2}{27p^2}\right)\\&\times e^{-3pq}\left[\sum_{k=0}^{+\infty}x^k\sum_{i=k+1}^{+\infty}\frac{1}{i!}\left(\frac{3p}{q}\right)^i\right]\\&-\left(\frac{1+2p}{9q^2}+\frac{3p-2q}{9p}\tilde{e}_2\right)\\&-\left(\frac{2p}{3q}-\frac{1}{3}\tilde{e}_2\right)x-\frac{p^2}{3q^2}x^2.\end{aligned} \tag{38}$$

So, the degree distributions of RBDN for $0<p<1/2$, $m=3$ are as follows:

$$\Pi(k)=\begin{cases}c\cdot e^{-3p/q}\sum\limits_{i=k+1}^{+\infty}\frac{1}{i!}\left(\frac{3p}{q}\right)^i-a_0, & k=0,\\ c\cdot e^{-3p/q}\sum\limits_{i=k+1}^{+\infty}\frac{1}{i!}\left(\frac{3p}{q}\right)^i-a_1, & k=1,\\ c\cdot e^{-3p/q}\sum\limits_{i=k+1}^{+\infty}\frac{1}{i!}\left(\frac{3p}{q}\right)^i-a_2, & k=2,\\ c\cdot e^{-3p/q}\sum\limits_{i=k+1}^{+\infty}\frac{1}{i!}\left(\frac{3p}{q}\right)^i, & k\geq 3,\end{cases} \tag{39}$$

where

$$\begin{cases}a_0=\dfrac{1+2p}{9q^2}+\dfrac{3p-2q}{9p}\tilde{e}_2,\\ a_1=\dfrac{2p}{3q}-\dfrac{1}{3}\tilde{e}_2,\\ a_2=\dfrac{p^2}{3q^2},\\ c=\dfrac{6q^2-2q+6}{27pq}-\dfrac{2\tilde{e}_2q^2}{27p^2},\\ \tilde{e}_2=\dfrac{p(q-p)}{q^2}q^{2p/(q-p)}\left[2\left(1+2q+3q^2\right)p^{-2q/(q-p)}\right.\\ \quad\left.\times\displaystyle\int_0^{p/q}\frac{x^{1/(q-p)}}{1-x}\,dx-(2+3q)\,q^{-1/(q-p)}\right].\end{cases} \tag{40}$$

Figure 1 illustrates the exact solutions (es) and simulation results of the degree distributions for $0<p<1/2$, $m=3$, where the horizontal and vertical ordinates denote the degree of nodes and the probability respectively. Each simulation number is an average value of 1000 simulation results for $t=10^4$. As shown in Fig. 1, the exact solutions match perfectly with the numerical solutions (ns), verifying the correctness of our exact solutions.

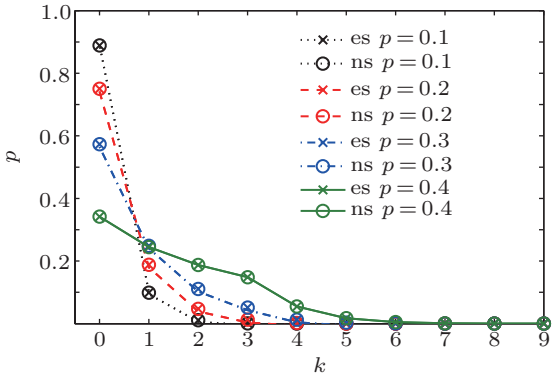


Fig. 1. (color online) Comparisons between exact solution and numerical solution ($t=10^4$) of degree distribution of RBDN for $p<1/2$, $m=3$.

4. Poisson tail

By Eq. (39), we may find that for the large k ($k\geq m+1$), the degree distribution of RBDN exhibits a backward accumulation form of Poisson distribution, that is,

$$\Pi(k)=\alpha\cdot\sum_{r=k+1}^{+\infty}\frac{\lambda^r}{r!},\quad k\geq m+1, \tag{41}$$

where α is a positive constant and $\lambda=mp/q$. For sufficiently large k, we have

$$\Pi(k)=\alpha\cdot\sum_{r=k+1}^{+\infty}\frac{\lambda^r}{r!}\sim\alpha\frac{\lambda^{k+1}}{(k+1)!}. \tag{42}$$

Thus in the case of $0<p<1/2$, the degree distribution of RBDN exhibits a Poisson tail. Here we employ the same approach as in Ref. [27] to verify this Poisson tail for the degree distribution.

Let

$$r(k)=\frac{\Pi(k)}{\Pi(k-1)}, \tag{43}$$

then using Eq. (42), we will have

$$r(k)\sim\frac{\lambda}{k};\quad \ln r(k)\sim\ln\lambda-\ln k, \tag{44}$$

that is, $r(k)$ is a line with slope -1 for large k in the two-logarithm axis diagram.

Figure 2 illustrates the Poisson tails of the degree distributions for various values of p ($0<p<1/2$). As we can see, in the case of $k\geq m+1$, the slopes of lines tend to be -1, showing the Poisson tails for the degree distributions of RBDN.

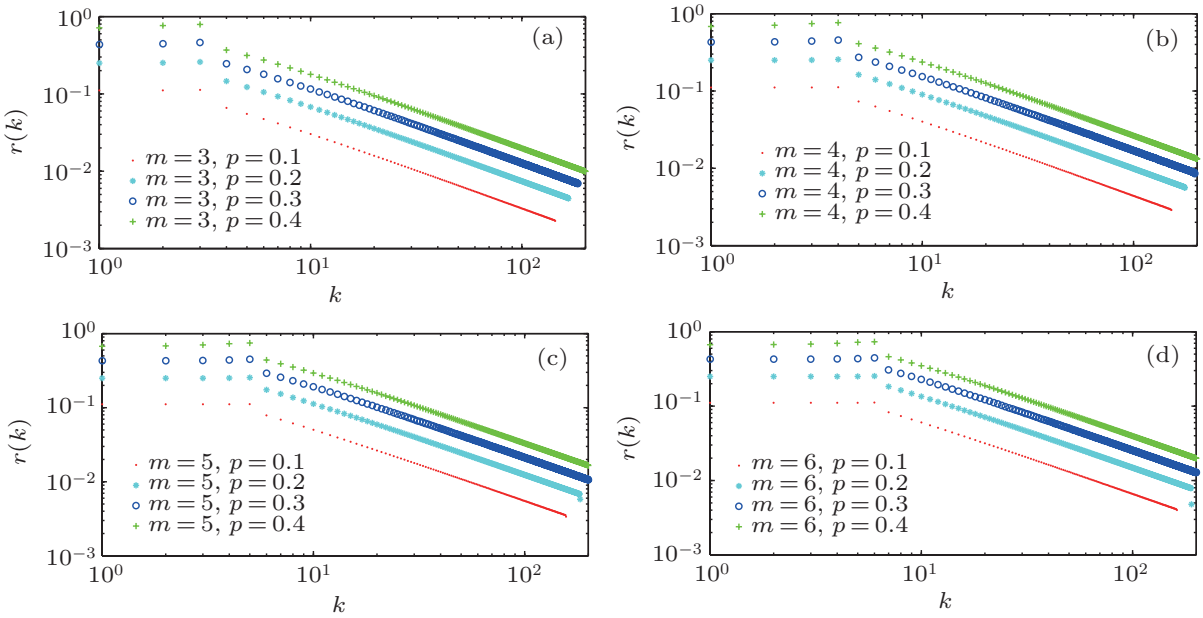


Fig. 2. (color online) Poisson tails of RBDN with $0 < p < 1/2$.

5. Conclusions

In this paper, we provide a general approach to obtaining the exact solutions of the degree distributions for RBDN in the case of $0 < p < 1/2$, $m \geq 3$. Specifically, taking $m = 3$ for example, we explain the detailed solving process and computer simulation is used to verify our degree distribution solutions. In addition, the characteristics of the degree distribution are discussed. Our findings suggest that the degree distributions will exhibit Poisson tail property for the declining RBDN.

Appendix A: State transformation equations

The state transformation equations of $NK(t)$ are as follows:

$$\begin{cases} \tilde{P}_{(1,0)}(t+1) = q\tilde{P}_{(1,0)}(t) + q\tilde{P}_{(2,0)}(t) + q\tilde{P}_{(2,1)}(t), \\ 2\tilde{P}_{(2,0)}(t+1) = 2q\tilde{P}_{(3,0)}(t) + q\tilde{P}_{(3,1)}(t), \\ \qquad\qquad \vdots \\ (m+1)\tilde{P}_{(m+1,0)}(t+1) = (m+1)q\tilde{P}_{(m+2,0)}(t) + q\tilde{P}_{(m+2,1)}(t), \\ (m+2)\tilde{P}_{(m+2,0)}(t+1) = (m+2)q\tilde{P}_{(m+3,0)}(t) + q\tilde{P}_{(m+3,1)}(t) + p\tilde{P}_{(m+1,0)}(t), \\ \qquad\qquad \vdots \\ n\tilde{P}_{(n,0)}(t+1) = nq\tilde{P}_{(n+1,0)}(t) + q\tilde{P}_{(n+1,1)}(t) + (n-m-1)p\tilde{P}_{(n-1,0)}(t), \\ \qquad\qquad \vdots . \end{cases} \tag{A1}$$

$$\begin{cases} 2\tilde{P}_{(2,1)}(t+1) = q\tilde{P}_{(3,1)}(t) + 2q\tilde{P}_{(3,2)}(t) + p\tilde{P}_{(1,0)}(t) + p\tilde{P}_{(1,0)}(t), \\ 3\tilde{P}_{(3,1)}(t+1) = 2q\tilde{P}_{(4,1)}(t) + 2q\tilde{P}_{(4,2)}(t) + 2p\tilde{P}_{(2,0)}(t), \\ \qquad\qquad \vdots \\ (m+1)\tilde{P}_{(m+1,1)}(t+1) = mq\tilde{P}_{(m+2,1)}(t) + 2q\tilde{P}_{(m+2,2)}(t) + mp\tilde{P}_{(m,0)}(t), \\ (m+2)\tilde{P}_{(m+2,1)}(t+1) = (m+1)q\tilde{P}_{(m+3,1)}(t) + 2q\tilde{P}_{(m+3,2)}(t) + mp\tilde{P}_{(m+1,0)}(t) + p\tilde{P}_{(m+1,1)}(t), \\ \qquad\qquad \vdots \\ n\tilde{P}_{(n,1)}(t+1) = (n-1)q\tilde{P}_{(n+1,1)}(t) + 2q\tilde{P}_{(n+1,2)}(t) + mp\tilde{P}_{(n-1,0)}(t) + (n-m-1)p\tilde{P}_{(n-1,1)}(t), \\ \qquad\qquad \vdots . \end{cases} \tag{A2}$$

$$\begin{cases} m\tilde{P}_{(m,m-1)}(t+1) = q\tilde{P}_{(m+1,m-1)}(t) + mq\tilde{P}_{(m+1,m)}(t) + (m-1)p\tilde{P}_{(m-1,m-2)}(t) + p\left[\sum_{i=0}^{m-2}\tilde{P}_{(m-1,i)}(t)\right], \\ (m+1)\tilde{P}_{(m+1,m-1)}(t+1) = 2q\tilde{P}_{(m+2,m-1)}(t) + mq\tilde{P}_{(m+2,m)}(t) + mp\tilde{P}_{(m,m-2)}(t), \\ (m+2)\tilde{P}_{(m+2,m-1)}(t+1) = 3q\tilde{P}_{(m+3,m-1)}(t) + mq\tilde{P}_{(m+3,m)}(t) + mp\tilde{P}_{(m+1,m-2)}(t) + (m+2-m-1)p\tilde{P}_{(m+1,m-1)}(t), \\ \vdots \\ n\tilde{P}_{(n,m-1)}(t+1) = (n-m+1)q\tilde{P}_{(n+1,m-1)}(t) + mq\tilde{P}_{(n+1,m)}(t) + mp\tilde{P}_{(n-1,m-2)}(t) + (n-m-1)p\tilde{P}_{(n-1,m-1)}(t), \\ \vdots. \end{cases} \tag{A3}$$

$$\begin{cases} (m+1)\tilde{P}_{(m+1,m)}(t+1) = q\tilde{P}_{(m+2,m)}(t) + (m+1)q\tilde{P}_{(m+2,m+1)}(t) + mp\tilde{P}_{(m,m-1)}(t) + p\left[\sum_{i=0}^{m-1}\tilde{P}_{(m,i)}(t)\right], \\ (m+2)\tilde{P}_{(m+2,m)}(t+1) = 2q\tilde{P}_{(m+3,m)}(t) + (m+1)q\tilde{P}_{(m+3,m+1)}(t) + mp\tilde{P}_{(m+1,m-1)}(t) + p\tilde{P}_{(m+1,m)}(t) + p\left[\sum_{i=0}^{m}\tilde{P}_{(m+1,i)}(t)\right], \\ \vdots \\ n\tilde{P}_{(n,m)}(t+1) = (n-m)q\tilde{P}_{(n+1,m)}(t) + (m+1)q\tilde{P}_{(n+1,m+1)}(t) + mp\tilde{P}_{(n-1,m-1)}(t) \\ \qquad + (n-m-1)p\tilde{P}_{(n-1,m)}(t) + p\left[\sum_{i=0}^{n-2}\tilde{P}_{(n-1,i)}(t)\right], \\ \vdots. \end{cases} \tag{A4}$$

$$\begin{cases} (r+1)\tilde{P}_{(r+1,r)}(t+1) = q\tilde{P}_{(r+2,r)}(t) + (r+1)q\tilde{P}_{(r+2,r+1)}(t) + mp\tilde{P}_{(r,r-1)}(t), \\ (r+2)\tilde{P}_{(r+2,r)}(t+1) = 2q\tilde{P}_{(r+3,r)}(t) + (r+1)q\tilde{P}_{(r+3,r+1)}(t) + mp\tilde{P}_{(r+1,r-1)}(t) + (r+2-m-1)p\tilde{P}_{(r+1,r)}(t), \\ \vdots \\ n\tilde{P}_{(n,r)}(t+1) = (n-r)q\tilde{P}_{(n+1,r)}(t) + (r+1)q\tilde{P}_{(n+1,r+1)}(t) + mp\tilde{P}_{(n-1,r-1)}(t) + (n-m-1)p\tilde{P}_{(n-1,r)}(t), \\ \vdots. \end{cases} \quad r \geq m+1 \tag{A5}$$

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