Stability of weighted spectral distribution in a pseudo tree-like network model*

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(Received 23 September 2015; revised manuscript received 26 December 2015; published online 1 April 2016)

The comparison of networks with different orders strongly depends on the stability analysis of graph features in evolving systems. In this paper, we rigorously investigate the stability of the weighted spectral distribution (i.e., a spectral graph feature) as the network order increases. First, we use deterministic scale-free networks generated by a pseudo tree-like model to derive the precise formula of the spectral feature, and then analyze the stability of the spectral feature based on the precise formula. Except for the scale-free feature, the pseudo tree-like model exhibits the hierarchical and small-world structures of complex networks. The stability analysis is useful for the classification of networks with different orders and the similarity analysis of networks that may belong to the same evolving system.

Keywords: weighted spectral distribution, pseudo tree-like model, deterministic network, scale-free and small-world network

PACS: 89.75.Hc, 05.40.Fb, 05.10.–a **DOI:** 10.1088/1674-1056/25/5/058901

1. Introduction

The order (defined as the number of nodes) of many scale-free networks grows over time, e.g., social networks, airport networks, and Internet topologies.[1] Thus, it is inevitable to compare networks with different orders for the study of these evolving systems. Network comparison has been widely used for the classification and similarity analysis of diverse networks. Specifically, the comparison of networks with different orders strongly depends on the stability features of the evolving systems. This paper focuses on the stability analysis of a spectral graph feature of scale-free evolving systems, i.e., the weighted spectral distribution (WSD).

Graph spectra incorporate all eigenvalues of graph matrices, including adjacency and normalized Laplacian matrices. The natural connectivity is a spectral graph feature defined on the adjacency spectrum.[2] Moreover, Wu *et al.*[3] rigorously demonstrated that the feature is asymptotically independent of the order of regular graphs in which each node has an identical expected degree. However, our recent work[4,5] found numerically that the stability of the natural connectivity does not work in scale-free networks. The WSD is another spectral graph feature defined on the normalized Laplacian spectrum that strongly corresponds to the distribution of random N-cycles in a network.[6] Fay *et al.*[7] designed a calculation algorithm for the WSD using Sylvester's law of inertia which is restricted to networks with less than 30000 nodes. Using 4-cycle structures, Jiao *et al.*[4] proposed a fast algorithm by which the WSD can be accurately calculated in scale-free networks with more than millions of nodes. The WSD captures the local connectivity of diverse networks.[8,9] Additionally, Jiao *et al.*[5] numerically observed that the WSD increases sublinearly with the network order in a stochastic scale-free network model. However, the stochastic models are not suitable for the rigorous analysis of the WSD's stability since it is impossible to derive the WSD's precise formula in these models. Therefore, there is a need to rigorously investigate the stability of the WSD for better understanding the performance of the spectral graph feature in evolving scale-free networks.

In contrast to stochastic models, deterministic models help us to understand the construction rules embedded in randomized structures and allow for rigorous computation of many graph features, e.g., clustering coefficients, diameter, betweenness, path length, and mean first-passage time.[10–20] The primary deterministic scale-free model was proposed by Barabasi *et al.*[19] However, this model lacks the small-world effect that exists in most real networks, i.e., its clustering coefficient is too small while the average length path remains relatively large. To remedy this weakness, Chen *et al.*[20] constructed a pseudo tree-like (PTL) deterministic model having both the small-world and the scale-free characteristics. Moreover, in Chen's model,[20] the distribution of clustering coefficients $C(k)$ with respect to degree k has the power-law form $C(k) \propto k^{-1}$ which exists in most hierarchical systems.[21] Therefore, our contribution includes the rigorous analysis of the WSD's stability using Chen's deterministic model.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 61402485, 61303061, and 71201169).
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2. Background

2.1. WSD

Let $G=(V,E)$ be a simple and undirected graph, where V and E are the node set and the edge set, respectively. Let d_v be the degree of a node v in graph G. Then, the normalized Laplacian matrix of graph G can be expressed as[6]

$$\boldsymbol{L}(G)(u,v)=\begin{cases}1, & \text{if } u=v \text{ and } d_v\neq 0,\\ -\dfrac{1}{\sqrt{d_u\cdot d_v}}, & \text{if } u \text{ and } v \text{ are adjacent},\\ 0, & \text{otherwise.}\end{cases} \quad (1)$$

All eigenvalues of matrix $\boldsymbol{L}(G)$ (i.e., $0=\lambda_1\leq\cdots\leq\lambda_n\leq 2$) form the normalized Laplacian spectrum. The WSD is defined as follows:[7]

$$W(G,N)=\sum_{i=1,2,\ldots,n}(1-\lambda_i)^N. \quad (2)$$

Additionally, the WSD counts the sum over all N-cycles in graph G[7]

$$\sum_{i=1,2,\ldots,n}(1-\lambda_i)^N=\sum_{C}\frac{1}{d_{u_1}d_{u_2}\cdots d_{u_N}}, \quad (3)$$

where $C=u_1u_2\cdots u_N$ is all N-cycles in graph G. The parameter N represents many graph properties. One important property is the number of 4-cycles. For a graph of its density, it is quasi-random.[22,23] In general, the sum over the 4-cycles represents the quasi-randomness of the graph in a non-regular case.[22,23] Therefore, four is considered to be the suitable value of the parameter N.[4–6]

2.2. Pseudo tree-like model

Let T_{n+1} denote the $n+1$ hierarchical-level graph generated by the PTL model. Specifically, T_{n+1} is generated recursively with level t increased from 1 to $n+1$ as follows (Fig. 1).[20]

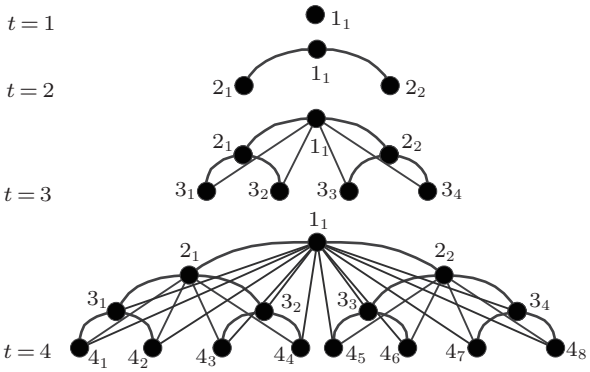


Fig. 1. Generation process of the deterministic PTL network model.

At $t=1$, start with a single node 1_1, which is the main root of the whole network.

At $t=2$, two leaf nodes 2_1 and 2_2 are added and connected to the main root 1_1.

At $t=3$, four leaf nodes 3_1–3_4 are added and each node 3_j $(1\leq j\leq 4)$ is connected to the main root 1_1 and the subordinate root 2_1 (for $j=1,2$) or 2_2 (for $j=3,4$).

In general, at level $t\in\{1,2,\ldots,n+1\}$, 2^{t-1} leaf nodes $t_j(1\leq j\leq 2^{t-1})$ are added and each node t_j is connected to its $t-1$ roots $(t-k)_{\lceil j/2^k\rceil}$ $(k=1,2,\ldots,t-1)$, where $\lceil x\rceil$ rounds the value of x to the nearest integer towards infinity.

Chen *et al.*[20] demonstrated that the PTL model has the scale-free, the small-world, and the hierarchical characteristics of most complex networks.

3. Precise formula of the WSD in T_{n+1}

Equations (2) and (3) show that the WSD counts the sum of the products of degree reciprocals embedded in all N-cycles ($N=4$ is commonly selected) of graph G. For each node $v\in V$, let $N(v)=\{1,2,\ldots,r\}$ denote the set of neighbors of node v and $N(i)=\{v,v_1^i,v_2^i,\ldots,v_{s_i}^i\}$ denote the set of neighbors of node $i\in\{1,2,\ldots,r\}$. Let $C(i,j)=\{v,c_1,c_2,\ldots,c_{t_{ij}}\}$ denote the set of common neighbors of nodes i and j with $i,j\in N(v)$ and $i\neq j$. Then, the WSD of graph G can be calculated as follows (see Appendix A):

$$\mathrm{WSD}=\sum_{v\in V}(\mathrm{WSD}_{12}(v)+\mathrm{WSD}_{34}(v)), \quad (4)$$

$$\mathrm{WSD}_{12}(v)=\sum_{i\in N(v)}\sum_{j\in N(i)}\frac{1}{d_v\cdot(d_i)^2\cdot d_j}, \quad (5)$$

$$\mathrm{WSD}_{34}(v)=2\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\sum_{k\in C(i,j)}\frac{1}{d_v\cdot d_i\cdot d_j\cdot d_k}. \quad (6)$$

According to Eqs. (4)–(6), the neighbor set of each node and the common neighbor set of each two nodes are crucial for the calculation of the WSD in T_{n+1}.

At level t $(1\leq t\leq n+1)$ of T_{n+1}, 2^{t-1} nodes $t_1,t_2,\ldots,t_{2^{t-1}}$ are attached to other nodes in the same manner, i.e., node t_1 is equivalent to each node of other $2^{t-1}-1$ nodes at level t. Hence, we only consider the neighbor set (i.e., $N(t_1)$) of node t_1 and the common neighbor set (i.e., $C(t_1,s_k)=N(t_1)\cap N(s_k)$) of two nodes t_1 and $s_k(\neq t_1)$, where $1\leq s\leq n+1$ and $1\leq k\leq 2^{s-1}$. Without loss of generality, we assume $s\geq t$.

According to the construction rules listed in Subsection 2.2,

$$N(t_1)=\mathrm{rot}(t_1)\cup\mathrm{des}(t_1), \quad (7)$$

where $\mathrm{rot}(t_1)=\{(t-1)_1,(t-2)_1,\ldots,1_1\}$ includes all roots of node t_1 (one main root 1_1 and $t-2$ subordinate roots) and $\mathrm{des}(t_1)=\bigcup_{i=1}^{n-t+1}\{(t+i)_1,(t+i)_2,\ldots,(t+i)_{2^i}\}$ includes all descendants of node t_1 (i.e., all nodes which have node t_1 as one root).

If s_k is a descendant of node t_1, $\mathrm{des}(s_k) \subseteq \mathrm{des}(t_1)$ and t_1 is one root of node s_k (i.e., $\mathrm{rot}(s_k) = \mathrm{rot}(t_1) \cup \{t_1\} \cup (\mathrm{rot}(s_k) \cap \mathrm{des}(t_1))$). For this case,

$$C(t_1, s_k) = \mathrm{des}(s_k) \cup (\mathrm{rot}(s_k) - \{t_1\}) = N(s_k) - \{t_1\}. \quad (8)$$

If $s_k (s \geq t)$ is not a descendant of node t_1, $\mathrm{des}(s_k) \cap (\mathrm{des}(t_1) \cup \mathrm{rot}(t_1)) = \Phi$ and $\mathrm{rot}(s_k) \cap \mathrm{des}(t_1) = \Phi$. For this case,

$$C(t_1, s_k) = \mathrm{rot}(t_1) \cap \mathrm{rot}(s_k). \quad (9)$$

Specifically, $N(s_k) = \mathrm{rot}(s_k) \cup \mathrm{des}(s_k)$, where $\mathrm{rot}(s_k)$ includes all roots of node s_k and $\mathrm{des}(s_k)$ includes all descendants of node s_k.

In T_{n+1}, the network order can be expressed as

$$N_{n+1}^T = \sum_{t=1}^{n+1} 2^{t-1} = 2^{n+1} - 1, \quad (10)$$

and the degree of node $t_j (1 \leq j \leq 2^{t-1})$ at level $t \in \{1, 2, \ldots, n+1\}$ is

$$d_t = 2 \cdot (2^{n-t+1} - 1) + (t-1). \quad (11)$$

The sum of the degree reciprocals of nodes in $N(t_1)$ can be expressed as

$$S_{N(t)}^d = \sum_{i=1}^{n-t+1} 2^i \cdot \frac{1}{d_{t+i}} + \sum_{i=1}^{t-1} \frac{1}{d_{t-i}}. \quad (12)$$

The equivalence between node t_1 and node t_j $(2 \leq j \leq 2^{t-1})$ means that $S_{N(t)}^d$ represents the sum of degree reciprocals of nodes in $N(t_j)$ for $\forall j \in \{1, 2, \ldots, 2^{t-1}\}$. According to Eqs. (5) and (7), we can obtain

$$\mathrm{WSD}_{12}(t_1) = \frac{1}{d_t} \left(\sum_{j=1}^{t-1} \frac{1}{d_{t-j}^2} \cdot S_{N(t-j)}^d + \sum_{j=1}^{n-t+1} 2^j \cdot \frac{1}{d_{t+j}^2} \cdot S_{N(t+j)}^d \right). \quad (13)$$

Let $S_{t+1}^1 = \{(t+1)_1\} \cup \mathrm{des}((t+1)_1)$ and $S_{t+1}^2 = \{(t+1)_2\} \cup \mathrm{des}((t+1)_2)$, where $\mathrm{des}((t+1)_1)$ and $\mathrm{des}((t+1)_2)$ include all descendants of nodes $(t+1)_1$ and $(t+1)_2$, respectively. Then, the pairs of nodes in $N(t_1) = \mathrm{rot}(t_1) \cup S_{t+1}^1 \cup S_{t+1}^2$ can be classified into four cases: case 1, pairs of nodes in $\mathrm{rot}(t_1)$; case 2, one node in $\mathrm{rot}(t_1)$ and the other node in $S_{t+1}^1 \cup S_{t+1}^2$; case 3, one node in S_{t+1}^1 and the other node in S_{t+1}^2; and case 4, pairs of nodes in S_{t+1}^1, which is equivalent to pairs of nodes in S_{t+1}^2.

Let $P_{N(t_1)}$ denote the set of all pairs of nodes in $N(t_1)$, where for $\forall i, j \in N(t_1) \wedge i \neq j$, (i, j) and (j, i) are two different node pairs. Then, $P_{N(t_1)} = S_1 \cup S_2 \cup S_3 \cup S_4^1 \cup S_4^2$, where S_1, S_2, S_3, and $S_4^1 \cup S_4^2$ are node pair sets associated with cases 1–4, respectively. Specifically, S_4^1 and S_4^2 correspond to S_{t+1}^1 and S_{t+1}^2, respectively.

According to Eq. (6), we can obtain

$$\mathrm{WSD}_{34}(t_1) = \frac{1}{d_t} \cdot \left(\sum_{l=1}^{3} \sum_{(i,j) \in S_l} \sum_{k \in C(i,j)} \frac{1}{d_i \cdot d_j \cdot d_k} + 2 \cdot \sum_{(i,j) \in S_4} \sum_{k \in C(i,j)} \frac{1}{d_i \cdot d_j \cdot d_k} \right). \quad (14)$$

Due to the equivalence between S_{t+1}^1 and S_{t+1}^2, both S_4^1 and S_4^2 associated with case 4 are substituted by node pair set S_4 in Eq. (14).

For each pair of nodes (i.e., (i_1, j_1) with $1 \leq i, j \leq t-1$) in $\mathrm{rot}(t_1) = \{(t-1)_1, (t-2)_1, \ldots, 1_1\}$, if $i < j$, node j_1 is a descendant of node i_1. According to Eq. (8), $C(i_1, j_1) = N(j_1) - \{i_1\}$. Hence, for node pair set S_1 associated with case 1,

$$\sum_{(i_1,j_1) \in S_1} \sum_{k \in C(i_1,j_1)} \frac{1}{d_i \cdot d_j \cdot d_k} = 2 \cdot \sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} \frac{1}{d_i \cdot d_j} \cdot \left(S_{N(j)}^d - \frac{1}{d_i} \right). \quad (15)$$

Please note that (i_1, j_1) and (j_1, i_1) are two different node pairs included in S_1.

According to Eq. (12), we can obtain

$$S_{N(j)}^d - \frac{1}{d_i} = \sum_{k=1}^{n-j+1} 2^k \cdot \frac{1}{d_{j+k}} + \sum_{k=1}^{j-i-1} \frac{1}{d_{j-k}} + \sum_{k=j-i+1}^{j-1} \frac{1}{d_{j-k}}. \quad (16)$$

For each pair of nodes $(i_1, (t+j)_l)$ with $i_1 \in \mathrm{rot}(t_1) = \{(t-1)_1, (t-2)_1, \ldots, 1_1\}$ and $(t+j)_l \in S_{t+1}^1 \cup S_{t+1}^2 = \bigcup_{j=1}^{n-t+1} \{(t+j)_1, (t+j)_2, \ldots, (t+j)_{2^j}\}$, node $(t+j)_l$ is a descendant of node i_1. According to Eq. (8), $C((i_1, (t+j)_l) = N((t+j)_l) - \{i_1\}$. Hence, for node pair set S_2 associated with case 2,

$$\sum_{(i_1,(t+j)_l) \in S_2} \sum_{k \in C(i_1,(t+j)_l)} \frac{1}{d_i \cdot d_{t+j} \cdot d_k} + \sum_{((t+j)_l,i_1) \in S_2} \sum_{k \in C((t+j)_l,i_1)} \frac{1}{d_{t+j} \cdot d_i \cdot d_k} = 2 \cdot \sum_{i=1}^{t-1} \sum_{j=1}^{n-t+1} \frac{1}{d_i} \cdot \left[2^j \cdot \frac{1}{d_{t+j}} \cdot \left(S_{N(t+j)}^d - \frac{1}{d_i} \right) \right]. \quad (17)$$

For each pair of nodes $((t+i)_x, (t+j)_y)$ with $(t+i)_x \in S_{t+1}^1 = \bigcup_{i=1}^{n-t+1} \{(t+i)_1, (t+i)_2, \ldots, (t+i)_{2^{i-1}}\}$ and $(t+j)_y \in S_{t+1}^2 = \bigcup_{j=1}^{n-t+1} \{(t+j)_{2^{j-1}+1}, (t+j)_{2^{j-1}+2}, \ldots, (t+j)_{2^j}\}$, nodes $t_1, (t-1)_1, \ldots, 1_1$ are the common roots of nodes $(t+i)_x$ and $(t+j)_y$, i.e., $\mathrm{rot}((t+i)_x) \cap \mathrm{rot}((t+j)_y) = \{t_1, (t-1)_1, \ldots, 1_1\}$. Additionally, there is no root-descendant relationship between nodes $(t+i)_x$ and $(t+j)_y$. According to Eq. (9), $C((t+i)_x, (t+j)_y) = \{t_1, (t-1)_1, \ldots, 1_1\}$. Hence, for node pair set S_3 associated with case 3,

$$\sum_{((t+i)_x,(t+j)_y) \in S_3} \sum_{k \in C((t+i)_x,(t+j)_y)} \frac{1}{d_{t+i} \cdot d_{t+j} \cdot d_k}$$

$$+\sum_{((t+j)_y,(t+i)_x)\in S_3}\ \sum_{k\in C((t+j)_y,(t+i)_x)}\frac{1}{d_{t+j}\cdot d_{t+i}\cdot d_k}$$
$$=2\cdot\sum_{i=1}^{n-t+1}2^{i-1}\cdot\frac{1}{d_{t+i}}\cdot\sum_{j=1}^{n-t+1}2^{j-1}\cdot\frac{1}{d_{t+j}}\cdot\sum_{k=1}^{t}\frac{1}{d_k}. \quad (18)$$

Let S_{t+1}^{d,S_3} denote the sum of the products of degree reciprocals associated with node pairs in S_3, i.e., S_{t+1}^{d,S_3} is equal to Eq. (18) which can be expressed as

$$S_{t+1}^{d,S_3}=2\cdot\left(\sum_{k=1}^{n-t+1}2^{k-1}\cdot\frac{1}{d_{t+k}}\right)^2\cdot\sum_{k=1}^{t}\frac{1}{d_k}. \quad (19)$$

Due to the equivalence between S_{t+1}^1 and S_{t+1}^2, we only consider node pair set S_4 generated by node set $S_{t+1}^1=\{(t+1)_1\}\cup S_{t+2}^1\cup S_{t+2}^2$. As the extension of S_{t+1}^1 and S_{t+1}^2, $S_{t+2}^1=S_{(t+1)+1}^1=\{(t+2)_1\}\cup\mathrm{des}((t+2)_1)$ and $S_{t+2}^2=S_{(t+1)+1}^2=\{(t+2)_2\}\cup\mathrm{des}((t+2)_2)$. Hence, node pair set S_4 can be classified into three parts: part 1, one node $(t+1)_1$ and the other node in $S_{t+2}^1\cup S_{t+2}^2$; part 2, one node in S_{t+2}^1 and the other node in S_{t+2}^2; part 3, pairs of nodes in S_{t+2}^1, which is equivalent to pairs of nodes in S_{t+2}^2.

As is well known,

$$S_{t+2}^1=\bigcup_{i=1}^{n-t}\{(t+1+i)_1,(t+1+i)_2,\ldots,(t+1+i)_{2^{i-1}}\}.$$

Let S_{t+1}^{d,S_4} denote the sum of the products of degree reciprocals associated with node pairs in S_4. Similar to the calculation for Eqs. (15)–(19), we can obtain

$$\begin{aligned}S_{t+1}^{d,S_4}&=\sum_{(i,j)\in S_4}\ \sum_{k\in C(i,j)}\frac{1}{d_i\cdot d_j\cdot d_k}\\&=2\cdot\frac{1}{d_{t+1}}\cdot\sum_{i=1}^{n-t}2^i\cdot\frac{1}{d_{t+1+i}}\cdot\left(S_{N(t+1+i)}^d-\frac{1}{d_{t+1}}\right)\\&\quad+S_{t+2}^{d,S_3}+2\cdot S_{t+2}^{d,S_4},\end{aligned} \quad (20)$$

where the three terms of the right-hand side of Eq. (20) correspond to parts 1–3, respectively. Specifically, $S_{t+2}^{d,S_3}=S_{(t+1)+1}^{d,S_3}$ and $S_{t+2}^{d,S_4}=S_{(t+1)+1}^{d,S_4}$.

According to Eq. (16), we can obtain

$$\begin{aligned}S_{N(t+1+i)}^d-\frac{1}{d_{t+1}}&=\sum_{k=1}^{n-t-i}2^k\cdot\frac{1}{d_{t+1+i+k}}+\sum_{k=1}^{i-1}\frac{1}{d_{t+1+i-k}}\\&\quad+\sum_{k=i+1}^{t+i}\frac{1}{d_{t+1+i-k}}.\end{aligned} \quad (21)$$

According to Appendix B, we can obtain

$$\begin{aligned}S_{t+1}^{d,S_4}&=\sum_{i=1}^{n-t}2^{i+1}\cdot\frac{1}{d_{t+1+i}}\cdot\left(\sum_{k=1}^{n-t-i}2^k\cdot\frac{1}{d_{t+1+i+k}}\right.\\&\quad\times\sum_{j=1}^{i}\frac{1}{d_{t+j}}+\sum_{j=1}^{i}\frac{1}{d_{t+j}}\cdot\sum_{k=1}^{i-j}\frac{1}{d_{t+1+i-k}}\\&\quad\left.+\sum_{j=1}^{i}\frac{1}{d_{t+j}}\cdot\sum_{k=i-j+2}^{t+i}\frac{1}{d_{t+1+i-k}}\right)\\&\quad+\sum_{j=1}^{n-t}2^{j-1}\cdot S_{t+j+1}^{d,S_3},\end{aligned} \quad (22)$$

where $S_{t+j+1}^{d,S_3}=S_{(t+j)+1}^{d,S_3}$ which can be derived using Eq. (19).

In T_{n+1}, node set $V=\bigcup_{t=1}^{n+1}\{t_1,t_2,\ldots,t_{2^{t-1}}\}$. Therefore, according to Eq. (4) and the equivalence between node t_1 and each node t_j $(2\le j\le 2^{t-1})$, the precise formula of the WSD in T_{n+1} can be derived as

$$\mathrm{WSD}_{T_{n+1}}=\sum_{t=1}^{n+1}2^{t-1}\cdot(\mathrm{WSD}_{12}(t_1)+\mathrm{WSD}_{34}(t_1)). \quad (23)$$

Using Eqs. (12), (13), and (14)–(22), we can obtain

$$\begin{aligned}&\mathrm{WSD}_{12}(t_1)\\&=\frac{1}{d_t}\left[\sum_{j=1}^{t-1}\frac{1}{d_{t-j}^2}\cdot\left(\sum_{i=1}^{n-t+j+1}2^i\cdot\frac{1}{d_{t-j+i}}\right.\right.\\&\quad\left.+\sum_{i=1}^{t-j-1}\frac{1}{d_{t-j-i}}\right)\\&\quad+\sum_{j=1}^{n-t+1}2^j\cdot\frac{1}{d_{t+j}^2}\cdot\left(\sum_{i=1}^{n-t-j+1}2^i\cdot\frac{1}{d_{t+j+i}}\right.\\&\quad\left.\left.+\sum_{i=1}^{t+j-1}\frac{1}{d_{t+j-i}}\right)\right],\end{aligned} \quad (24)$$

$$\begin{aligned}&\mathrm{WSD}_{34}(t_1)\\&=\frac{1}{d_t}\cdot\left[2\cdot\sum_{i=1}^{t-2}\sum_{j=i+1}^{t-1}\frac{1}{d_i\cdot d_j}\right.\\&\quad\times\left(\sum_{k=1}^{n-j+1}2^k\cdot\frac{1}{d_{j+k}}+\sum_{k=1}^{j-i-1}\frac{1}{d_{j-k}}\right.\\&\quad\left.+\sum_{k=j-i+1}^{j-1}\frac{1}{d_{j-k}}\right)+\sum_{i=1}^{t-1}\sum_{j=1}^{n-t+1}\frac{2^{j+1}}{d_i\cdot d_{t+j}}\\&\quad\times\left(\sum_{k=1}^{n-t-j+1}2^k\cdot\frac{1}{d_{t+j+k}}+\sum_{k=1}^{t+j-i-1}\frac{1}{d_{t+j-k}}\right.\\&\quad\left.+\sum_{k=t+j-i+1}^{t+j-1}\frac{1}{d_{t+j-k}}\right)+2\cdot\left(\sum_{k=1}^{n-t+1}2^{k-1}\cdot\frac{1}{d_{t+k}}\right)^2\\&\quad\times\sum_{k=1}^{t}\frac{1}{d_k}+\sum_{i=1}^{n-t}2^{i+2}\cdot\frac{1}{d_{t+1+i}}\\&\quad\times\left(\sum_{k=1}^{n-t-i}2^k\cdot\frac{1}{d_{t+1+i+k}}\cdot\sum_{j=1}^{i}\frac{1}{d_{t+j}}\right.\\&\quad+\sum_{j=1}^{i}\frac{1}{d_{t+j}}\cdot\sum_{k=1}^{i-j}\frac{1}{d_{t+1+i-k}}+\sum_{j=1}^{i}\frac{1}{d_{t+j}}\\&\quad\left.\times\sum_{k=i-j+2}^{t+i}\frac{1}{d_{t+1+i-k}}\right)+\sum_{j=1}^{n-t}2^{j+1}\\&\quad\times\left.\left(\sum_{k=1}^{n-t-j+1}2^{k-1}\cdot\frac{1}{d_{t+j+k}}\right)^2\cdot\sum_{k=1}^{t+j}\frac{1}{d_k}\right].\end{aligned} \quad (25)$$

4. Stability of the WSD in T_{n+1}

Equation (11) shows that, for $\forall t\in\{1,2,\ldots,n+1\}$, the degree of node t_j $(1\le j\le 2^{t-1})$ at level $t\in\{1,2,\ldots,n+1\}$ in T_{n+1} is $d_t=2\cdot(2^{n-t+1}-1)+(t-1)$. If all node degrees of Eqs. (23)–(25) are substituted using Eq. (11), it is difficult to derive the sum formula of $\mathrm{WSD}_{T_{n+1}}$. But we can derive two sum formulas to bound the $\mathrm{WSD}_{T_{n+1}}$.

Theorem 1 For $\forall t\in\{1,2,\ldots,n+1\}$ and $\forall\alpha>1$, with $n\to+\infty$,

$$2^{n-t+1}\le d_t=2\cdot(2^{n-t+1}-1)+(t-1)\le 2^{\alpha\cdot n-t+3}. \quad (26)$$

Proof $t \geq 1 \Rightarrow d_t \geq 2\cdot(2^{n-t+1}-1) = 2^{n-t+1}+(2^{n-t+1}-2)$. If $t \in \{1,2,\ldots,n\}$, $2^{n-t+1} \geq 2$ (i.e., $d_t \geq 2^{n-t+1}$). If $t = n+1$, $d_t = n$ and $2^{n-t+1} = 1$ (i.e., $d_t \geq 2^{n-t+1}$).

With increasing t, $\varphi(t) = t-1$ monotonically increases and $\psi(t) = 2^{\alpha\cdot n-t+1}$ monotonically decreases. For $\forall\alpha > 1$, $\varphi(n+1) = n \leq \psi(n+1) = 2^{(\alpha-1)\cdot n}$ with $n \to +\infty$. Hence, for $\forall t \in \{1,2,\ldots,n+1\}$, with $n \to +\infty$, $\varphi(t) \leq \psi(t)$ (i.e., $t-1 \leq 2^{\alpha\cdot n-t+1}$). That is, $d_t \leq 2\cdot(2^{n-t+1}-1)+2^{\alpha\cdot n-t+1} \leq 3\cdot 2^{\alpha\cdot n-t+1} \leq 2^{\alpha\cdot n-t+3}$. The proof is completed.

Corollary 1 For $\forall t \in \{1,2,\ldots,n+1\}$ and $\forall\alpha > 1$, with $n \to +\infty$,

$$2^{-\alpha\cdot n+t-3} \leq \frac{1}{d_t} \leq 2^{-n+t-1}. \tag{27}$$

If $1/d_t (t \in \{1,2,\ldots,n\})$ of Eqs. (23)–(25) is substituted using $2^{-\alpha\cdot n+t-3}$ or 2^{-n+t-1}, the sum formula of $\mathrm{WSD}_{T_{n+1}}$ is easy to derive using the geometric progression's sum formula. Therefore, for $\forall\alpha > 1$, with $n \to +\infty$, using Eq. (27), we can obtain two boundaries of $\mathrm{WSD}_{12}(t_1)$ and $\mathrm{WSD}_{34}(t_1)$ as follows:

$$f_{12}^{\min} \leq \mathrm{WSD}_{12}(t_1) \leq f_{12}^{\max}, \tag{28}$$

$$f_{34}^{\min} \leq \mathrm{WSD}_{34}(t_1) \leq f_{34}^{\max}, \tag{29}$$

where

$$\begin{aligned} f_{12}^{\min} &= \frac{97}{630}\cdot 2^{-4\alpha\cdot n+4t-11} + \frac{17}{21}\cdot 2^{-4\alpha\cdot n+3t-11} \\ &\quad - \frac{1}{9}\cdot 2^{(2-4\alpha)\cdot n+2t-8} + \frac{1}{21}\cdot 2^{-4\alpha\cdot n+t-6} \\ &\quad - \frac{1}{3}\cdot 2^{(2-4\alpha)\cdot n+t-7} + \frac{1}{45}\cdot 2^{(4-4\alpha)\cdot n-2} \\ &\quad - \frac{1}{7}\cdot 2^{(3-4\alpha)\cdot n-5}, \end{aligned} \tag{30}$$

$$\begin{aligned} f_{12}^{\max} &= \frac{97}{630}\cdot 2^{-4n+4t-3} + \frac{17}{21}\cdot 2^{-4n+3t-3} \\ &\quad - \frac{1}{9}\cdot 2^{-2n+2t} + \frac{1}{21}\cdot 2^{-4n+t+2} \\ &\quad - \frac{1}{3}\cdot 2^{-2n+t+1} + \frac{1}{45}\cdot 2^{6} - \frac{1}{7}\cdot 2^{-n+3}, \end{aligned} \tag{31}$$

$$\begin{aligned} f_{34}^{\min} &= \frac{29}{27}\cdot 2^{(2-4\alpha)\cdot n+2t-6} + \frac{1}{3}\cdot 2^{(2-3\alpha)\cdot n+t-4} \\ &\quad - \frac{1}{3}\cdot\left(n-t+\frac{683}{9765}\right)\cdot 2^{-4\alpha\cdot n+4t-11} \\ &\quad - \frac{5}{21}\cdot 2^{(3-4\alpha)\cdot n+t-5} - \frac{1}{3}\cdot 2^{-3\alpha\cdot n+3t-6} \\ &\quad - \frac{1}{9}\cdot\left(n-t-\frac{3}{5}\right)\cdot 2^{(4-4\alpha)\cdot n-4} \\ &\quad + \frac{1}{3}\cdot\left(n-t+\frac{1371}{21}\right)\cdot 2^{-4\alpha\cdot n+3t-10} \\ &\quad - \frac{1}{9}\cdot 2^{-4\alpha\cdot n+2t-7} - \frac{1}{63}\cdot 2^{-4\alpha\cdot n+t-2} \\ &\quad + \frac{1}{9}\cdot(2^{-t+3}-3t+5)\cdot 2^{(2-4\alpha)\cdot n+t-6} \\ &\quad + \frac{1}{3}\cdot 2^{-3\alpha\cdot n+2t-6} \\ &\quad - \frac{1}{3}\cdot 2^{(2-3\alpha)\cdot n-4} - \frac{1}{9}\cdot 2^{(4-4\alpha)\cdot n-t-4}, \end{aligned} \tag{32}$$

$$\begin{aligned} f_{34}^{\max} &= \frac{29}{27}\cdot 2^{-2n+2t+2} + \frac{1}{3}\cdot 2^{-n+t+2} \\ &\quad - \frac{1}{3}\cdot\left(n-t+\frac{683}{9765}\right)\cdot 2^{-4n+4t-3} \\ &\quad - \frac{5}{21}\cdot 2^{-n+t+3} - \frac{1}{3}\cdot 2^{-3n+3t} - \frac{1}{9}\cdot\left(n-t-\frac{3}{5}\right)\cdot 2^{4} \\ &\quad + \frac{1}{3}\cdot\left(n-t+\frac{1371}{21}\right)\cdot 2^{-4n+3t-2} \\ &\quad - \frac{1}{9}\cdot 2^{-4n+2t+1} - \frac{1}{63}\cdot 2^{-4n+t+6} \\ &\quad + \frac{1}{9}\cdot(2^{-t+3}-3t+5)\cdot 2^{-2n+t+2} + \frac{1}{3}\cdot 2^{-3n+2t} \\ &\quad - \frac{1}{3}\cdot 2^{-n+2} - \frac{1}{9}\cdot 2^{-t+4}. \end{aligned} \tag{33}$$

With $\alpha \to 1$, we can obtain

$$\begin{aligned} f_{12}^{\min} \to g_{12}^{\min} &= \frac{97}{630}\cdot 2^{-4n+4t-11} + \frac{17}{21}\cdot 2^{-4n+3t-11} \\ &\quad - \frac{1}{9}\cdot 2^{-2n+2t-8} + \frac{1}{21}\cdot 2^{-4n+t-6} \\ &\quad - \frac{1}{3}\cdot 2^{-2n+t-7} + \frac{1}{45}\cdot 2^{-2} - \frac{1}{7}\cdot 2^{-n-5}, \end{aligned} \tag{34}$$

$$\begin{aligned} f_{34}^{\min} \to g_{34}^{\min} &= \frac{29}{27}\cdot 2^{-2n+2t-6} + \frac{1}{3}\cdot 2^{-n+t-4} \\ &\quad - \frac{1}{3}\cdot\left(n-t+\frac{683}{9765}\right)\cdot 2^{-4n+4t-11} \\ &\quad - \frac{5}{21}\cdot 2^{-n+t-5} - \frac{1}{3}\cdot 2^{-3n+3t-6} \\ &\quad - \frac{1}{9}\cdot\left(n-t-\frac{3}{5}\right)\cdot 2^{-4} \\ &\quad + \frac{1}{3}\cdot\left(n-t+\frac{1371}{21}\right)\cdot 2^{-4n+3t-10} \\ &\quad - \frac{1}{9}\cdot 2^{-4n+2t-7} - \frac{1}{63}\cdot 2^{-4n+t-2} \\ &\quad + \frac{1}{9}\cdot(2^{-t+3}-3t+5)\cdot 2^{-2n+t-6} + \frac{1}{3}\cdot 2^{-3n+2t-6} \\ &\quad - \frac{1}{3}\cdot 2^{-n-4} - \frac{1}{9}\cdot 2^{-t-4}. \end{aligned} \tag{35}$$

Hence, $g_{12}^{\min} + g_{34}^{\min} \leq \mathrm{WSD}_{12}(t_1) + \mathrm{WSD}_{34}(t_1) \leq f_{12}^{\max} + f_{34}^{\max}$.

Using Eq. (23), with $n \to +\infty$, we can obtain

$$\begin{aligned} &\frac{203363}{33635}\cdot 2^{n-3} + \mathrm{O}(2^n) \leq \mathrm{WSD}_{T_{n+1}} \\ &\leq \frac{301681}{100905}\cdot 2^{n+6} + \mathrm{O}(2^n), \end{aligned} \tag{36}$$

where $\mathrm{O}(2^n)/2^n \to 0$ with $n \to +\infty$. According to Eq. (10), with $n \to +\infty$, we can obtain

$$\frac{203363}{33635}\cdot 2^{-4} \leq \mathrm{WSD}_{T_{n+1}}/N_{n+1}^{\mathrm{T}} \leq \frac{301681}{100905}\cdot 2^{5}. \tag{37}$$

Therefore, $\mathrm{WSD}_{T_{n+1}}/N_{n+1}^{\mathrm{T}}$ is bounded as the network order N_{n+1}^{T} increases, which exhibits the stability of the WSD in T_{n+1}.

5. Application of the WSD's stability

The classification of networks with different orders is an important application of the stability features embedded in evolving systems. This section first selects several simulated and real-world networks with different orders, and then classifies these networks into diverse evolving systems using the WSD's stability.

The primary scale-free stochastic model was proposed by Barabasi and Albert.[24] At each time step of the model's growth process, a new node is attached to m different old nodes that are already present in the existing system. Furthermore, Zou *et al.*[25] constructed another scale-free stochastic model that is an extension of Barabasi–Albert model. Specifically, the newly added node at each time step of Barabasi–Albert model is replaced by a newly added module generated by Watts–Strogatz model[26] in Zou's model. The module shows the morphology feature of many real networks,[25] thus we select Zou's model to generate diverse simulated networks.

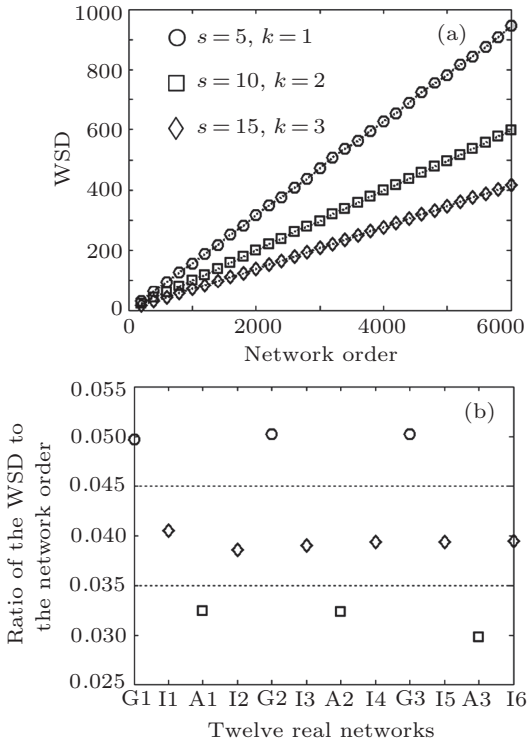


Fig. 2. Classification of diverse simulated and real-world networks: (a) WSD vs. network order in Zou's model with different modules (decided by the inputs s and k where other inputs of the model are set as $m=3$ and $\alpha=1$). (b) Ratio of the WSD to the network order in 12 real-world networks whose descriptions are included in Table 1. Note that each quantity in panel (a) is an average over 10 realizations.

The module generated by Watts–Strogatz model includes s nodes. Specifically, each node in the module is first connected with its $2k$ neighbors, and then each edge of the node is randomly rewired with a probability α. As shown in Fig. 2(a), when $m=3$ and $\alpha=1$, by using different inputs s and k, Zou's model can simulate diverse evolving networks with different modules. Figure 2(a) shows that for a certain module corresponding to unaltered inputs s and k, the WSD grows sublinearly as the network order increases. Additionally, figure 2(a) shows that the ratio of the WSD to the network order is a good indicator which can be used to distinguish evolving systems with different modules.

Furthermore, we choose 12 real-world networks with different orders,[1] as shown in Table 1. Specifically, in Table 1, G_1, G_2, and G_3 correspond to a sequence of snapshots of the Gnutella peer-to-peer file sharing network, I_1, I_2,..., I_6 correspond to a sequence of snapshots of the autonomous-system-level Internet topology, and A_1, A_2, and A_3 are networks that were collected by trawling Amazon website.

Table 1. Twelve real-world networks with different orders.[1]

Item	Node number	Edge number	Snapshot time
G_1	22687	54705	Aug 25 2002
I_1	16301	65910	Jan 05 2004
A_1	400727	3200440	Mar 12 2003
I_2	17848	74344	Sep 06 2004
G_2	36682	88328	Aug 30 2002
I_3	19846	80970	Jul 04 2005
A_2	410236	3356824	May 05 2003
I_4	21901	89176	May 01 2006
G_3	62586	147892	Aug 31 2002
I_5	24454	99660	Mar 05 2007
A_3	403394	3387388	Jun 01 2003
I_6	26475	106762	Nov 05 2007

As shown in Fig. 2(b), we calculate the ratio of the WSD to the network order in the twelve real-world networks listed in Table 1. Figure 2(b) shows that the ratio indeed is a good indicator which distinguishes evolving systems.

6. Conclusion

Our recent work[4,5] numerically verifies that the WSD increases sublinearly in a stochastic evolving scale-free network model as the network order increases. However, the stochastic models using interactive growth mechanisms are not suitable for the rigorous analysis of the scaling feature of the WSD. In this paper, our contributions focus on the derivation of the WSD's precise formula in a deterministic model which captures the scale-free, small-world, and hierarchical features of many real-world networks. The precise formula rigorously explains the WSD's stability (i.e., the ratio of the WSD to the network order is bounded as the network order increases). The stability indicates that the WSD can be applied to compare different order networks and to distinguish diverse evolving systems whose network orders grow over time.

Appendix A

Equations (2) and (3) represent that the WSD counts the sum of the products of degree reciprocals embedded in all N-cycles ($N=4$ is commonly selected) of graph G. Specially, the

same node can appear in a 4-cycle more than once. Figure A1 shows four patterns of 4-cycles with start node v, where the multiple edges between two nodes only represent the walks on 4-cycles and correspond to one edge in an undirected and simple graph. We can verify that any 4-cycle in graph G must be an instance of four patterns shown in Fig. A1, whereas the pattern in Fig. A1 may be related to more than one 4-cycle: pattern 1 corresponds to one 4-cycle v–i–v–j–v, pattern 2 corresponds to one 4-cycle v–i–k–i–v, pattern 3 corresponds to two 4-cycles v–i–v–j–v and v–j–v–i–v, and pattern 4 corresponds to two 4-cycles v–i–k–j–v and v–j–k–i–v. Additionally, different start nodes are associated with different 4-cycles. For example, 4-cycle i–v–i–k–i (with start node i) in Fig. A1(b) belongs to pattern 3 (does not belong to pattern 2).

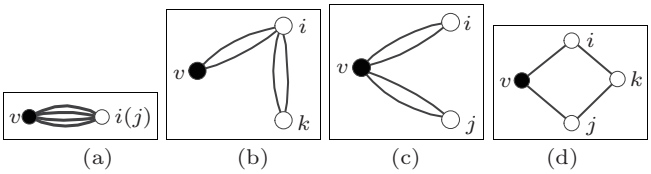


Fig. A1. All patterns of 4-cycles with start node v: (a) pattern 1, (b) pattern 2, (c) pattern 3, (d) pattern 4.[4]

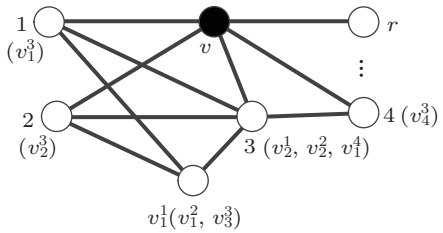


Fig. A2. Enumeration of all 4-cycles with start node v. One node may be labeled more than once. For example node 3 is labeled by v_2^1, v_2^2, and v_1^4, because it is a neighbor of nodes 1, 2, and 4.

For each node $v \in V$, let $N(v) = \{1,2,\ldots,r\}$ denote the set of neighbors of node v and $N(i) = \{v, v_1^i, v_2^i, \ldots, v_{s_i}^i\}$ denote the set of neighbors of node $i \in \{1,2,\ldots,r\}$. Let $C(i,j) = N(i) \cap N(j) = \{v, c_1, c_2, \ldots, c_{t_{ij}}\}$ denote the set of common neighbors of nodes i and j with $i,j \in N(v)$ and $i \neq j$. As shown in Fig. A2, 4-cycles v–i–j–i–v with $i \in \{1,2,\ldots,r\}$ and $j \in N(i)$ are all instances of patterns 1 and 2 (please note that $v \in N(i)$). Hence, the sum of the products of degree reciprocals embedded in 4-cycles with start node v and belonging to patterns 1 and 2 can be calculated as

$$\mathrm{WSD}_{12}(v) = \sum_{i\in N(v)} \sum_{j\in N(i)} \frac{1}{d_v \cdot (d_i)^2 \cdot d_j}. \tag{A1}$$

As shown in Fig. A2, 4-cycles v–i–k–j–v with $i,j \in \{1,2,\ldots,r\}$, $i \neq j$ and $k \in C(i,j)$ are all instances of patterns 3 and 4 (please note that $v \in C(i,j)$). Hence, the sum of the products of degree reciprocals embedded in 4-cycles with start node v and belonging to patterns 3 and 4 can be calculated as

$$\mathrm{WSD}_{34}(v) = \sum_{\substack{i,j\in N(v)\\ i\neq j}} \sum_{k\in C(i,j)} \frac{1}{d_v \cdot d_i \cdot d_j \cdot d_k}$$

$$= 2\sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \sum_{k\in C(i,j)} \frac{1}{d_v \cdot d_i \cdot d_j \cdot d_k}. \tag{A2}$$

With all start nodes $v \in V$, the WSD of graph G can be calculated as

$$\mathrm{WSD} = \sum\nolimits_{v\in V} (\mathrm{WSD}_{12}(v) + \mathrm{WSD}_{34}(v)). \tag{A3}$$

Appendix B

According to Eqs. (20) and (21),

$$S_{t+1}^{d,S_4} = \frac{1}{d_{t+1}} \cdot \sum\nolimits_{i=1}^{n-t} 2^{i+1} \cdot \frac{1}{d_{t+1+i}} \cdot \left(\sum\nolimits_{k=1}^{n-t-i} 2^k \cdot \frac{1}{d_{t+1+i+k}} + \sum\nolimits_{k=1}^{i-1} \frac{1}{d_{t+1+i-k}} + \sum\nolimits_{k=i+1}^{t+i} \frac{1}{d_{t+1+i-k}} \right) + S_{t+2}^{d,S_3} + 2 \cdot S_{t+2}^{d,S_4}. \tag{B1}$$

Using $t+1$ to substitute t in Eq. (B1), we can obtain

$$S_{t+2}^{d,S_4} = \frac{1}{d_{t+2}} \cdot \sum\nolimits_{i=1}^{n-t-1} 2^{i+1} \cdot \frac{1}{d_{t+2+i}} \cdot \left(\sum\nolimits_{k=1}^{n-t-1-i} 2^k \cdot \frac{1}{d_{t+2+i+k}} + \sum\nolimits_{k=1}^{i-1} \frac{1}{d_{t+2+i-k}} + \sum\nolimits_{k=i+1}^{t+1+i} \frac{1}{d_{t+2+i-k}} \right) + S_{t+3}^{d,S_3} + 2 \cdot S_{t+3}^{d,S_4}. \tag{B2}$$

Using $i-1$ to substitute i in Eq. (B2), we can obtain

$$2 \cdot S_{t+2}^{d,S_4} = \frac{1}{d_{t+2}} \cdot \sum\nolimits_{i=2}^{n-t} 2^{i+1} \cdot \frac{1}{d_{t+1+i}} \cdot \left(\sum\nolimits_{k=1}^{n-t-i} 2^k \cdot \frac{1}{d_{t+1+i+k}} + \sum\nolimits_{k=1}^{i-2} \frac{1}{d_{t+1+i-k}} + \sum\nolimits_{k=i}^{t+i} \frac{1}{d_{t+1+i-k}} \right) + 2 \cdot S_{t+3}^{d,S_3} + 2^2 \cdot S_{t+3}^{d,S_4}. \tag{B3}$$

Using $t+1$ to substitute t and $i-1$ to substitute i in Eq. (B3), we can obtain

$$2^2 \cdot S_{t+3}^{d,S_4} = \frac{1}{d_{t+3}} \cdot \sum\nolimits_{i=3}^{n-t} 2^{i+1} \cdot \frac{1}{d_{t+1+i}} \cdot \left(\sum\nolimits_{k=1}^{n-t-i} 2^k \cdot \frac{1}{d_{t+1+i+k}} + \sum\nolimits_{k=1}^{i-3} \frac{1}{d_{t+1+i-k}} + \sum\nolimits_{k=i-1}^{t+i} \frac{1}{d_{t+1+i-k}} \right) + 2^2 \cdot S_{t+4}^{d,S_3} + 2^3 \cdot S_{t+4}^{d,S_4}. \tag{B4}$$

Then we can derive the general formula as follows:

$$2^{j-1} \cdot S_{t+j}^{d,S_4} = \frac{1}{d_{t+j}} \cdot \sum\nolimits_{i=j}^{n-t} 2^{i+1} \cdot \frac{1}{d_{t+1+i}} \cdot \left(\sum\nolimits_{k=1}^{n-t-i} 2^k \cdot \frac{1}{d_{t+1+i+k}} + \sum\nolimits_{k=1}^{i-j} \frac{1}{d_{t+1+i-k}} + \sum\nolimits_{k=i-j+2}^{t+i} \frac{1}{d_{t+1+i-k}} \right) + 2^{j-1} \cdot S_{t+j+1}^{d,S_3} + 2^j \cdot S_{t+j+1}^{d,S_4}, \tag{B5}$$

where $j \in \{1,2,\ldots,n-t\}$.

Specifically, when $j = n-t$, we can obtain

$$2^{n-t-1} \cdot S_n^{d,S_4}$$

$$= \frac{1}{d_n} \cdot \sum_{i=n-t}^{n-t} 2^{i+1} \cdot \frac{1}{d_{t+1+i}} \cdot \left(\sum_{k=1}^{n-t-i} 2^k \cdot \frac{1}{d_{t+1+i+k}} + \sum_{k=1}^{i-n+t} \frac{1}{d_{t+1+i-k}} + \sum_{k=i-n+t+2}^{t+i} \frac{1}{d_{t+1+i-k}} \right) + 2^{n-t-1} \cdot S_{n+1}^{d,S_3} + 2^{n-t} \cdot S_{n+1}^{d,S_4}. \quad \text{(B6)}$$

According to the definition in Section 3, S_{n+1}^{d,S_4} represents the sum of the products of degree reciprocals associated with node pairs in $S_4 = \{(n+1)_1\}$. There is no node pair in $\{(n+1)_1\}$, which induces $S_{n+1}^{d,S_4} = 0$.

According to the sum of Eq. (B5) with j from 1 to $n-t$,

$$S_{t+1}^{d,S_4} = \sum_{i=1}^{n-t} 2^{i+1} \cdot \frac{1}{d_{t+1+i}} \cdot \left(\sum_{k=1}^{n-t-i} 2^k \cdot \frac{1}{d_{t+1+i+k}} \cdot \sum_{j=1}^{i} \frac{1}{d_{t+j}} + \sum_{j=1}^{i} \frac{1}{d_{t+j}} \cdot \sum_{k=1}^{i-j} \frac{1}{d_{t+1+i-k}} + \sum_{j=1}^{i} \frac{1}{d_{t+j}} \cdot \sum_{k=i-j+2}^{t+i} \frac{1}{d_{t+1+i-k}} \right) + \sum_{j=1}^{n-t} 2^{j-1} \cdot S_{t+j+1}^{d,S_3}. \quad \text{(B7)}$$

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