Bianchi type I in $f(T)$ gravitational theories*

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A tetrad field that is homogeneous and anisotropic which contains two unknown functions $A(t)$ and $B(t)$ of cosmic time is applied to the field equations of $f(T)$, where T is the torsion scalar, $T = T^{\mu}{}_{\nu\rho}S_{\mu}{}^{\nu\rho}$. We calculate the equation of continuity and rewrite it as a product of two brackets, the first is a function of $f(T)$ and the second is a function of the two unknowns $A(t)$ and $B(t)$. We use two different relations between the two unknown functions $A(t)$ and $B(t)$ in the second bracket to solve it. Both of these relations give constant scalar torsion and solutions coincide with the de Sitter one. So, another assumption related to the contents of the matter fields is postulated. This assumption enables us to drive a solution with a non-constant value of the scalar torsion and a form of $f(T)$ which represents ΛCDM.

Keywords: $f(T)$ gravitational theory, Bianchi type I, special solutions

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1. Introduction

Recently, it has been found that there is a contradiction between cosmological observations and Friedmann–Robertson–Walker (FRW) cosmology. The observations show that the universe is in an accelerating expansion era, but the FRW cosmology of Einstein's general relativity (GR) shows that it is not. This accelerating expansion of the universe is suggested to be due to a mysterious type of energy with negative pressure that is known as dark energy (DE). The evidence of the existence of this type of energy comes from the observation of supernovae type Ia,[1–7] cosmic microwave background (CMB) anisotropies measured with Wilkinson microwave anisotropy probe (WMAP),[8] and the large scale structures.[9–11] These observations suggest that more than two-thirds of our universe consists of DE and the remaining consists of relativistic dark matter and baryons.[12]

Modified theories of gravity have gained a lot of interest because of their possible explanation of DE.[13,14] This unknown energy, DE, which has negative pressure, may be physically equivalent to vacuum energy and is almost equally distributed in the universe. It has been used as an ingredient factor in a recent attempt to formulate a cyclic model of the universe. In its developing process, the universe passes through several eras corresponding to several values of ω which is the parameter of the equation of state ($p = \omega\rho$): the stiff fluid era ($\omega = 1$), the radiation dominated era ($\omega = 1/3$), the dust dominated era ($\omega = 0$), and the transition era ($\omega = -1/3$); then it tends to the DE dominated era ($\omega = -1$). In GR, the cosmological constant can be considered as the simplest candidate for DE, but it suffers from two theoretical problems, coincidence and cosmological constant.[15] A number of models with DE to explain the late-time cosmic acceleration without the cosmological constant have been proposed (for more details of this topic, readers are advised to read review[16] and references therein).

Einstein proposed the idea of teleparallelism in a trail to unify gravity and electromagnetism into a unified field theory in 1928.[17] Teleparallel space–time is characterized by an affine connection whose curvature is vanishing, but has torsion. In the past two decades, the geometry of absolute parallelism (AP) has attracted the attention of many researchers in two directions. The first comprises the development of the geometry,[18,19] while the second focuses on the physical applications of this geometry.[20,21] The main reason for the name of teleparallel, which means "parallel at a distance",[22] is that parallel transport of a vector depends on the path whose corresponding curvature is identically zero. It has been established that GR can, in fact, be re-constructed in teleparallel language.[23–29] The theory is known as the teleparallel equivalent of general relativity (TEGR). An interesting formulation of TEGR as a higher gauge theory can be found in Ref. [30]. To understand the acceleration of the universe, many theories of modified gravity have been introduced, among which is the attempt to generalize TEGR to $f(T)$ theory similar to the idea of generalizing GR to $f(R)$ gravity.[31–36]

For gravity theories which are invariant under both local Lorentz and diffeomorphism transformations, the tetrad and metric formulations are equivalent. The Lagrangian of TEGR is equivalent, up to a total divergence term, to the GR La-

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grangian and thus the two theories are classically equivalent. Moreover, T, which is the scalar of the torsion, is local Lorentz invariant only up to a divergence. However, the $f(T)$ theory is not local Lorentz invariant.[37]

Recently, many researchers have studied the Bianchi type I (BI) model in the presence of anisotropic DE. Rodrigues[38] constructed a BI ΛCDM cosmological model whose DE component preserves the non-dynamical character but yields an anisotropic vacuum pressure. Koivisto and Mota[39] proposed a different approach to resolve the CMB anisotropy problem; the earlier isotropy of the universe could be distorted by the direction-dependent acceleration of the later universe. Mota[40] investigated the BI cosmological model containing interacting DE fluid with non-dynamical anisotropic equation of state (EoS) and perfect fluid component. They suggested that if the EoS is anisotropic, the expansion rate of the universe becomes direction dependent at late time and the cosmological models with anisotropic EoS can explain some of the observed anomalies in CMB. The exact solution of BI is derived assuming some constraints on the coefficient of the second derivative of $f(T)$.[41] This solution gives a constant torsion scalar. Here in this study, we apply the field equations of $f(T)$ to the anisotropic homogenous model. Using the continuity equation, we derive a solution whose scalar torsion is constant.

It is the aim of the present study to apply the field equations of the $f(T)$ gravitational theory to a tetrad field having homogeneity and anisotropy and try to solve some of the above mentioned problems that cannot be solved within the framework of GR. In Section 2, a brief review of the $f(T)$ gravitational theory is presented. In Section 3, the tetrad field that has homogeneity and anisotropy is given and the application of the field equations of $f(T)$ to this tetrad is done. The resulting differential equations are solved using two different methods in Section 4. In Section 5, we use two fluids and try to find a solution to $f(T)$ and discuss some cosmological consequences. The final section is devoted to discussion.

2. Brief review of $f(T)$ formalism

The mathematical concept of the $f(T)$ gravity theory is based on the Weitzenböck geometry. Our convention and nomenclature are as follows. The Latin indices describe the components of the tangent space to the manifold (spacetime), while the Greek ones describe the components of the spacetime. For a general spacetime metric, we write the line element as $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ij}e^i{}_\mu e^j{}_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu$, where $\eta_{ij} = (-1,+1,+1,+1)$ is the Minkowski metric. $e^i{}_\mu$ is the covariant vector fields and its inverse $e_i{}^\mu$ is the contravariant vector fields, satisfying the orthogonality and unitary conditions $e^i{}_\mu e_j{}^\mu = \delta^i{}_j$ and $e^i{}_\mu e_i{}^\nu = \delta_\mu{}^\nu$. In a spacetime with absolute parallelism, the parallel vector fields $e_i{}^\mu$ define the nonsymmetric affine connection[42] $\Gamma^\lambda{}_{\mu\nu} = e_i{}^\lambda e^i{}_{\mu,\nu}$, where $e^i{}_{\mu,\nu} = \partial_\nu e^i{}_\mu$. The curvature tensor defined by the Weitzenböck connection, $\Gamma^\lambda{}_{\mu\nu}$, is identically vanishing. The torsion and the contortion components are defined as $T^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu} - \Gamma^\alpha{}_{\mu\nu} = e_a{}^\alpha\left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu\right)$ and $K^{\mu\nu}{}_\alpha = -\left(T^{\mu\nu}{}_\alpha - T^{\nu\mu}{}_\alpha - T_\alpha{}^{\mu\nu}\right)/2$.

The skew symmetric tensor $S_\mu{}^{\nu\rho}$ has the form $S_\mu{}^{\nu\rho} = \left(K^{\nu\rho}{}_\mu + \delta^\nu_\mu T^{\lambda\rho}{}_\lambda - \delta^\rho_\mu T^{\lambda\nu}{}_\lambda\right)/2$ and the torsion scalar is given by $T = T^\mu{}_{\nu\rho}S_\mu{}^{\nu\rho} = (1/4)T^{\mu\nu\rho}T_{\mu\nu\rho} + (1/2)T^{\mu\nu\rho}T_{\rho\nu\mu} - T_{\mu\nu}{}^\mu T^{\rho\nu}{}_\rho$. The action of the $f(T)$ theory is given by

$$\mathcal{S}(e^a{}_\mu,\Phi_A) = \int \mathrm{d}^4x\, e\left[\frac{M_{\mathrm{Pl}}^2}{2}(f(T)-2\Lambda) + \mathcal{L}_{\mathrm{Matter}}(\Phi_A)\right], \quad (1)$$

where $e = \sqrt{-g} = \det\left(e^a{}_\mu\right)$, $\mathcal{L}_{\mathrm{Matter}}$ is the Lagrangian of the matter field, Λ is the cosmological constant, and Φ_A are the matter fields. Here, M_{Pl} is the reduced Planck mass, which is related to the gravitational constant G by $M_{\mathrm{Pl}} = \sqrt{\hbar c/8\pi G}$. Similar to the $f(R)$ theory, one can define the action of the $f(T)$ theory as a function of the tetrad fields $e^a{}_\mu$, and by putting the variation of the function with respect to the tetrad fields $e^a{}_\mu$ to be vanishing, one can obtain the following equations of motion:

$$\begin{aligned} &S_\mu{}^{\nu\rho}T_{,\rho}\; f(T)_{TT} \\ &+\left[e^{-1}e^a{}_\mu\partial_\rho\left(ee_a{}^\alpha S_\alpha{}^{\nu\rho}\right) - T^\alpha{}_{\mu\lambda}S_\alpha{}^{\nu\lambda}\right]f(T)_T \\ &+\frac{1}{2}\delta^\nu_\mu(f(T)-2\Lambda) = 8\pi\mathcal{T}_\mu{}^\nu, \end{aligned} \quad (2)$$

where

$$T_{,\rho} = \frac{\partial T}{\partial x^\rho}, \quad f(T)_T = \frac{\partial f(T)}{\partial T}, \quad f(T)_{TT} = \frac{\partial^2 f(T)}{\partial T^2},$$

and $\mathcal{T}_\mu{}^\nu$ is the energy–momentum tensor.

Now we are going to rewrite the field equations (2) in another form. The field equations (2) are written in terms of the tetrad and its partial derivatives. These equations appear to be different from Einstein's field equations. Following Refs. [43]–[45], one can obtain an equation relating the scalar torsion T to the Ricci scalar R. These will make the equivalence between teleparallel gravity and GR clear. On the other hand, the tetrad cannot be eliminated completely in favor of the metric in Eq. (2), because of the lack of the local Lorentz symmetry, but one can show that the latter can be brought in a form that closely resembles Einstein's equation. This form is more suitable for constructing analytic solutions in the $f(T)$ theory. To write the field equations in a covariant version, we must replace the partial derivatives in the tensors by covariant derivatives compatible with the metric $g_{\mu\nu}$, i.e., ∇_σ, where $\nabla_\sigma g_{\mu\nu} = 0$. Thus, the definition of the torsion tensor can be written as

$$T^\mu{}_{\nu\rho} = e_a{}^\mu(\nabla_\nu e^a{}_\rho - \nabla_\rho e^a{}_\nu). \quad (3)$$

Using Eq. (3) in the definition of contorsion and the skew symmetric tensor $S^{\mu\nu}{}_{\rho}$, one can obtain

$$\begin{aligned} K^{\mu}{}_{\nu\rho} &= e_a{}^{\mu}\nabla_{\rho}e^{a}{}_{\nu}, \\ S^{\mu\nu}{}_{\rho} &= \eta^{ij}e_i{}^{\mu}\nabla_{\rho}e_j{}^{\nu}+\delta^{\nu}{}_{\rho}\eta^{ij}e_i{}^{\sigma}\nabla_{\sigma}e_j{}^{\mu} \\ &\quad -\delta^{\mu}{}_{\rho}\eta^{ij}e_i{}^{\sigma}\nabla_{\sigma}e_j{}^{\nu}. \end{aligned} \tag{4}$$

On the other hand, from the relation between the Weitzenböck connection and the Levi–Civita connection, one can write the Riemann tensor for the Levi–Civita connection, $\{{}^{\rho}_{\mu\nu}\}$, in terms of the non-symmetric connection, $\Gamma^{\rho}{}_{\mu\nu}$, in the form

$$\begin{aligned} R^{\rho}{}_{\mu\lambda\nu} &= \partial_{\lambda}\Gamma^{\rho}{}_{\mu\nu}-\partial_{\nu}\Gamma^{\rho}{}_{\mu\lambda}+\Gamma^{\rho}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu}-\Gamma^{\rho}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda} \\ &= \nabla_{\lambda}K^{\rho}{}_{\mu\nu}-\nabla_{\nu}K^{\rho}{}_{\mu\lambda}+K^{\rho}{}_{\sigma\lambda}K^{\sigma}{}_{\mu\nu} \\ &\quad -K^{\rho}{}_{\sigma\nu}K^{\sigma}{}_{\mu\lambda}. \end{aligned} \tag{5}$$

The associated Ricci tensor can then be written as

$$R_{\mu\nu}=\nabla_{\nu}K^{\rho}{}_{\mu\rho}-\nabla_{\rho}K^{\rho}{}_{\mu\nu}+K^{\rho}{}_{\sigma\nu}K^{\sigma}{}_{\mu\rho}-K^{\rho}{}_{\sigma\rho}K^{\sigma}{}_{\mu\nu}. \tag{6}$$

Now, by using the definition of the contorsion along with the relations $K^{(\mu\nu)\sigma}=T^{\mu(\nu\sigma)}=S^{\mu(\nu\sigma)}=0$ and considering that $S^{\mu}{}_{\rho\mu}=2K^{\mu}{}_{\rho\mu}=-T^{\mu}{}_{\rho\mu}$, one has[46–52]

$$\begin{aligned} R_{\mu\nu} &= -\nabla^{\lambda}S_{\nu\lambda\mu}-g_{\mu\nu}\nabla^{\lambda}T^{\rho}{}_{\lambda\rho}-S^{\rho\sigma}_{\mu}K_{\sigma\rho\nu}, \\ R &= -T-2\nabla^{\mu}T^{\rho}{}_{\mu\rho}. \end{aligned} \tag{7}$$

Equation (7) implies that the torsion scalar T and Ricci scalar R differ only by a covariant divergence of a space–time vector. Therefore, the Einstein–Hilbert action and the teleparallel action (i.e. $\mathcal{L}_{\rm TEGR}=\int \mathrm{d}^4x|e|T$) will both lead to the same field equations and are dynamically equivalent theories. However, this divergence term is the main reason that makes the field equations of $f(T)$ non invariance under local Lorentz transformation (LLT). Let us explain this for some specific form of $f(T)$. If

$$\begin{aligned} f(R) &= R+\varepsilon R^2\equiv\left[-T-2\nabla^{\mu}T^{\rho}{}_{\mu\rho}\right] \\ &\quad +\varepsilon\left[-T-2\nabla^{\mu}T^{\rho}{}_{\mu\rho}\right]^2 \\ &= -[T-\varepsilon T^2]-(1+\varepsilon)\nabla^{\mu}T^{\rho}{}_{\mu\rho} \\ &\quad +4\varepsilon\left\{\left[\nabla^{\mu}T^{\rho}{}_{\mu\rho}\right]^2+T\nabla^{\mu}T^{\rho}{}_{\mu\rho}\right\}, \end{aligned} \tag{8}$$

the last term in the right hand side of Eq. (8) is not a total derivative term and therefore it is responsible for making $f(R)=R+R^2$ when written in terms of T and T^2 not invariant under LLT in contrast to the linear case, i.e., the form of Eq. (7). The same discussion can be applied to the general forms of $f(R)$ and $f(T)$, which shows in general a difference between $f(R)$ and $f(T)$ gravitational theories. It is known that the field equations of $f(R)$ are of fourth order and invariant under LLT while the field equations of $f(T)$ are of second order and not invariant under LLT. Therefore, if the divergence term vanishes, then $R=-T$, hence one can show that any solution in TEGR can be a solution to GR; and for a general case, one can also show that any solution of $f(T)$ can be a solution to $f(R)$. This is one the the aims of this study, which is to show that in the spherically symmetric case, any solution of $f(T)$ whose divergence term vanishes will also be a solution to the $f(R)$ theory.

By using the equations listed above and after some algebraic manipulations, one can obtain

$$G_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T=-\nabla^{\rho}S_{\nu\rho\mu}-S^{\sigma\rho}{}_{\mu}K_{\rho\sigma\nu}, \tag{9}$$

where $G_{\mu\nu}=R_{\mu\nu}-g_{\mu\nu}R/2$ is the Einstein tensor. Finally, by using Eq. (9), the field equations of the $f(T)$ gravity, equation (2), can be rewritten in the form

$$\begin{aligned} &f(T)_T G_{\mu\nu}+\frac{1}{2}\left(f(T)-Tf(T)_T\right)g_{\mu\nu} \\ &+S_{\mu}{}^{\rho\nu}T_{,\rho}\, f(T)_{TT}=4\pi T^{\nu}{}_{\mu}. \end{aligned} \tag{10}$$

Equation (10) can be taken as the starting point of the $f(T)$ modified gravity model and it has a structure similar to the field equations of the $f(R)$ gravity. Note that in the more general case with $f(T)\neq T$, the field equations are in covariant form. Nevertheless, the theory is not local Lorentz invariant. In case of $f(T)=T$ and constant torsion, $f(T_0)$, GR is recovered and the field equations are covariant and the theory is Lorentz invariant. Since the principle of homogeneity and isotropy is not valid near the big bang, we will apply the modified field equations (10) to a homogenous and anisotropic model and discuss their relevant physics in the next sections.

3. Homogenous and anisotropic tetrad field

The spatially homogeneous and anisotropic, Bianchi type I (BI), universe which has transverse direction x and two equivalent longitudinal directions y and z is given by[53]

$$\mathrm{d}s^2=\mathrm{d}t^2-A^2(t)\mathrm{d}x^2-B^2(t)(\mathrm{d}y^2+\mathrm{d}z^2), \tag{11}$$

where $A(t)$ and $B(t)$ are the cosmic scale factors. For $A(t)=B(t)=a(t)$, this reduces to the flat FRW spacetime. The diagonal tetrad components of BI have the form[13]

$$h^{i}{}_{\mu}=\mathrm{diag}(1,A(t),B(t),B(t)). \tag{12}$$

The dual of Eq. (12) has the form

$$h^{\mu}{}_{i}=\mathrm{diag}(1,A(t)^{-1},B(t)^{-1},B(t)^{-1}). \tag{13}$$

Using Eqs. (3), (4), and (12), we obtain the torsion scalar in the form

$$T=-2\left(2\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}\right), \tag{14}$$

where $\dot{A}\equiv \mathrm{d}A(t)/\mathrm{d}t$ and $\dot{B}\equiv \mathrm{d}B(t)/\mathrm{d}t$. We define some physical quantities for the BI model which are important in cosmological observations. The average scale factor R, the mean

Hubble parameter H, and the anisotropic parameter have the forms

$$R=(AB^2)^{1/3},\quad H=\frac{1}{3}\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right),$$
$$\Delta=\frac{1}{3}\Sigma_{i=1}^{3}\left(\frac{H_i-H}{H}\right)^2, \tag{15}$$

where $H_1=\dot{A}/A$ and $H_2=\dot{B}/B=H_3$ are the directional Hubble parameters along x, y, and z axes. We assume the energy–momentum tensor $T^{\mu}{}_{\nu}$ to be

$$T^{\mu}{}_{\nu}=\mathrm{diag}(-\rho_{\mathrm{m}},P_{\mathrm{m1}},P_{\mathrm{m2}},P_{\mathrm{m2}}), \tag{16}$$

where ρ_{m}, P_{m1}, P_{m2}, and P_{m2} are the energy density and the pressures of matter along x, y, and z directions. The non-vanishing components of the field equations (10) are

$$f(T)-2Tf_T=2\kappa^2\rho_{\mathrm{m}}, \tag{17}$$
$$-f(T)-4\left(\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}+\frac{\ddot{B}}{B}\right)f_T$$
$$-4\frac{\dot{B}}{B}\dot{T}f_{TT}=2\kappa^2P_{\mathrm{m1}}, \tag{18}$$
$$-f(T)-2\left(\frac{3\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}+\frac{\ddot{B}}{B}+\frac{\ddot{A}}{A}\right)f_T$$
$$-2\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)\dot{T}f_{TT}=2\kappa^2P_{\mathrm{m2}}. \tag{19}$$

Equations (17)–(19) reduce to those of TEGR when $f(T)=T$ and reduce to the FRW model when $A=B=R(t)$. Equations (17)–(19) tell us that we have six unknowns ρ_{m}, P_{m1}, P_{m2}, $f(T)$, $A(t)$, and $B(t)$, therefore, we need three extra conditions to make the system closed. This will be discussed in the next section by using two different methods.

4. Solution using continuity equation

Using Eqs. (17)–(19) in the continuity equation, $T^{\mu\nu}{}_{;\nu}=0$, we obtain

$$\dot{\rho_{\mathrm{m}}}+\left[\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right]\rho_{\mathrm{m}}+\frac{\dot{A}}{A}P_{\mathrm{m1}}+2\frac{\dot{B}}{B}P_{\mathrm{m2}},$$
$$=\left[4\left\{\frac{\dot{B}^2}{B^2}+2\frac{\dot{A}\dot{B}}{AB}\right\}f_{TT}+f_T\right]$$
$$\times\left(\frac{\dot{B}^2\dot{A}}{B^2A}-\frac{\dot{B}\ddot{B}}{B^2}-\frac{\dot{B}^3}{B^3}-\frac{\dot{B}\dot{A}^2}{A^2B}-\frac{\dot{A}\ddot{B}-\ddot{A}\dot{B}}{AB}\right)=0. \tag{20}$$

The second bracket contains two unknown functions $A(t)$ and $B(t)$, therefore, we need an extra condition to solve it. Here we will use two different procedures that have been used in the literature. Other conditions may give interesting results, which may be considered in future work.

4.1. First procedure assuming $A(t)=B(t)^m$

For a spatial homogeneous metric, the normal congruence to homogeneous expansion implies that the expansion scalar θ is proportional to the shear scalar σ[54–62]

$$\theta\propto\sigma, \tag{21}$$

where

$$\theta=\frac{\dot{A}}{A}+2\frac{\dot{B}}{B},\quad \sigma^2=\frac{1}{2}\left(\sum_{i=1}^{3}H^2{}_i-\frac{1}{3}\theta^2\right). \tag{22}$$

Using Eq. (22), we obtain the expansion scalar and the shear scalar for the BI universe in the following forms

$$\theta=\frac{\dot{A}}{A}+2\frac{\dot{B}}{B},\quad \sigma=\frac{1}{\sqrt{3}}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right), \tag{23}$$

which lead to

$$A(t)=B(t)^m. \tag{24}$$

By using the above condition, the anisotropy parameter of the expansion is found to be

$$\Delta=2\frac{(m-1)^2}{(m+1)^2}. \tag{25}$$

Equation (25) shows that when $m=1$, we obtain $A(t)=B(t)$, i.e., the isotropic case. The isotropic behavior of the expanding universe is obtained for $\Delta=0$, i.e., $m=1$.

By using Eq. (24), the solution of the second bracket of Eq. (20) for the unknown functions $A(t)$ and $B(t)$ has the form

$$B(t)=\mathrm{e}^{\frac{(t-c_3)\sqrt{2c_4}}{\sqrt{2m+1}}},\quad A(t)=\mathrm{e}^{\frac{m(t-c_3)\sqrt{2c_4}}{\sqrt{2m+1}}}, \tag{26}$$

where c_3 and c_4 are another two constants of integration. Using Eq. (26) in Eq. (14), we obtain a constant torsion scalar. Substituting Eq. (26) into the field equations (17)–(19), we obtain

$$P_{\mathrm{m1}}=P_{\mathrm{m2}}=-\rho_{\mathrm{m}}=\text{constant}, \tag{27}$$

which is similar to the de Sitter solution. Equation (27) informs us that this model possesses a dark energy since $\omega=-1$, where

$$\omega=\frac{\omega_1+\omega_2}{2},\quad \omega_1=\frac{P_{\mathrm{m1}}}{\rho_{\mathrm{m}}},\quad \text{and}\quad \omega_2=\frac{P_{\mathrm{m2}}}{\rho_{\mathrm{m}}}. \tag{28}$$

4.2. Second procedure assuming $A(t)=c_1'\mathrm{e}^{3H_0t}/B^2$

Recall the definition of the mean Hubble parameter mentioned in Eq. (15)

$$H=\frac{1}{3}\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)=\frac{1}{3}\frac{\dot{V}}{V}, \tag{29}$$

where V is the spatial volume of the universe

$$V=AB^2. \tag{30}$$

Using the following volumetric expansion law:[61]

$$V=c_1'\,\mathrm{e}^{3H_0t}, \tag{31}$$

where c_1' and H_0 are two positive constants, we obtain

$$A(t)=\frac{c_1'\,\mathrm{e}^{3H_0t}}{B^2}. \tag{32}$$

Equation (32) reduces to the isotropic case when $A(t) = B(t)$. Using Eq. (32) in the second bracket of Eq. (20), we obtain

$$B(t) = \mathrm{e}^{\frac{1}{3}(t-c_2')(3H_0+\sqrt{9H_0^2+6c_1'})}, \tag{33}$$

where c_2' is another constant of integration. Substituting Eq. (33) into Eq. (32), we obtain

$$A(t) = c_1' \mathrm{e}^{3H_0 t+\frac{2}{3}(c_2'-t)(3H_0+\sqrt{9H_0^2+6c_1'})}. \tag{34}$$

Following the same steps of the first procedure, we obtain

$$\omega = -1. \tag{35}$$

The density and the pressures of this model are also constants.

In the above two models, we have many problems: (i) constant scalar torsion which is responsible for making the energy density and the pressure constant, (ii) violation of conservation in spite of using Eq. (20). So we will use another method to try to overcome the above problems.

5. Solution with two fluids

Here, we will use two equations of state, i.e., two fluids, and the relation between the scale factors given by Eq. (24) is as follows:

$$P_{\mathrm{m1}} = \omega_1 \rho_{\mathrm{m}}, \quad P_{\mathrm{m2}} = \omega_2 \rho_{\mathrm{m}}, \tag{36}$$

where ω_1 and ω_2 can take the values 0 ,1, or -1. For each value of ω_1, ω_2 can take the values -1 ,0 ,1, through that, we can obtain different models. Here, we are interested in the physical case in which $\omega_1 = -1$ and $\omega_2 = 0$. Substituting Eq. (36) into the field equations (17)–(19), we obtain three differential equations in three unknowns $[\rho_{\mathrm{m}}, f(T), B(t)]$

$$\begin{aligned}
&f(T) - 2T f_T = 2\kappa^2 \rho_{\mathrm{m}},\\
&-2\left(\frac{\dot{B}^2}{B^2} + 2\frac{\ddot{B}}{B}\right) f_T - 4\frac{\dot{B}}{B}\dot{T} f_{TT} = 0,\\
&f(T) - 2(m+1)\left[\left((m+1)\frac{\dot{B}^2}{B^2} + \frac{\ddot{B}}{B}\right) f_T + 2\frac{\dot{B}}{B}\dot{T} f_{TT}\right] = 0.
\end{aligned} \tag{37}$$

As $f(T)$ in Bianchi type I spacetime is a function of time $f(T \to t)$, one easily can show that

$$f_T = \dot{f}/\dot{T}, \quad f_{TT} = \left(\dot{T}\ddot{f} - \ddot{T}\dot{f}\right)/\dot{T}^3. \tag{38}$$

Substituting Eq. (38) into Eq. (37), then solving to $f(T \to t)$, and after some manipulations, we obtain the solution of Eq. (37) in the form

$$B(t) = \left(\frac{1}{(m+1)(c_2''t + c_3'')}\right)^{\frac{-1}{m+1}}, \tag{39}$$

$$T(t) = -\frac{2c''(2m+1)}{(m+1)^2(c_2''t + c_3'')^2},$$

$$f(t) = c'''\left(\frac{1}{(m+1)(c_2''t + c_3'')}\right)^{\frac{2}{m+1}}, \tag{40}$$

$$\rho_{\mathrm{m}} = \frac{c_1''(m-1)}{4(m+1)}\left(\frac{1}{(m+1)(c_2''t + c_3'')}\right)^{\frac{2}{m+1}}, \tag{41}$$

where c''', c''', c_2'', and c_3'' are three constants of integration. Substituting into Eqs. (18) and (19), we obtain

$$P_{\mathrm{m1}} = -\frac{c_1''(m-1)}{4(m+1)}\left(\frac{1}{(m+1)(c_2''t + c_3'')}\right)^{\frac{2}{m+1}}, \tag{42}$$

$$P_{\mathrm{m2}} = 0. \tag{43}$$

Using the relation between scale factors $A(t)$ and $B(t)$, i.e., Eq. (22), we obtain

$$A(t) = \left(\frac{1}{(m+1)(c_2''t + c_3'')}\right)^{\frac{-m}{m+1}}. \tag{44}$$

So far, all the six unknowns are obtained.

To see if the above model is consistent with the observations, i.e., it has an acceleration, we calculate the deceleration parameter and obtain

$$q = -\left[\frac{A(t)\ddot{A}(t)}{\dot{A}^2(t)}\left(\frac{B(t)\ddot{B}(t)}{\dot{B}^2(t)}\right)^2\right]^{1/3} = \sqrt[3]{m}. \tag{45}$$

Equation (45) informs us that the deceleration will depend on the parameter m and this parameter is not allowed to take -1 and $-1/2$, i.e., $m \neq -1$ or $-1/2$, because these values make the scale factors and the scalar torsion ill defined. Therefore, to obtain an accelerated model with two fluids, the parameter m can take any negative values except -1 or $-1/2$.

6. Results and discussion

In the present research, we have studied a spatially homogeneous and anisotropic BI universe. We use the continuity equation and write it as a product of two brackets. We assume the vanishing of the second bracket, which contains two unknown functions $A(t)$ and $B(t)$. To solve this bracket, we assume two different relations between these two unknown functions. First, we use the relation $A(t) = B(t)^m$, where $m \geqslant 2$, and derive a solution for $B(t)$ and consequently $A(t)$. Second, we use the relation $A(t) = c_1{}' \mathrm{e}^{3H_0 t}/B(t)^2$ and repeat the same procedure to obtain the two unknown functions $A(t)$ and $B(t)$.

The most important property of the derived solutions of $A(t)$ and $B(t)$ is that the torsion scalar is a constant. The density and the pressures of these solutions are constants. We have calculated some cosmological parameters, such as the deceleration parameter. We show that our model is accelerating and its EoS parameter $\omega = -1$. This means that $f(T)$ in both cases can be regarded as a cosmological constant.[62] The above cosmological parameters show that the scale factors (26), (33), and (34) grow exponentially with time; however, the universe constituents do not change with time. This does not allow the universe to evolve. However, the universe shows an accelerated expansion. Equation (27) and continuity (20) lead to the

conclusion that the total density has a constant value, nevertheless, the universe is expanding! This leads directly to a violation of the conservation principle of energy.

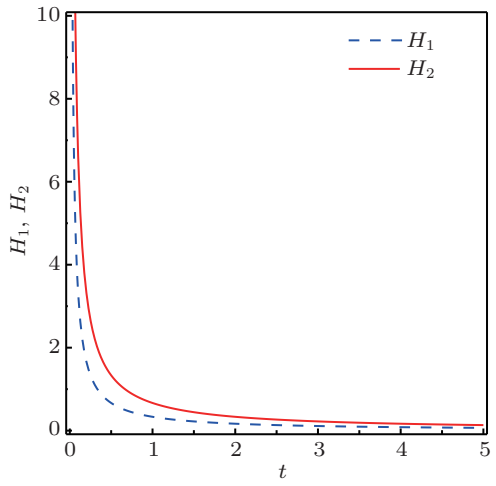


Fig. 1. (color online) Evolution of Hubble parameters $H_1 = \dot{A}(t)/A(t)$ and $H_2 = \dot{B}(t)/B(t)$, where $m = 2$, $c_2'' = 1$, and $c_3'' = 0$.

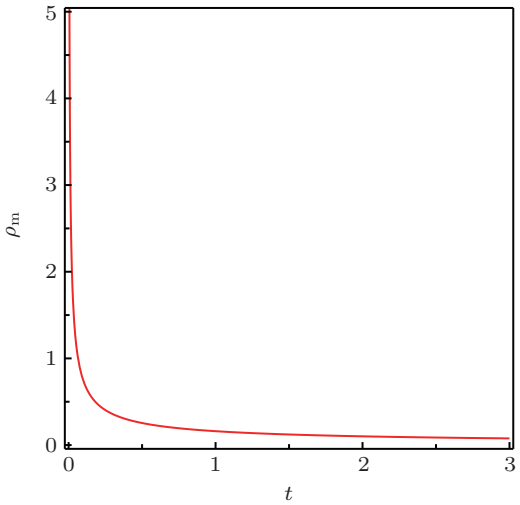


Fig. 2. (color online) Evolution of the density $\rho_{\rm m}$ in Eq. (39), where $m = 2$, $c_1'' = 1$, $c_2'' = 1$, and $c_3'' = 0$.

Another method has been considered using two different equations of state and a relation between the scale factors $A(t)$ and $B(t)$. This method enables us to derive the forms of $f(T)$, $A(t)$, $B(t)$, $\rho_{\rm m}$, $P_{\rm m1}$, and $P_{\rm m2}$ as in Eqs. (39)–(44). These forms depend on the cosmic time and have evolution as shown in Figs. 1–3. One can notice that density is decreasing with time and the pressure is increasing but with negative sign, which indicates the existence of pressure against gravity causing accelerating expansion as shown in the recent observations.[4,7,63–66] Equation (38) enables us to write the time mathematically in terms of the torsion scalar, i.e., $t = \tau_0 + \frac{\sqrt{-2(2m+1)}}{\sqrt{T(m+1)}}$; then we re-express Eq. (40) in terms of the torsion scalar as inverse power-law of T as

$$f(T) = c\left(\frac{1}{\sqrt{-2T(2m+1)}}\right)^{2/(1+m)}. \qquad (46)$$

Note that we give $f(T)$ in terms of the torsion scalar to explore $f(T)$ in its usual form. But all the calculations in this work are performed using the time dependent form (38). Equation (46) reduces to the isotropic case when $m = 1$.[32] For the anisotropic case, $f(T)$ can represent CDM when $m \geq 2$. Bengochea *et al.*[28] have investigated the observational information for the model $f(T) = T - \alpha/(-T^n)$ by using the most recent SN Ia + BAO + CMB[64–66] data and they found that the values lie in the range $n \in [0.23, 0.03]$. When $m = 3$, the behavior of $f(T)$ given by Eq. (46) will be generally coincident with that of Ref. [29].

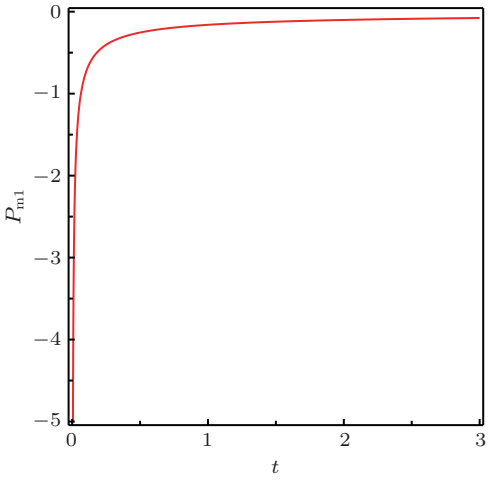


Fig. 3. (color online) Evolution of the pressure $P_{\rm m1}$ in Eq. (42), where $m = 2$, $c_1'' = 1$, $c_2'' = 1$, and $c_3'' = 0$.

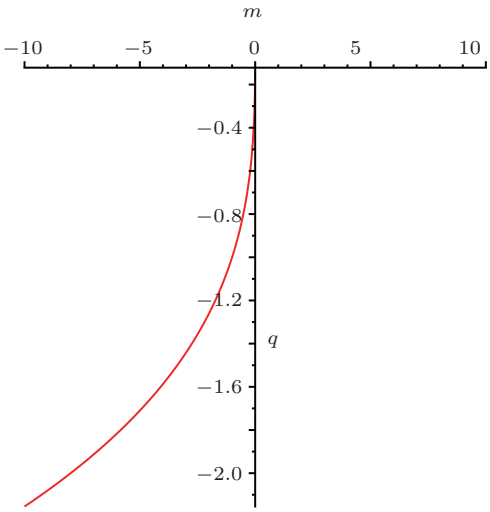


Fig. 4. (color online) Behavior of the deceleration parameter. It is important to note that the parameter m must not take the values -1 and $-1/2$.

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