Path integral approach to electron scattering in classical electromagnetic potential*

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(Received 23 August 2015; revised manuscript received 18 December 2015; published online 4 April 2016)

As is known to all, the electron scattering in classical electromagnetic potential is one of the most widespread applications of quantum theory. Nevertheless, many discussions about electron scattering are based upon single-particle Schrodinger equation or Dirac equation in quantum mechanics rather than the method of quantum field theory. In this paper, by using the path integral approach of quantum field theory, we perturbatively evaluate the scattering amplitude up to the second order for the electron scattering by the classical electromagnetic potential. The results we derive are convenient to apply to all sorts of potential forms. Furthermore, by means of the obtained results, we give explicit calculations for the one-dimensional electric potential.

Keywords: electron scattering, path integral approach, quantum field theory

PACS: 03.65.–w, 03.70.+k **DOI:** 10.1088/1674-1056/25/5/050303

1. Introduction

The original concept of the path integral approach may stem from the Wiener integral, which was proposed by Norbert Wiener in order to study the stochastic process of Brownian motion. In 1933, Dirac[1] extended Wiener's idea to bring in quantum mechanics for establishing the action principle and its Lagrangian formalism.

The thorough and complete path integral approach was developed by Richard Feynman[2] in 1948. In 1963, Feynman expressed the gravitational interaction of two particles by means of an interchange of a virtual graviton.[3] The mathematical technique of the path integral approach was first applied to non-Abelian gauge fields. In 1967, Faddeev and Popov[4] utilized the path integral approach to obtain the detailed quantization rules which are observed to calculate the contribution from arbitrary Feynman diagrams in the theory of SU(2) gauge invariant fields. In 1971, with the help of the path integral approach and dimensional regularization method, Hooft[5] successfully demonstrated that massive Lagrangians for non-Abelian gauge theories without anomalies are renormalizable. At present, the Higgs boson,[6–9] which is the last elementary particle with a mass near 125 GeV predicted by the standard model of particle physics, had been discovered in the ATLAS and CMS experiments at CERN's Large Hadron Collider on 4 July 2012, and all observational evidence points to the fact that quantum field theory provides an accurate description of all known elementary particles. In retrospect, it is clearly shown that the path integral approach is of great significance for the development of quantum field theory.

On the other hand, since the first ruby laser was invented in 1960 by Maiman at Hughes Laboratories, particularly with the rapid development of chirped pulse amplification, laser power has achieved tremendous growth. Nowadays, it has been reported that the present world highest laser intensity is the HERCULES laser produced by the University of Michigan, whose focused laser intensity has achieved as high as 2×10^{22} W/cm^2.[10] Consequently, a large number of theoretical and experimental phenomena which are involved in the processes of the electron scattering in the strong laser field, such as acceleration of electrons,[11,12] Thomson backscattering of optical laser light off relativistic electrons,[13–16] and nonlinear Compton scattering,[17–21] need to be discussed further.

Therefore, seeing that many new operator methods have witnessed a boom in quantum mechanics, such as the operator-Hermite-polynomial method,[22] as far as the application of the method is concerned, we believe that it is crucial to treat the classical problem of the electron scattering with the path integral approach which plays a central role in quantum field theory. The main merit of the path integral approach is that it provides a comparatively convenient road to quantization and to expression for correlation function, which is closely related to amplitudes for electron scattering processes.

This paper is organized as follows. In Section 2, we introduce the Grassmann algebra, generating functional and correlation functions in the path integral approach of quantum field theory. In Section 3, we apply the perturbation technique to calculate the correlation function up to the second order.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11374360, 11405266, and 11505285) and the National Basic Research Program of China (Grant No. 2013CBA01504).

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Based on Lehmann–Symanzik–Zimmermann (LSZ) reduction formula, we derive a general result for the electron scattering off the classical electromagnetic potential. In Section 4, we shall present an explicit calculation that is the electron scattering off the one-dimensional time-independent electric potential to illustrate the use of the results presented in Section 3. In Section 5, we shall discuss a heuristic application of the Gaussian barrier. Finally, in Section 6, we shall give a brief summary.

2. Path integral treatment

The foundational component of path integral formalism is the generating functional of correlation functions.[23] For the spinor field, it reads

$$Z[\bar{\eta},\eta] = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi \exp\{\mathrm{i}S_0[\bar{\psi},\psi] + \mathrm{i}S_{\mathrm{int}}[\bar{\psi},\psi] + \mathrm{i}S_{\mathrm{source}}[\bar{\psi},\psi,\bar{\eta},\eta]\}, \quad (1)$$

where S_0, S_{int}, and S_{source} are free action, interactional action, and source action, respectively, ψ and η are Grassmann fields whose values are anticommuting numbers. For the purpose of describing the interaction of spinor field and electromagnetic field, we choose $S_{\mathrm{int}}[\bar{\psi},\psi]$ as

$$S_{\mathrm{int}}[\bar{\psi},\psi] = -e\int \mathrm{d}^4 z A_\mu(z)\bar{\psi}(z)\gamma^\mu\psi(z). \quad (2)$$

To begin, we need to refer to the definition of the functional derivatives, $\delta/\delta\eta$ and $\delta/\delta\bar{\eta}$, as follows. With respect to a four-dimensional case, the functional derivatives obey the conditions

$$\frac{\delta}{\delta\eta(x)}\eta(y) = \delta^{(4)}(x-y), \quad (3)$$

and

$$\frac{\delta}{\delta\bar{\eta}(x)}\bar{\eta}(y) = \delta^{(4)}(x-y). \quad (4)$$

In addition, if η and $\bar{\eta}$ are Grassmann numbers, they satisfy

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\eta}\eta = -\frac{\mathrm{d}}{\mathrm{d}\eta}\eta\bar{\eta} = -\bar{\eta}. \quad (5)$$

To evaluate the path integral $Z[\bar{\eta},\eta]$ more generally, we must split up the exponential into the interaction and free term including source. Meanwhile, to fix the problem, we can use a trick that replaces the fields $\bar{\psi}$ and ψ in the interaction part by functional derivatives. It would be nice if we can pull the interaction part out of the integral and perform the remaining integral. Then, equation (1) becomes

$$Z[\bar{\eta},\eta] = \exp\left\{\mathrm{i}S_{\mathrm{int}}\left[+\mathrm{i}\frac{\delta}{\delta\eta},-\mathrm{i}\frac{\delta}{\delta\bar{\eta}}\right]\right\}Z_0[\bar{\eta},\eta], \quad (6)$$

where $Z_0[\bar{\eta},\eta]$ is the generating functional of the free spinor field

$$Z_0[\bar{\eta},\eta] = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi \exp\left\{\mathrm{i}\int \mathrm{d}^4x[\bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi + \eta\bar{\psi} + \psi\bar{\eta}]\right\} = Z_0[0,0]\exp\left\{-\int \mathrm{d}^4x\mathrm{d}^4y\bar{\eta}(x)S_{\mathrm{F}}(x-y)\eta(y)\right\}. \quad (7)$$

Notice that $Z_0[0,0]$ actually is the Gaussian integral and can be performed precisely. So as to simplify the subsequent calculations, we will drop it directly. Therefore, we redefine it as

$$Z_0[\bar{\eta},\eta] = \exp\left\{-\int \mathrm{d}^4x\mathrm{d}^4y\bar{\eta}(x)S_{\mathrm{F}}(x-y)\eta(y)\right\}, \quad (8)$$

where $S_{\mathrm{F}}(x-y)$ is called the Feynman propagator for the spinor field. In position space, it is written as

$$S_{\mathrm{F}}(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(p\!\!\!/ + m)}{p^2 - m^2 + \mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}p(x-y)}. \quad (9)$$

On the basis of path integral formulism, the two-point correlation function for the spinor field is given by

$$\begin{aligned}&\langle\Omega|T\psi_{\mathrm{H}}(x_1)\bar{\psi}_{\mathrm{H}}(x_2)|\Omega\rangle \\ &= \frac{\int \mathscr{D}\bar{\psi}\mathscr{D}\psi\ \psi(x_1)\bar{\psi}(x_2)\exp\{\mathrm{i}S[\bar{\psi},\psi]\}}{\int \mathscr{D}\bar{\psi}\mathscr{D}\psi\exp\{\mathrm{i}S[\bar{\psi},\psi]\}} \\ &= \frac{(-\mathrm{i}\delta/\delta\bar{\eta}(x_1))(+\mathrm{i}\delta/\delta\eta(x_2))Z[\bar{\eta},\eta]}{Z[\bar{\eta},\eta]}\bigg|_{\bar{\eta},\eta=0}, \end{aligned} \quad (10)$$

where $|\Omega\rangle$ denotes the vacuum state of interacting theory, T is called time-ordered product, which instructs us to place the operators that follow in order with the latest to the left. The subscript H means working in the Heisenberg picture.

Consider Eq. (10), in fact, the disconnected diagrams in the numerator can be just canceled by the denominator, only the connected diagrams can make contributions to the correlation function. In other words, for the two-point correlation function, we can simply sum all connected diagrams with two external points. Now it reads

$$\begin{aligned}&\langle\Omega|T\psi_{\mathrm{H}}(x_1)\bar{\psi}_{\mathrm{H}}(x_2)|\Omega\rangle \\ &= \left(-\mathrm{i}\frac{\delta}{\delta\bar{\eta}(x_1)}\right)\left(+\mathrm{i}\frac{\delta}{\delta\eta(x_2)}\right)Z[\bar{\eta},\eta]\bigg|_{\bar{\eta},\eta=0}^{\mathrm{connected}} \\ &= \frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)} \\ &\quad\times\exp\left\{\mathrm{i}S_{\mathrm{int}}\left[+\mathrm{i}\frac{\delta}{\delta\eta},-\mathrm{i}\frac{\delta}{\delta\bar{\eta}}\right]\right\}Z_0[\bar{\eta},\eta]\bigg|_{\bar{\eta},\eta=0}^{\mathrm{connected}}. \end{aligned} \quad (11)$$

3. Perturbative expansion for the generating functional

To compute $\langle\Omega|T\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle$, we would like to expand the generating functional $Z[\bar{\eta},\eta]$ perturbatively with respect to the powers of S_{int},

$$Z[\bar{\eta},\eta] = \left\{1 + \mathrm{i}S_{\mathrm{int}}\left[+\mathrm{i}\frac{\delta}{\delta\eta},-\mathrm{i}\frac{\delta}{\delta\bar{\eta}}\right]\right.$$

$$-\frac{1}{2}S_{\rm int}\left[+{\rm i}\frac{\delta}{\delta\eta},-{\rm i}\frac{\delta}{\delta\bar{\eta}}\right]^2+\cdots\Big\}Z_0[\bar{\eta},\eta]. \quad (12)$$

The zero-order term in the expansion of Eq. (11) is given by

$$\begin{aligned}&\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}Z_0[\bar{\eta},\eta]\Big|^{\rm connected}_{\bar{\eta},\eta=0}\\&=\frac{\delta}{\delta\bar{\eta}(x_1)}\left[\int{\rm d}^4x\bar{\eta}(x)S_{\rm F}(x-x_2)\right]Z_0[\bar{\eta},\eta]\Big|^{\rm connected}_{\bar{\eta},\eta=0}\\&=S_{\rm F}(x_1-x_2).\end{aligned} \quad (13)$$

It is nothing but the Feynman propagator.

For the first-order term, it reads

$$\begin{aligned}&\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}({\rm i}S_{\rm int}[\bar{\psi},\psi])Z_0[\bar{\eta},\eta]\Big|^{\rm connected}_{\bar{\eta},\eta=0}\\&=-{\rm i}e\int{\rm d}^4zA_\mu(z)\frac{\delta}{\delta\bar{\eta}(x_1)}\\&\quad\times\frac{\delta}{\delta\eta(x_2)}\bar{\psi}(z)\gamma^\mu\psi(z)Z_0[\bar{\eta},\eta]\Big|^{\rm connected}_{\bar{\eta},\eta=0}.\end{aligned} \quad (14)$$

We replace the fields $\bar{\psi}$ and ψ with functional derivatives. Thus, equation (14) becomes

$$\begin{aligned}&-{\rm i}e\int{\rm d}^4zA_\mu(z)\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\\&\quad\times\left(+{\rm i}\frac{\delta}{\delta\eta(z)}\right)\gamma^\mu\left(-{\rm i}\frac{\delta}{\delta\bar{\eta}(z)}\right)Z_0[\bar{\eta},\eta]\Big|^{\rm connected}_{\bar{\eta},\eta=0}\\&=-{\rm i}e\int{\rm d}^4zA_\mu(z)\gamma^\mu\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\\&\quad\times\frac{\delta}{\delta\eta(z)}\frac{\delta}{\delta\bar{\eta}(z)}Z_0[\bar{\eta},\eta]\Big|^{\rm connected}_{\bar{\eta},\eta=0}\\&=-{\rm i}e\int{\rm d}^4zA_\mu(z)\gamma^\mu\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\frac{\delta}{\delta\eta(z)}\frac{\delta}{\delta\bar{\eta}(z)}\\&\quad\times\exp\left\{-\int{\rm d}^4x{\rm d}^4y\bar{\eta}(x)S_{\rm F}(x-y)\eta(y)\right\}\Big|^{\rm connected}_{\bar{\eta},\eta=0}\\&={\rm i}e\int{\rm d}^4zA_\mu(z)S_{\rm F}(x_1-z)\gamma^\mu S_{\rm F}(z-x_2).\end{aligned} \quad (15)$$

For the second-order term, it becomes complicated. We need to calculate the following quantity:

$$\begin{aligned}&\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}(-\frac{1}{2}S^2_{\rm int}[\bar{\psi},\psi])Z_0[\bar{\eta},\eta]\Big|^{\rm connected}_{\bar{\eta},\eta=0}\\&=-\frac{1}{2}e^2\int{\rm d}^4z{\rm d}^4wA_\mu(z)A_\nu(w)\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\\&\quad\times\bar{\psi}(z)\gamma^\mu\psi(z)\bar{\psi}(w)\gamma^\nu\psi(w)Z_0[\bar{\eta},\eta]\Big|^{\rm connected}_{\bar{\eta},\eta=0}.\end{aligned} \quad (16)$$

Repeat using the above trick, i.e., replace the fields $\bar{\psi}$ and ψ by functional derivatives. Thus, equation (16) becomes

$$\begin{aligned}&-\frac{1}{2}e^2\int{\rm d}^4z{\rm d}^4wA_\mu(z)A_\nu(w)\gamma^\mu\gamma^\nu\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\\&\quad\times\frac{\delta}{\delta\eta(z)}\frac{\delta}{\delta\bar{\eta}(z)}\frac{\delta}{\delta\eta(w)}\frac{\delta}{\delta\bar{\eta}(w)}Z_0[\bar{\eta},\eta]\Big|^{\rm connected}_{\bar{\eta},\eta=0}\\&=-\frac{1}{2}e^2\int{\rm d}^4z{\rm d}^4wA_\mu(z)A_\nu(w)\gamma^\mu\gamma^\nu\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\\&\quad\times\Big\{-S_{\rm F}(z-w)S_{\rm F}(w-z)-S_{\rm F}(z-z)S_{\rm F}(w-w)\\&\quad-S_{\rm F}(z-z)\int{\rm d}^4xS_{\rm F}(x-w)\bar{\eta}(x)\int{\rm d}^4y\\&\quad\times S_{\rm F}(w-y)\eta(y)-S_{\rm F}(z-w)\int{\rm d}^4xS_{\rm F}(x-w)\bar{\eta}(x)\\&\quad\times\int{\rm d}^4yS_{\rm F}(z-y)\eta(y)+S_{\rm F}(z-w)\int{\rm d}^4y\\&\quad\times S_{\rm F}(w-y)\eta(y)+S_{\rm F}(w-w)\int{\rm d}^4xS_{\rm F}(x-z)\bar{\eta}(x)\\&\quad+\int{\rm d}^4xS_{\rm F}(x-w)\bar{\eta}(x)\int{\rm d}^4yS_{\rm F}(w-y)\\&\quad\times\eta(y)\int{\rm d}^4xS_{\rm F}(x-z)\bar{\eta}(x)\int{\rm d}^4yS_{\rm F}(z-y)\eta(y)\Big\}\\&\quad\times\exp\left\{-\int{\rm d}^4x{\rm d}^4y\bar{\eta}(x)S_{\rm F}(x-y)\eta(y)\right\}\Big|^{\rm connected}_{\bar{\eta},\eta=0}\\&=-e^2\int{\rm d}^4z{\rm d}^4wA_\mu(z)A_\nu(w)S_{\rm F}(x_1-w)\\&\quad\times\gamma^\mu S_{\rm F}(w-z)\gamma^\nu S_{\rm F}(z-x_2).\end{aligned} \quad (17)$$

By applying the generalized LSZ reduction formula[24] for the spinor field, we establish the connection between the two-point correlation function and the $\boldsymbol{S}$ matrix. Namely, we have

$$\begin{aligned}&(\sqrt{Z})^2\langle\boldsymbol{p}|S|\boldsymbol{k}\rangle\\&=\int{\rm d}^4x_1\,{\rm e}^{{\rm i}px_1}\int{\rm d}^4x_2\,{\rm e}^{-{\rm i}kx_2}\bar{u}(p)(-{\rm i})({\rm i}\partial\!\!\!/_{x_1}-m)\\&\quad\times\langle\Omega|T\psi_{\rm H}(x_1)\bar{\psi}_{\rm H}(x_2)|\Omega\rangle(-{\rm i})(-{\rm i}\overleftarrow{\partial\!\!\!/}_{x_2}-m)u(k),\end{aligned} \quad (18)$$

where u is the spinor wave function of the electron, $\bar{u}$ is the Dirac conjugation of u, and the quantity Z that appears in this equation is a c number, known as field-strength renormalization factor. Since we only work up to the second-order term, we set $Z=1$.

To isolate the interaction part of the $\boldsymbol{S}$ matrix, we define the $\boldsymbol{T}$ matrix as

$$\boldsymbol{S}=\boldsymbol{1}+{\rm i}\boldsymbol{T}. \quad (19)$$

It is easy to see that, for the identity matrix, it indubitably links the zero-order term which is trivial.

The lowest non-trivial order contribution for $\boldsymbol{T}$ matrix is the first-order term. Hence, combine the result of Eq. (15), the only thing we need to do is to calculate the quantity

$$\begin{aligned}&{\rm i}e\int{\rm d}^4x_1\,{\rm e}^{{\rm i}px_1}\int{\rm d}^4x_2\,{\rm e}^{-{\rm i}kx_2}\int{\rm d}^4zA_\mu(z)\\&\quad\times\bar{u}(p)(-{\rm i})({\rm i}\partial\!\!\!/_{x_1}-m)S_{\rm F}(x_1-z)\\&\quad\times\gamma^\mu S_{\rm F}(z-x_2)(-{\rm i})(-{\rm i}\overleftarrow{\partial\!\!\!/}_{x_2}-m)u(k)\\&=-{\rm i}e\int{\rm d}^4x_1\,{\rm e}^{{\rm i}px_1}\int{\rm d}^4x_2\,{\rm e}^{-{\rm i}kx_2}\end{aligned}$$

$$\times \int \mathrm{d}^4 z A_\mu(z)\bar{u}(p)\delta^4(x_1-z)\gamma^\mu\delta^4(z-x_2)u(k)$$
$$= -\mathrm{i}e\bar{u}(p)\gamma^\mu u(k)\int \mathrm{d}^4 z\,\mathrm{e}^{\mathrm{i}z(p-k)}A_\mu(z). \quad (20)$$

Substituting the above result into Eq. (18), we have

$$\langle \boldsymbol{p}|\mathrm{i}T|\boldsymbol{k}\rangle^{(1)} = -\mathrm{i}e\bar{u}(p)\gamma^\mu u(k)\tilde{A}_\mu(p-k), \quad (21)$$

where $\tilde{A}_\mu(p-k)$ is the four-dimensional Fourier transform of $A_\mu(z)$,

$$\tilde{A}_\mu(p-k) = \int \mathrm{d}^4 z\,\mathrm{e}^{\mathrm{i}z(p-k)}A_\mu(z). \quad (22)$$

Continuing to consider the situation of the second-order term, we obtain

$$\begin{aligned}
&-e^2\int \mathrm{d}^4x_1\,\mathrm{e}^{\mathrm{i}px_1}\int \mathrm{d}^4x_2\,\mathrm{e}^{-\mathrm{i}kx_2}\int \mathrm{d}^4z\mathrm{d}^4w\\
&\times A_\mu(z)A_\nu(w)\bar{u}(p)(-\mathrm{i})(\mathrm{i}\overrightarrow{\partial}_{x_1}-m)S_\mathrm{F}(x_1-w)\gamma^\mu\\
&\times S_\mathrm{F}(w-z)\gamma^\nu S_\mathrm{F}(z-x_2)(-\mathrm{i})(-\mathrm{i}\overleftarrow{\partial}_{x_2}-m)u(k)\\
=\ &-e^2\int \mathrm{d}^4x_1\,\mathrm{e}^{\mathrm{i}px_1}\int \mathrm{d}^4x_2\,\mathrm{e}^{-\mathrm{i}kx_2}\int \mathrm{d}^4\mathrm{d}^4wA_\mu(z)\\
&\times A_\nu(w)\bar{u}(p)\delta^4(x_1-w)\gamma^\mu S_\mathrm{F}(w-z)\gamma^\nu\delta^4(z-x_2)u(k)\\
=\ &-e^2\int \mathrm{d}^4x_1\,\mathrm{e}^{\mathrm{i}px_1}\int \mathrm{d}^4x_2\,\mathrm{e}^{-\mathrm{i}kx_2}A_\mu(x_2)\\
&\times A_\nu(x_1)\bar{u}(p)\gamma^\mu S_\mathrm{F}(x_1-x_2)\gamma^\nu u(k)\\
=\ &-e^2\int \frac{\mathrm{d}^4q}{(2\pi)^4}\bar{u}(p)\gamma^\mu\tilde{A}_\mu(q-k)\\
&\times\frac{\mathrm{i}(\not{q}+m)}{q^2-m^2+\mathrm{i}\varepsilon}\gamma^\nu\tilde{A}_\nu(p-q)u(k). \quad (23)
\end{aligned}$$

Substituting the result of Eq. (23) into Eq. (19), we have

$$\begin{aligned}
&\langle \boldsymbol{p}|\mathrm{i}T|\boldsymbol{k}\rangle^{(2)}\\
=\ &-e^2\int \frac{\mathrm{d}^4q}{(2\pi)^4}\bar{u}(p)\gamma^\mu\tilde{A}_\mu(q-k)\tilde{S}_\mathrm{F}(q)\gamma^\nu\tilde{A}_\nu(p-q)u(k), \quad (24)
\end{aligned}$$

where $\tilde{S}_\mathrm{F}(q)$ is the Feynman propagator in momentum space, it has the form

$$\tilde{S}_\mathrm{F}(q) = \frac{\mathrm{i}(\not{q}+m)}{q^2-m^2+\mathrm{i}\varepsilon}. \quad (25)$$

Therefore, summing up the above, we finally have

$$\begin{aligned}
\langle \boldsymbol{p}|\mathrm{i}T|\boldsymbol{k}\rangle &= \langle \boldsymbol{p}|\mathrm{i}T|\boldsymbol{k}\rangle^{(1)} + \langle \boldsymbol{p}|\mathrm{i}T|\boldsymbol{k}\rangle^{(2)}\\
&= -\mathrm{i}e\bar{u}(p)\gamma^\mu u(k)\tilde{A}_\mu(p-k)\\
&\quad -e^2\int \frac{\mathrm{d}^4q}{(2\pi)^4}\bar{u}(p)\gamma^\mu\tilde{A}_\mu(q-k)\tilde{S}_\mathrm{F}(q)\\
&\quad \times\gamma^\nu\tilde{A}_\nu(p-q)u(k). \quad (26)
\end{aligned}$$

So far, we have derived the amplitude of electron scattering up to the second order which is completely general, it can be used to calculate electron scattering with arbitrary form of electromagnetic potential.

4. Explicit calculation for the one-dimensional case

In this section, we have now shown that we utilize the result obtained in Eq. (26) to apply to a simple case of one-dimensional classical electric potential scattering. For the sake of simplicity, we shall suppose that the electron travels along the positive z^3 axis, then let us set

$$\begin{aligned}
&k^\mu = (k^0,0,0,k^3), \quad p^\mu = (p^0,0,0,p^3),\\
&A_\mu = (A_0(z^3),0,0,0), \quad (27)
\end{aligned}$$

where $k^0 = \sqrt{(k^3)^2+m^2}$, $p^0 = \sqrt{(p^3)^2+m^2}$, k^3 and p^3 are the initial and final momentum of electron, respectively. Meanwhile, we set $k^3 > 0$. Then, at the level of the first order, the T-matrix element becomes

$$\begin{aligned}
&\langle p^3|\mathrm{i}T|k^3\rangle^{(1)}\\
=\ &-\mathrm{i}e\bar{u}(p^3)\gamma^0u(k^3)\int \mathrm{d}^2z\,\mathrm{e}^{\mathrm{i}z(p-k)}A_0(z)\\
=\ &-\mathrm{i}eu^\dagger(p^3)u(k^3)\int \mathrm{d}z^0\,\mathrm{e}^{\mathrm{i}z^0(p^0-k^0)}\int \mathrm{d}z^3\,\mathrm{e}^{-\mathrm{i}z^3(p^3-k^3)}A_0(z^3)\\
=\ &-\mathrm{i}u^\dagger(p^3)u(k^3)(2\pi)\delta(p^0-k^0)\\
&\times\int \mathrm{d}z^3\,\mathrm{e}^{-\mathrm{i}z^3(p^3-k^3)}V(z^3). \quad (28)
\end{aligned}$$

In the last line, we define $V(z^3) = eA_0(z^3)$.

To describe the electron scattering process completely, we have to integrate over the final momentum p^3 with the Lorentz-invariant measure,

$$\begin{aligned}
&\int \frac{\mathrm{d}p^3}{2\pi}\frac{1}{2E(p^3)}\langle p^3|\mathrm{i}T|k^3\rangle^{(1)}\\
=\ &-\mathrm{i}\int \frac{\mathrm{d}p^3}{2\pi}\frac{1}{2E(p^3)}u^\dagger(p^3)u(k^3)\int \mathrm{d}z^0\,\mathrm{e}^{\mathrm{i}z^0(p^0-k^0)}\\
&\times\int \mathrm{d}z^3\,\mathrm{e}^{-\mathrm{i}z^3(p^3-k^3)}V(z^3)\\
=\ &-\mathrm{i}\int \frac{\mathrm{d}p^3}{2E(p^3)}u^\dagger(p^3)u(k^3)\delta(p^0-k^0)\\
&\times\int \mathrm{d}z^3\,\mathrm{e}^{-\mathrm{i}z^3(p^3-k^3)}V(z^3). \quad (29)
\end{aligned}$$

In order to make our subsequent calculations easily, it is convenient to write the normalization condition for spinor $u(p)$ as

$$u^{s\dagger}(\boldsymbol{p})u^r(\boldsymbol{p}) = 2E_{\boldsymbol{p}}\xi^{s\dagger}\xi^r = 2E_{\boldsymbol{p}}\delta^{sr} \quad (30)$$

and

$$u^{s\dagger}(-\boldsymbol{p})u^r(\boldsymbol{p}) = 2m\xi^{s\dagger}\xi^r = 2m\delta^{sr}, \quad (31)$$

where ξ^s is a two-component constant spinor and s is the spinor index.

With the aim of performing the integration over p^3, we use the identity equation[25] as

$$\delta(p^0-k^0) = \frac{k^0}{k^3}[\delta(p^3-k^3)+\delta(p^3+k^3)]. \quad (32)$$

Then, equation (29) will be split into two separate parts.

Namely, for the reflection part of electron scattering, $p^3 = -k^3$, at the level of first order, we have

$$\begin{aligned}\mathscr{R}^{(1)} &= -\mathrm{i}\int \frac{\mathrm{d}p^3}{2E(p^3)}u^{\dagger}(p^3)u(k^3)\frac{k^0}{k^3}\delta(p^3+k^3)\\ &\quad\times\int \mathrm{d}z^3\,\mathrm{e}^{-\mathrm{i}z^3(p^3-k^3)}V(z^3)\\ &= -\mathrm{i}\frac{u^{\dagger}(-k^3)u(k^3)}{2E(k^3)}\frac{k^0}{k^3}\int \mathrm{d}z^3\,\mathrm{e}^{\mathrm{i}2z^3k^3}V(z^3)\\ &= -\mathrm{i}\frac{m}{k^3}\int \mathrm{d}z^3\,\mathrm{e}^{\mathrm{i}2z^3k^3}V(z^3). \end{aligned}\tag{33}$$

Analogously, for the transmission part, $p^3 = k^3$, we have

$$\begin{aligned}\mathscr{T}^{(1)} &= -\mathrm{i}\int \frac{\mathrm{d}p^3}{2E(p^3)}u^{\dagger}(p^3)u(k^3)\frac{k^0}{k^3}\delta(p^3-k^3)\\ &\quad\times\int \mathrm{d}z^3\,\mathrm{e}^{-\mathrm{i}z^3(p^3-k^3)}V(z^3)\\ &= -\mathrm{i}\frac{u^{\dagger}(k^3)u(k^3)}{2E(k^3)}\frac{k^0}{k^3}\int \mathrm{d}z^3V(z^3)\\ &= -\mathrm{i}\frac{k^0}{k^3}\int \mathrm{d}z^3V(z^3). \end{aligned}\tag{34}$$

At the level of second order, the T-matrix element has

$$\begin{aligned}&\langle p^3|\mathrm{i}T|k^3\rangle^{(2)}\\ &= -e^2\int\frac{\mathrm{d}^2q}{(2\pi)^2}\bar{u}(p^3)\gamma^0\tilde{A}_0(q-k)\\ &\quad\times\frac{\mathrm{i}(\gamma_0q^0+\gamma_3q^3+m)}{(q^0)^2-(q^3)^2-m^2+\mathrm{i}\varepsilon}\gamma^0\tilde{A}_0(p-q)u(k^3)\\ &= -\int\frac{\mathrm{d}q^3}{2\pi}\int\mathrm{d}z^0\,\mathrm{e}^{-\mathrm{i}z^0k^0}\int\mathrm{d}w^0\,\mathrm{e}^{\mathrm{i}w^0p^0}\int\mathrm{d}z^3\\ &\quad\times\mathrm{e}^{-\mathrm{i}z^3(q^3-k^3)}V(z^3)\int\mathrm{d}w^3\,\mathrm{e}^{-\mathrm{i}w^3(p^3-q^3)}V(w^3)\\ &\quad\times\int\frac{\mathrm{d}q^0}{2\pi}\,\mathrm{e}^{\mathrm{i}q^0(z^0-w^0)}u^{\dagger}(p^3)\frac{\mathrm{i}(\gamma_0q^0+\gamma_3q^3+m)}{(q^0)^2-(q^3)^2-m^2+\mathrm{i}\varepsilon}\gamma^0u(k^3)\\ &= -\delta(p^0-k^0)\int\mathrm{d}z^3\,\mathrm{e}^{\mathrm{i}z^3k^3}V(z^3)\\ &\quad\times\int\mathrm{d}w^3\,\mathrm{e}^{-\mathrm{i}w^3p^3}V(w^3)u^{\dagger}(p^3)\\ &\quad\times\int\mathrm{d}q^3\frac{\mathrm{i}(\gamma_0p^0+\gamma_3q^3+m)}{(p^3)^2-(q^3)^2+\mathrm{i}\varepsilon}\,\mathrm{e}^{\mathrm{i}q^3(w^3-z^3)}\gamma^0u(k^3). \end{aligned}\tag{35}$$

Here, to complete the integral over q^3, we draw support from the method of residue theorem and choose the contour satisfying the Feynman prescription to deal with it. Namely, when $w^3 > z^3$ we can perform the q^3 integral by closing the contour below the axis. When $w^3 < z^3$ we close the contour above the axis. Then, equation (35) becomes

$$\begin{aligned}&\langle p^3|\mathrm{i}T|k^3\rangle^{(2)}\\ &= -\delta(p^0-k^0)\int\mathrm{d}z^3\,\mathrm{e}^{\mathrm{i}z^3k^3}V(z^3)\int\mathrm{d}w^3\,\mathrm{e}^{-\mathrm{i}w^3p^3}V(w^3)u^{\dagger}(p^3)\\ &\quad\times\Upsilon(w^3,z^3;p^0,p^3)\gamma^0u(k^3), \end{aligned}\tag{36}$$

where the function $\Upsilon(w^3,z^3;p^0,p^3)$ is defined as

$$\begin{aligned}&\Upsilon(w^3,z^3;p^0,p^3)\\ &= \theta(w^3-z^3)\frac{\pi}{|p^3|}(\gamma_0p^0+\gamma_3|p^3|+m)\,\mathrm{e}^{\mathrm{i}|p^3|(w^3-z^3)}\\ &\quad+\theta(z^3-w^3)\frac{\pi}{|p^3|}(\gamma_0p^0-\gamma_3|p^3|+m)\,\mathrm{e}^{\mathrm{i}|p^3|(z^3-w^3)}. \end{aligned}\tag{37}$$

Similarly, we have to implement the integral over p^3, again, by using Eq. (32), for reflection, $p^3 = -k^3$, at the level of second order, we obtain

$$\begin{aligned}\mathscr{R}^{(2)} &= -\int\frac{\mathrm{d}p^3}{2\pi}\frac{1}{2E(p^3)}\frac{k^0}{k^3}\delta(p^3+k^3)\\ &\quad\times\int\mathrm{d}z^3\,\mathrm{e}^{\mathrm{i}z^3k^3}V(z^3)\int\mathrm{d}w^3\,\mathrm{e}^{-\mathrm{i}w^3p^3}V(w^3)u^{\dagger}(p^3)\\ &\quad\times\Upsilon(w^3,z^3;p^0,p^3)\gamma^0u(k^3)\\ &= -\frac{1}{4\pi k^3}\int\mathrm{d}z^3\,\mathrm{e}^{\mathrm{i}z^3k^3}V(z^3)\int\mathrm{d}w^3\,\mathrm{e}^{\mathrm{i}w^3k^3}V(w^3)u^{\dagger}(-k^3)\\ &\quad\times\Upsilon(w^3,z^3;k^0,-k^3)\gamma^0u(k^3)\\ &= -\frac{mk^0}{(k^3)^2}\int\mathrm{d}z^3\mathrm{d}w^3V(z^3)V(w^3)[\theta(w^3-z^3)\\ &\quad\times\mathrm{e}^{\mathrm{i}2k^3w^3}+\theta(z^3-w^3)\,\mathrm{e}^{\mathrm{i}2k^3z^3}]. \end{aligned}\tag{38}$$

For transmission, $p^3 = k^3$. At the level of second order, we obtain that

$$\begin{aligned}\mathscr{T}^{(2)} &= -\int\frac{\mathrm{d}p^3}{2\pi}\frac{1}{2E(p^3)}\frac{k^0}{k^3}\delta(p^3-k^3)\\ &\quad\times\int\mathrm{d}z^3\,\mathrm{e}^{\mathrm{i}z^3k^3}V(z^3)\int\mathrm{d}w^3\,\mathrm{e}^{-\mathrm{i}w^3p^3}V(w^3)u^{\dagger}(p^3)\\ &\quad\times\Upsilon(w^3,z^3;p^0,p^3)\gamma^0u(k^3)\\ &= -\frac{1}{4\pi k^3}\int\mathrm{d}z^3\,\mathrm{e}^{\mathrm{i}z^3k^3}V(z^3)\int\mathrm{d}w^3\,\mathrm{e}^{-\mathrm{i}w^3k^3}V(w^3)u^{\dagger}(k^3)\\ &\quad\times\Upsilon(w^3,z^3;k^0,k^3)\gamma^0u(k^3)\\ &= -\frac{1}{(k^3)^2}\int\mathrm{d}z^3\mathrm{d}w^3V(z^3)V(w^3)[\theta(w^3-z^3)(k^0)^2\\ &\quad+\theta(z^3-w^3)m^2\,\mathrm{e}^{\mathrm{i}2k^3(z^3-w^3)}]. \end{aligned}\tag{39}$$

5. Application in analyzing Gaussian barrier

As a kind of concrete application, we choose $V(z^3)$ as the Gaussian barrier given that it has a bearing on the strong laser field in optics, which is a radially symmetrical distribution whose potential variation is given by

$$V(z^3) = V\,\mathrm{e}^{-\alpha(z^3)^2},\quad -\infty < z^3 < +\infty, \tag{40}$$

where V and α are constant real numbers, and $V > 0$, $\alpha > 0$. Meanwhile, we assume that the magnitude of momentum of incident electron satisfies the condition $k^3 \ll \alpha$, because experimentally the initial velocity of incident electron is typically small.

For convenience, much emphasis is placed on the transmission part. Our calculations presented in Section 4 show that equations (34) and (39) are able to give a clear physical picture for scattering by the Gaussian barrier on condition that the approximation up to the second order is accurate enough.

Straightforwardly, we have

$$\mathscr{T}^{(1)} = -\mathrm{i}\frac{k^0 V}{k^3}\int_{-\infty}^{+\infty}\mathrm{d}z^3\,\mathrm{e}^{-\alpha(z^3)^2} = -\mathrm{i}\frac{k^0 V}{k^3}\left(\frac{\pi}{\alpha}\right)^{1/2} \tag{41}$$

and

$$\begin{aligned}
\mathscr{T}^{(2)} &= -\frac{V^2}{(k^3)^2}\left[(k^0)^2\int_{-\infty}^{+\infty}\mathrm{d}w^3\,\mathrm{e}^{-\alpha(w^3)^2}\int_{-\infty}^{w^3}\mathrm{d}z^3\,\mathrm{e}^{-\alpha(z^3)^2}\right.\\
&\quad +m^2\int_{-\infty}^{+\infty}\mathrm{d}w^3\,\mathrm{e}^{-\alpha(w^3)^2}\,\mathrm{e}^{-\mathrm{i}2k^3w^3}\\
&\quad \left.\times\int_{w^3}^{+\infty}\mathrm{d}z^3\,\mathrm{e}^{-\alpha(z^3)^2}\,\mathrm{e}^{\mathrm{i}2k^3z^3}\right]\\
&= -\frac{V^2}{(k^3)^2}\left[(k^0)^2\left(\frac{\pi}{2\alpha}\right)+m^2\,\mathrm{e}^{-\frac{2(k^3)^2}{\alpha}}\right.\\
&\quad \left.\times\int_{-\infty}^{+\infty}\mathrm{d}w^3\,\mathrm{e}^{-\alpha\left(w^3+\mathrm{i}\frac{k^3}{\alpha}\right)^2}\int_{w^3}^{+\infty}\mathrm{d}z^3\,\mathrm{e}^{-\alpha\left(z^3-\mathrm{i}\frac{k^3}{\alpha}\right)^2}\right]\\
&= -\frac{V^2}{(k^3)^2}\left[(k^0)^2\left(\frac{\pi}{2\alpha}\right)+m^2\,\mathrm{e}^{-\frac{2(k^3)^2}{\alpha}}\right.\\
&\quad \left.\times\int_{-\infty}^{+\infty}\mathrm{d}w^3\,\mathrm{e}^{-\alpha(w^3)^2}\int_{w^3-\mathrm{i}\frac{2k^3}{\alpha}}^{+\infty}\mathrm{d}z^3\,\mathrm{e}^{-\alpha(z^3)^2}\right]\\
&= -\frac{V^2}{(k^3)^2}\left\{(k^0)^2\left(\frac{\pi}{2\alpha}\right)+m^2\,\mathrm{e}^{-\frac{2(k^3)^2}{\alpha}}\right.\\
&\quad \times\left[\left(\frac{\pi}{2\alpha}\right)-\int_{-\infty}^{+\infty}\mathrm{d}w^3\,\mathrm{e}^{-\alpha(w^3)^2}\right.\\
&\quad \left.\left.\times\int_{w^3-\mathrm{i}\frac{2k^3}{\alpha}}^{w^3}\mathrm{d}z^3\,\mathrm{e}^{-\alpha(z^3)^2}\right]\right\}\\
&\approx -\frac{V^2}{(k^3)^2}\left\{(k^0)^2\left(\frac{\pi}{2\alpha}\right)+m^2\,\mathrm{e}^{-\frac{2(k^3)^2}{\alpha}}\right.\\
&\quad \left.\times\left[\left(\frac{\pi}{2\alpha}\right)-\mathrm{i}\frac{2k^3}{\alpha}\int_{-\infty}^{+\infty}\mathrm{d}w^3\,\mathrm{e}^{-2\alpha(w^3)^2}\right]\right\}\\
&= -\frac{V^2}{(k^3)^2}\left\{(k^0)^2\left(\frac{\pi}{2\alpha}\right)+m^2\,\mathrm{e}^{-\frac{2(k^3)^2}{\alpha}}\right.\\
&\quad \left.\times\left[\left(\frac{\pi}{2\alpha}\right)-\mathrm{i}\frac{2k^3}{\alpha}\left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}}\right]\right\}.
\end{aligned} \tag{42}$$

Notice that in the second line from the bottom of Eq. (42), we take an approximation that is

$$\int_{w^3-\mathrm{i}\frac{2k^3}{\alpha}}^{w^3}\mathrm{d}z^3\,\mathrm{e}^{-\alpha(z^3)^2} \approx \mathrm{i}\frac{2k^3}{\alpha}\,\mathrm{e}^{-\alpha(w^3)^2}, \tag{43}$$

under the condition of $k^3 \ll \alpha$.

Then, the total transmission amplitude including the trivial part of the zero order is given as follows:

$$\begin{aligned}
\mathscr{T} &= \mathbf{1}-\frac{\pi}{2\alpha}\frac{V^2}{(k^3)^2}\left[(k^0)^2+m^2\,\mathrm{e}^{-\frac{2(k^3)^2}{\alpha}}\right]\\
&\quad -\mathrm{i}\frac{V}{k^3}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\left(k^0-\frac{\sqrt{2}Vm^2}{\alpha}\,\mathrm{e}^{-\frac{2(k^3)^2}{\alpha}}\right).
\end{aligned} \tag{44}$$

6. Summary

In the previous five sections, by using path integral approach, we have calculated the first- and second-order contributions of quantum field theory to the process of the electron scattering. Indeed, according to the general principle of quantum field theory, we need to quantize the spinor field and electromagnetic field simultaneously. However, to compare our final results directly with those of the single-particle Schrodinger equation or Dirac equation, we no longer quantize the electromagnetic field. Instead, we treat the field as a given, classical four-dimensional potential $A_\mu(x)$ throughout.

Usually, it is sufficient to solve most scattering problems using single-particle theory. Yet, sometimes one needs to consider more complicated situations which should use field theory like scattering problems in the strong laser fields. Therefore, the purpose of this paper is to show that the expression we derive in Eq. (26) is able to give a map describing the phenomena that contain kinds of scattering problems between quantum field theory and quantum mechanics. Moreover, we note that the path integral treatment of the one-dimensional electron scattering in classical electric potential may be related to give a possible explanation of the Klein paradox.[26–32]

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