Phase sensitivity of two nonlinear interferometers with inputting entangled coherent states*

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We investigate the phase sensitivity of the SU(1,1) interfereometer [SU(1,1)I] and the modified Mach–Zehnder interferometer (MMZI) with the entangled coherent states (ECS) as inputs. We consider the ideal case and the situations in which the photon losses are taken into account. We find that, under ideal conditions, the phase sensitivity of both the MMZI and the SU(1,1)I can beat the shot-noise limit (SNL) and approach the Heisenberg limit (HL). In the presence of photon losses, the ECS can beat the coherent and squeezed states as inputs in the SU(1,1)I, and the MMZI is more robust against internal photon losses than the SU(1,1)I.

Keywords: quantum optics, phase sensitivity, quantum metrology

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1. Introduction

Phase estimation is one of the most important issues in quantum metrology and has been widely studied.[1] Optical interferometers are the main devices for metrology and weak signal detection.[2] The ultimate goal of phase estimation is to enhance the phase sensitivity of the interferometers and to attain the Heisenberg limit (HL).[3] Theoretically, there are two main ways to enhance the phase sensitivity of interferometers. One way is using some specific quantum states, such as the squeezed states, the superposition and entanglement states to decrease the quantum noise.[4–9] The other way is using the nonlinear effects to increase the phase signal. As is shown in Ref. [10], because of the gain in the optical parametric amplifiers (OPAs), the maximum output intensity of the interferometer can be much bigger than the input intensity as well as the intensity inside the interferometer, and has better sensitivity than the traditional linear interferometer. Therefore, we can enhance the phase sensitivity with the help of the OPA. For example, the phase sensitivity and signal-to-noise ratio of the SU(1,1) interferometer (SU(1,1)I)[11] can be improved greatly due to the gain of the OPAs. Many of these theories for improving the phase sensitivity of interferometers have been confirmed by experiments in recent years.[12,13]

In this paper, we discuss a nonlinear interferometer which is composed of a beam splitter (BS) and an OPA.[14] It can be treated as a variation of the Mach–Zehnder interferometer (MZI) which is composed of two BSs. For simplicity in nomenclature, we name this nonlinear interferometer as the modified Mach–Zehnder interferometer (MMZI). It can also be treated as a variation of the SU(1,1)I which is composed of two OPAs. It is well known that the OPAs not only amplify the signal light but also amplify the noise, so we replace the second OPA in the SU(1,1)I with a BS to balance the signal and noise, and this forms the MMZI. We use the entangled coherent state (ECS)[15] as the input of the interferometer since the ECS can reduce the quantum noise and is robust against photon losses.[16] In real experiments, the photon losses are inevitable and have great effects on the phase sensitivity, therefore, we will also discuss these effects in the MMZI and in the SU(1,1)I.

The organization of this paper is as follows. In Section 2, we propose a special nonlinear optical interferometer in which there is a black box. If the black box is an OPA, then the interferometer is the so-called SU(1,1)I, and if the black box is a BS, then the interferometer is the so-called MMZI. Afterwards we study the phase sensitivity of these two interferometers with the ECS as inputs and make a comparision with the linear MZI. In Section 3 we study the effects of the photon losses on the phase sensitivity of the interferometers, and we also make a comparison with other approaches. Section 4 presents the conclusions.

2. The phase sensitivity of the two interferometers

An important problem with the optical interferometers is to improve the phase sensitivity as far as possible. Here, we

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discuss a special nonlinear interferometer, as shown in Fig. 1. The whole setup can be divided into three parts. Part I is for preparing the state of the light which will be the input of the interferometer (part II), and part III is for detection. Here we use the ECS $|\Psi\rangle_0 = N_\alpha(|\alpha\rangle_{a_0}|0\rangle_{b_0} + |0\rangle_{a_0}|\alpha\rangle_{b_0})$ ($N_\alpha = [2(1+\mathrm{e}^{-|\alpha|^2})]^{-1/2}$ is the normalized factor) as the input of the interferometer. The ECS can be generated by inputting a coherent state ($|\alpha/\sqrt{2}\rangle$) in mode a and an even coherent state $N_\alpha(|\alpha/\sqrt{2}\rangle + |-\alpha/\sqrt{2}\rangle)$ in mode b into a 50:50 BS, as shown in part I of Fig. 1, and this has been realized in experiments ($\alpha \sim 1.5$).[17] The light beams a_0 and b_0 in the ECS act as the signal beams. After the OPA, the energy of the pump light can be transferred to the signal beams. The signal beams carry the phase information after passing through a phase shifter $U(\phi) = \mathrm{e}^{\mathrm{i}b^\dagger b\phi}$. Then the signal beams enter into a black box, which can be a linear BS (for an MMZI) or a nonlinear OPA (for an SU(1,1)I), respectively. Finally, after the black box, the detection can be implemented at the output. There are several detection methods, for example, the homodyne detection, the parity detection, and so on. Here we use the homodyne detection at the port a_2. We define the quadrature operator

$$X = \frac{1}{\sqrt{2}}(a_2 + a_2^\dagger). \tag{1}$$

The phase sensitivity $\Delta\phi$ of an interferometer can be obtained from the error propagation formula[1]

$$(\Delta\phi)^2 = \frac{(\Delta X)^2}{|\partial\langle X\rangle/\partial\phi|^2}, \tag{2}$$

where $(\Delta O)^2 = \langle O^2\rangle - \langle O\rangle^2$ is the variance of O ($O = X, \phi$). In the following, we will use Eq. (2) to study the phase sensitivity of the SU(1,1)I and the MMZI with ECS as the inputs.

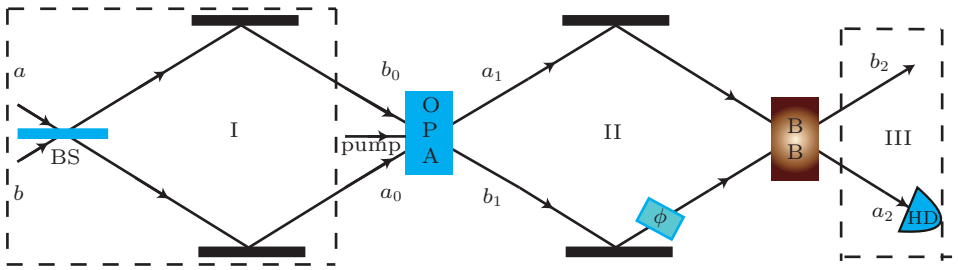


Fig. 1. (color online) A nonlinear optical interferometric setup which is composed of three parts. Part I is for preparing the input states to the main body of the interferometer (Part II), and Part III is for detection. The prepared states from Part I and the pump light go through an OPA. There is a phase shifter in one arm of the interferometer, and the light will carry phase information after passing through it. Before the detection, there is a black box BB, which may be a BS or an OPA, which represents an MMZI or an SU(1,1)I, respectively. The homodyne detection is implemented to the output mode a_2.

2.1. The SU(1,1) interferometer [SU(1,1)I]

The SU(1,1)I was proposed in Ref [18], and has been studied with different quantum states as inputs in recent years.[19–21] In an SU(1,1)I, the two BSs of a conventional Mach–Zehnder interferometer(MZI) are replaced by two OPAs which are characterized by the SU(1,1) group.[18,22] The OPA is a device that can transfer energy from a strong pump beam to two signal beams, and a detailed description can be found in Ref. [21]. The phase sensitivity of an SU(1,1)I can approach the HL in the condition of no photon losses. Here we use an ECS as the input of the SU(1,1)I. For the first OPA, the relationship between the operators of the output modes (a_1, b_1) and the input modes (a_0, b_0) is given by[23]

$$\begin{aligned} a_1 &= \cosh g_1 a_0 - \mathrm{e}^{\mathrm{i}\theta_1}\sinh g_1 b_0^\dagger, \\ b_1 &= \mathrm{e}^{\mathrm{i}\theta_1}\sinh g_1 a_0^\dagger + \cosh g_1 b_0, \end{aligned} \tag{3}$$

where g_1 and θ_1 are the gain factor and the phase shift of the parametric process, respectively. For a phase shifter, the transform relationship of the output modes and input modes is $b_1 \to b_1\mathrm{e}^{\mathrm{i}\phi}$. By repeated application of Eq. (3) and the transform relationship of the phase shifter, we can obtain the transform relation between the operators of the output modes (a_2, b_2) and the input modes (a_0, b_0) for an SU(1,1)I as

$$\begin{aligned} a_2 &= Ua_0 - Vb_0^\dagger, \\ b_2 &= \mathrm{e}^{\mathrm{i}\phi}(Ub_0 - Va_0^\dagger), \end{aligned} \tag{4}$$

with

$$\begin{aligned} U &= \cosh g_1\cosh g_2 + \mathrm{e}^{-i\phi}\mathrm{e}^{\mathrm{i}(\theta_2-\theta_1)}\sinh g_1\sinh g_2, \\ V &= \mathrm{e}^{\mathrm{i}\theta_1}\sinh g_1\cosh g_2 + \mathrm{e}^{-i\phi}\mathrm{e}^{\mathrm{i}\theta_2}\cosh g_1\sinh g_2. \end{aligned} \tag{5}$$

By using Eqs. (4), (5), and (1), and for an input ECS

$$|\Psi\rangle_0 = N_\alpha(|\alpha\rangle_{a_0}|0\rangle_{b_0} + |0\rangle_{a_0}|\alpha\rangle_{b_0})$$

($N_\alpha = [2(1+\mathrm{e}^{-|\alpha|^2})]^{-1/2}$), we can calculate the numerator and denominator of Eq. (2) as

$$(\Delta X)^2 = \left\{\frac{|\alpha|^2}{8}[\mathrm{e}^{2\mathrm{i}\theta_\alpha}(U+V^*)^2 + \mathrm{e}^{-2\mathrm{i}\theta_\alpha}(U^*+V)^2]\right.$$

$$+\frac{|\alpha|^2(1-\mathrm{e}^{-|\alpha|^2})}{4(1+\mathrm{e}^{-|\alpha|^2})}(U-V^*)(U^*-V)$$
$$+\frac{1}{2}(UU^*+VV^*)\Big\},$$
$$\left|\frac{\partial\langle X\rangle}{\partial\phi}\right|^2 = \frac{|\alpha|^2}{2}\sinh^2 g[\sinh g\sin(\phi+\theta_1-\theta_2-\theta_\alpha)$$
$$-\cosh g\sin(\phi-\theta_2+\theta_\alpha)]^2, \quad (6)$$

in which $\alpha=|\alpha|\,\mathrm{e}^{\mathrm{i}\theta_\alpha}$ and $g_1=g_2=g$ has been used. At the optimal phase point $\phi=0$, and when $\theta_2-\theta_1=\pi$, equation (6) reduces to

$$(\Delta X)^2 = \frac{|\alpha|^2}{8}(\mathrm{e}^{2\mathrm{i}\theta_\alpha}+\mathrm{e}^{-2\mathrm{i}\theta_\alpha})+\frac{|\alpha|^2(1-\mathrm{e}^{-|\alpha|^2})}{4(1+\mathrm{e}^{-|\alpha|^2})}+\frac{1}{2},$$
$$\left|\frac{\partial\langle X\rangle}{\partial\phi}\right|^2 = \frac{|\alpha|^2\sinh^2 g(\sinh g+\cosh g)^2}{2}. \quad (7)$$

Equation (7) will be used in the numerical calculations.

2.2. The modified MMZI

The MMZI can be obtained when the black box in Fig. 1 is a BS. For a 50:50 BS, the relationship between the operators of the output modes (a_2, b_2) and the input modes (a_1, $b_1\,\mathrm{e}^{\mathrm{i}\phi}$) is given by

$$a_2 = \frac{1}{\sqrt{2}}(a_1+b_1\,\mathrm{e}^{\mathrm{i}\phi}),$$
$$b_2 = \frac{1}{\sqrt{2}}(a_1-b_1\,\mathrm{e}^{\mathrm{i}\phi}). \quad (8)$$

Combining Eqs. (8) and (3) with $g_1=g$ and $\theta_1=\theta$ we can get the relationship between the output modes (a_2,b_2) and the input modes (a_0,b_0) of the MMZI as

$$a_2 = \frac{1}{\sqrt{2}}[\cosh g(a_0+\mathrm{e}^{\mathrm{i}\phi}b_0)$$
$$-\sinh g\,\mathrm{e}^{\mathrm{i}(\theta+\phi)}(a_0^\dagger+\mathrm{e}^{-\mathrm{i}\phi}b_0^\dagger)],$$
$$b_2 = \frac{1}{\sqrt{2}}[\cosh g(a_0-\mathrm{e}^{\mathrm{i}\phi}b_0)$$
$$+\sinh g\,\mathrm{e}^{\mathrm{i}(\theta+\phi)}(a_0^\dagger-\mathrm{e}^{-\mathrm{i}\phi}b_0^\dagger)]. \quad (9)$$

Therefore, we can calculate the numerator $(\Delta X)^2$ and denominator $|\partial\langle X\rangle/\partial\phi|^2$ of Eq. (2) for the input ECS. By taking $\phi=\theta=0$, $\theta_\alpha=\pi/2$, we can minimize the numerator and maximize the denominator, then $(\Delta X)^2$ and $|\partial\langle X\rangle/\partial\phi|^2$ reduce to

$$(\Delta X)^2 = \frac{|\alpha|^2}{2}\left[\frac{(1-\mathrm{e}^{-|\alpha|^2})}{1+\mathrm{e}^{-|\alpha|^2}}+\frac{1}{|\alpha|^2}\right](\cosh g-\sinh g)^2,$$
$$\left|\frac{\partial\langle X\rangle}{\partial\phi}\right|^2 = \frac{|\alpha|^2}{4}(\cosh g+\sinh g)^2. \quad (10)$$

To compare the optimal phase sensitivity of the two interferometers with the shot-noise limit (SNL) ($1/\sqrt{N_\mathrm{T}}$) and the HL ($1/N_\mathrm{T}$), we need to calculate the total average photon number inside the SU(1,1)I and MMZI. This total average photon number can be calculated as

$$N_\mathrm{T} = \left\langle a_1^\dagger a_1+b_1^\dagger b_1\right\rangle_{|\Psi\rangle_0}$$
$$= N_\alpha^2|\alpha|^2(1+2\sinh^2 g)+2\sinh^2 g. \quad (11)$$

According to Eqs. (2), (7), and (10), we plot the phase sensitivity as functions of the gain factor g and the coherent amplitude $|\alpha|$ in Figs. 2(a) and 2(b). It can be seen that these two interferometers can beat the SNL and approach the HL for large g and $|\alpha|$. The phase sensitivity of the SU(1,1)I is better than that of the MMZI for a wide range of g and $|\alpha|$, but is not as good as that of the MMZI for small g and $|\alpha|$. We find that the phase sensitivity of the SU(1,1)I and MMZI can be improved greatly with the increase of the gain factor g. In order to make a comparison with the linear MZI, we plot the phase sensitivity as a function of the total average photon number, as shown in Fig. 2(c), in which $g=1$ as used in the experiment.[24] We can see clearly that the MMZI and SU(1,1)I outperform the linear MZI for a large average photon number (about $N>3$).

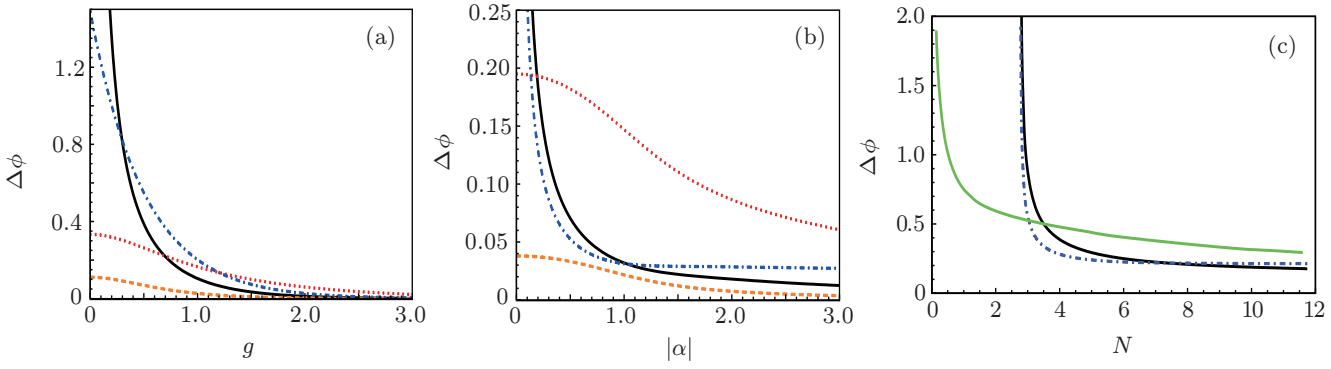


Fig. 2. (color online) (a) The phase sensitivity against the gain factor g for a given $|\alpha|=3$, and $\theta_\alpha=\pi/2$. (b) The phase sensitivity against the amplitude $|\alpha|$ for a given $g=2$ and $\theta_\alpha=\pi/2$. (c) The phase sensitivity against the total average photon number N. Black solid line: SU(1,1)I; blue dot-dashed line: MMZI; red dotted line: the shot-noise limit($1/\sqrt{N_\mathrm{T}}$); orange dashed line: the Heisenberg limit($1/N_\mathrm{T}$); green solid line: the linear MZI.

3. Effects of photon losses

In real experimental environments, the photon losses and detection efficiency have great effects on the precision and phase sensitivity.[25,26] In the presence of photon losses, especially with the internal losses, the phase sensitivity decoheres very quickly because of the amplification of the vacuum noise. For instance, the HL can be approached for the maximally path-entangled N00N state when there is no photon losses, but is very vulnerable under photon losses.[27,28] Therefore, it is necessary to enhance the robustness of interferometers against losses. Here we replace the second OPA of the SU(1,1)I with a BS to balance the signal and noise, and use four fictitious BSs to model photon losses in the two modes (Fig. 3). For convenience, we consider the same internal transmissivity T_1 and external transmissivity T_2 of the fictitious BSs in the two modes.

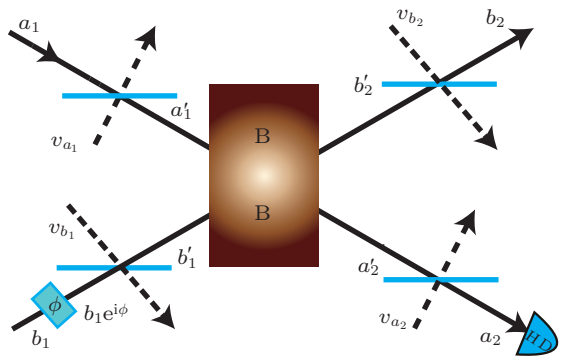


Fig. 3. (color online) (a) The schematic diagram of the photon losses after the phase shifter and before detection. The photon losses are modeled by four fictitious beam splitters.

For the lossy SU(1,1)I, the relations among the operators in Fig. 3 are given by

$$\begin{aligned} a_1' &= \sqrt{T_1}a_1 + \sqrt{1-T_1}v_{a_1}, \\ b_1' &= \sqrt{T_1}b_1 e^{i\phi} + \sqrt{1-T_1}v_{b_1}, \\ a_2' &= \cosh g a_1' - \sinh g e^{i\theta} b_1^{\dagger\prime}, \\ b_2' &= -\sinh g e^{i\theta} a_1^{\dagger\prime} + \cosh g b_1', \\ a_2 &= \sqrt{T_2}a_2' + \sqrt{1-T_2}v_{a_2}, \\ b_2 &= \sqrt{T_2}b_2' + \sqrt{1-T_2}v_{b_2}, \end{aligned} \tag{12}$$

where v_{a_1}, v_{b_1}, v_{a_2}, and v_{b_2} are the vacuum noise operators.

By combining Eq. (12) and Eq. (2), we can get the phase sensitivity of the lossy SU(1,1)I as

$$(\Delta\phi_{\rm L})^2 = (\Delta\phi)^2 + \Delta\phi_{\rm E}, \tag{13}$$

where we have chosen $\phi = 0, \theta_1 = 0$, $\theta_2 = \pi$, and $\theta_\alpha = \pi/2$. The first term represents the phase sensitivity when the photon losses are neglected, as discussed above. The second term

$$\Delta\phi_{\rm E} = \frac{(1-T_1)T_2\cosh 2g + (1-T_2)}{T_1T_2|\alpha|^2\sinh^2 g(\sinh g + \cosh g)^2} \tag{14}$$

is the additional term induced by the photon losses. When considering only the internal loss ($T_1 \neq 1, T_2 = 1$) or only the external losses ($T_1 = 1, T_2 \neq 1$), we can easily see that the influence of the internal loss is greater than the external loss when $\cosh 2g > 1$.

For the lossy MMZI, the relations among the operators in Fig. 3 are given by replacing the middle two equations in Eq. (13) with $a_2' = (a_1' + b_1')/\sqrt{2}$ and $b_2' = (a_1' - b_1')/\sqrt{2}$. The phase sensitivity is

$$(\Delta\phi_{\rm L})^2 = (\Delta\phi)^2 + \frac{2(1-T_1T_2)}{T_1T_2|\alpha|^2(\cosh g + \sinh g)^2}. \tag{15}$$

From the additional term in Eq. (15), we can see that the influence of the internal loss and that of the external loss are identical for the MMZI. This is because the BS is a passive linear optical instrument which can conserve the total energy.

Figure 4 shows the phase sensitivity $\Delta\phi$ as a function of the internal transmissivity T_1 for four different strategies. We can see that the phase sensitivity of the MMZI is most robust among the four strategies for a wide range of T_1. It is worth noting that the MMZI and SU(1,1)I with ECS inputs can beat the SU(1,1)I with $|\alpha\rangle_a|\xi\rangle_b$ inputs.

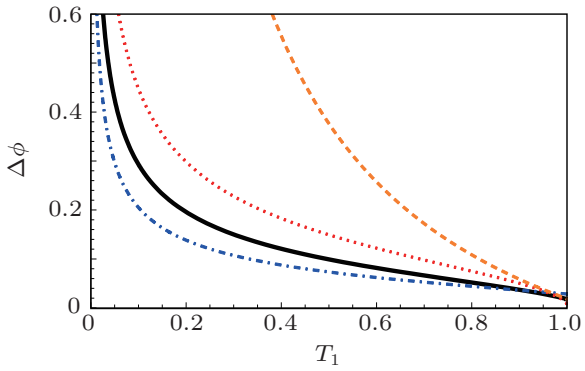


Fig. 4. (color online) The phase sensitivity $\Delta\phi$ as a function of the internal transmissivity rates T_1 with $|\alpha| = 3$, and $\theta_\alpha = \pi/2$. The black solid and blue dot-dashed lines represent the results of Eqs. (13) and (15) for $|\alpha| = 2$ and $g = 2$, respectively. The red dotted line: the result of Ref. [19] with a coherent state and a squeezed-vacuum state as inputs for $|\alpha| = 1.28436$, $r = 1.2$, and $g = 2$. The orange dashed line: the result of Ref. [20] with two coherent states as inputs with $|\alpha| = |\beta| = 1.40144$ and $g = 2$. Here the total average photon number inside the interferometer for the four schemes is the same and is $N_{\rm T} = 120.422$.

4. Conclusions

In conclusion, we investigated two nonlinear interferometers (MMZI and SU(1,1)I) and discussed their phase sensitivity with the ECS as inputs. We found that the phase sensitivity can beat the SNL and approach the HL for large gain factor g and amplitude $|\alpha|$, and the two nonlinear interferometers outperform the linear MZI for a larger average photon number in the absence of photon losses. Particularly, the phase sensitivity of the MMZI is more robust to photon losses than that of the SU(1,1)I in a wide region of losses. Both MMZI and SU(1,1)I with ECS inputs can beat the SU(1,1)I with $|\alpha\rangle|\xi\rangle$

inputs. Nevertheless, the phase sensitivity can still beat the SNL so long as the photon losses are small enough.

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