Kraus operator solutions to a fermionic master equation describing a thermal bath and their matrix representation*

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We solve the fermionic master equation for a thermal bath to obtain its explicit Kraus operator solutions via the fermionic state approach. The normalization condition of the Kraus operators is proved. The matrix representation for these solutions is obtained, which is incongruous with the result in the book completed by Nielsen and Chuang [*Quantum Computation and Quantum Information*, Cambridge University Press, 2000]. As especial cases, we also present the Kraus operator solutions to master equations for describing the amplitude-decay model and the diffusion process at finite temperature.

Keywords: Kraus operator solution, matrix representation, thermal bath, fermionic entangled state representation

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1. Introduction

Quantum theory of open systems is very significant in quantum optics since it provides a very useful tool for investigating the interaction rule of the system and its environment. When the system is immersed in or passes through its surrounding environment, such an interaction happens frequently and is governed by a quantum master equation characterizing the time-evolution of the system. Generally, the method of solving such master equations employs either the Glauber–Sudarshan P-representation or the Wigner function to convert them into the Fokker–Planck equations.[1–3] Recently, instead of this method, Fan *et al.*[4–6] have solved various quantum master equations and obtained the density operators' infinite operator-sum representation (or Kraus operator representation) using the newly developed thermo entangled state representation of which a fictitious mode is introduced as a counterpart of the system mode. As an example, let us review the interaction between the system and the thermal bath. For this case, the bosonic master equation describing the time-evolution of the system in the heat reservoir can be expressed as[7]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = g[2a^{\dagger}\rho a - aa^{\dagger}\rho - \rho aa^{\dagger}] + \kappa[2a\rho a^{\dagger} - a^{\dagger}a\rho - \rho a^{\dagger}a], \tag{1}$$

where ρ is the density operator of the system at time t, $(a^{\dagger}, a)$ is a pair of photon creation and annihilation operators, and the parameters g and κ are related to the heat bath. To solve this master equation (1), the two-mode bosonic entangled state is introduced, whose explicit form reads[8]

$$|\chi\rangle = \exp\Big(-\frac{1}{2}|\chi|^2 + \chi a^{\dagger} - \chi^{*}\tilde{a}^{\dagger} + a^{\dagger}\tilde{a}^{\dagger}\Big)|0\tilde{0}\rangle, \tag{2}$$

where $\tilde{a}^{\dagger}$ is a fictitious mode corresponding to the physical mode $a^{\dagger}$, $|\tilde{0}\rangle$ is annihilated by $\tilde{a}$, $[\tilde{a}, \tilde{a}^{\dagger}] = 1$. Further, the state $|\chi\rangle$ can be rewritten as the form $D(\chi)|\chi = 0\rangle$, where $D(\chi)$ is the displaced operator and $|\chi = 0\rangle = \exp(a^{\dagger}\tilde{a}^{\dagger})|0\tilde{0}\rangle$. On the basis of the state $|\chi = 0\rangle$, the corresponding relations of the operators between the real mode and the fictitious mode can be established, i.e., $a \Leftrightarrow \tilde{a}^{\dagger}$, $a^{\dagger} \Leftrightarrow \tilde{a}$, $a^{\dagger}a \Leftrightarrow \tilde{a}^{\dagger}\tilde{a}$. They can help one convert the density operator equation (1) into the time-evolution equation for the state vector $|\rho\rangle \equiv \rho|\chi = 0\rangle$ and thus solve this master equation concisely. Using this method, its solution of Eq. (1) is of the infinite operator-sum representation[4]

$$\rho = \sum_{i,j=0}^{\infty} M_{ij}\rho_0 M_{ij}^{\dagger}, \tag{3}$$

where ρ_0 is an initial quantum state and M_{ij} is the Kraus operator corresponding to a thermal reservoir, namely

$$M_{ij} = \sqrt{\frac{\kappa^i g^j T_3 T_1^{i+j}}{i!\,j!\,T_2^{2j}}}\,\mathrm{e}^{a^{\dagger}a\ln T_2}a^{\dagger j}a^i, \tag{4}$$

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with the parameters

$$T_1 = \frac{1-\mathrm{e}^{-2(\kappa-g)t}}{\kappa-g\,\mathrm{e}^{-2t(\kappa-g)}},$$
$$T_2 = \frac{(\kappa-g)\,\mathrm{e}^{-(\kappa-g)t}}{\kappa-g\,\mathrm{e}^{-2t(\kappa-g)}},$$
$$T_3 = \frac{\kappa-g}{\kappa-g\,\mathrm{e}^{-2t(\kappa-g)}}.$$

Given a quantum state ρ_0 as an initial state, the output state ρ can be directly calculated from Eq. (3). Next, it is natural to put forward an interesting question whether there exists a fermionic master equation corresponding to Eq. (1)? If yes, what are its Kraus operator solutions? By analogy with the bosonic master equation (1), we present the fermionic master equation of the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = g[2f^\dagger\rho f - ff^\dagger\rho - \rho ff^\dagger] + \kappa[2f\rho f^\dagger - f^\dagger f\rho - \rho f^\dagger f], \tag{5}$$

where $f^\dagger$ and f are fermionic creation and annihilation operators, respectively, $\{f^\dagger, f\} = 1$, $f^{\dagger 2} = f^2 = 0$. Thus, equation (5) refers to the fermionic master equation describing the system exposed to a thermal bath. To solve Eq. (5) and present its Kraus operator solutions, we need to make full use of the fermionic entangled state representation and the fermionic coherent state representation.

2. Fermionic entangled state representation

Note that here the two-mode fermionic entangled state is the form[9]

$$|\eta\rangle = \exp\left(-\frac{1}{2}\bar{\eta}\eta + f^\dagger\eta + \tilde{f}^\dagger\bar{\eta} + f^\dagger\tilde{f}^\dagger\right)|0\tilde{0}\rangle, \tag{6}$$

where $\tilde{f}$ is the fermionic annihilation operator of the fictitious mode representing the effect of environment as a counterpart of the system mode f. $(\eta, \bar{\eta})$ is a pair of Grassmann numbers (the superscript "$-$" refers to a completely independent additional Grassmann variable), $\{\bar{\eta}, \eta\} = 0$ and $\eta^2 = \bar{\eta}^2 = 0$. Actually, the fermionic entangled states $|\eta\rangle$ are just the common eigenvectors of the anti-commutative operators $(f - \tilde{f}^\dagger)$ and $(f^\dagger + \tilde{f})$, such that the corresponding eigen-equations read

$$(f - \tilde{f}^\dagger)|\eta\rangle = \eta|\eta\rangle, \quad (f^\dagger + \tilde{f})|\eta\rangle = \bar{\eta}|\eta\rangle. \tag{7}$$

In addition, using Eq. (6) and the operator identity

$$|00\rangle\langle 00| = :\mathrm{e}^{-f^\dagger f - \tilde{f}^\dagger\tilde{f}}:,$$

the symbol : : denotes normal ordering, we obtain the completeness relation

$$\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,|\eta\rangle\langle\eta|(-1)^{\tilde{f}^\dagger\tilde{f}} = 1$$

and the orthogonality

$$\langle\eta'|(-1)^{\tilde{f}^\dagger\tilde{f}}(f - \tilde{f}^\dagger)|\eta\rangle = \delta(\eta' - \eta)\delta(\bar{\eta}' - \bar{\eta}),$$

which cannot run as smooth as the bosonic case because of the presence of $(-1)^{\tilde{f}^\dagger\tilde{f}}$ as a result of the Pauli exclusive principle of fermions. More interestingly, through defining

$$|\eta = 0\rangle \equiv |I\rangle = \exp(f^\dagger\tilde{f}^\dagger)|0\tilde{0}\rangle = (1 + f^\dagger\tilde{f}^\dagger)|0\tilde{0}\rangle, \tag{8}$$

we find that the state $|I\rangle$ possesses the well-behaved properties

$$f|I\rangle = \tilde{f}^\dagger|I\rangle, \quad \tilde{f}|I\rangle = -f^\dagger|I\rangle, \quad (f^\dagger f)^n|I\rangle = (\tilde{f}^\dagger\tilde{f})^n|I\rangle, \tag{9}$$

which provides us with greater convenience for obtaining the Kraus operator solutions to Eq. (5).

3. Normally ordered form of the operator exp $\mathscr{H}$

Operating both sides of Eq. (5) on the state $|I\rangle$, and denoting $|\rho\rangle = \rho|I\rangle$, and using Eq. (9) we get the time-evolution equation for $|\rho\rangle$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle = [g(2f^\dagger\tilde{f}^\dagger - ff^\dagger - \tilde{f}\tilde{f}^\dagger) + \kappa(2\tilde{f}f - f^\dagger f - \tilde{f}^\dagger\tilde{f})]|\rho\rangle, \tag{10}$$

which is easily solved to yield

$$|\rho\rangle = \exp[g(2f^\dagger\tilde{f}^\dagger - ff^\dagger - \tilde{f}\tilde{f}^\dagger)t + \kappa(2\tilde{f}f - \tilde{f}^\dagger\tilde{f} - f^\dagger f)t]|\rho_0\rangle. \tag{11}$$

For convenience, letting

$$\mathscr{H} \equiv g(2f^\dagger\tilde{f}^\dagger - ff^\dagger - \tilde{f}\tilde{f}^\dagger)t + \kappa(2\tilde{f}f - \tilde{f}^\dagger\tilde{f} - f^\dagger f)t,$$

thus the exponential operator exp $\mathscr{H}$ implies quantum entanglement between the system and its surrounding environment. To disentangle the two-mode fermionic exponential operator exp $\mathscr{H}$, we write the operator exp $\mathscr{H}$ as

$$\exp\mathscr{H} = \mathrm{e}^{(\kappa+g)t}\exp\left(\frac{1}{2}B\Gamma\tilde{B}\right), \tag{12}$$

where B is defined as

$$B \equiv \begin{pmatrix} F^\dagger & F \end{pmatrix} \equiv (f^\dagger\ \tilde{f}^\dagger\ f\ \tilde{f}), \quad \tilde{B} \equiv \begin{pmatrix} \tilde{F}^\dagger \\ \tilde{F} \end{pmatrix}, \tag{13}$$

and Γ is a 4×4 matrix, i.e.,

$$\Gamma \equiv \begin{pmatrix} 2gtJ_2 & (g-\kappa)tI_2 \\ (\kappa-g)tI_2 & 2\kappa tJ_2^T \end{pmatrix}, \tag{14}$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_2^2 = -I_2. \tag{15}$$

On the the hand, the operator exp $\mathscr{H}$ can also be transformed into the two-mode fermionic coherent state representation as follows:

$$\exp\mathscr{H}=\frac{1}{\sqrt{\det Q}}\int\prod_{i=1}^{2}\mathrm{d}\bar{\alpha}_i\mathrm{d}\alpha_i\left|\begin{pmatrix}Q & -L\\ -N & P\end{pmatrix}\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right|, \tag{16}$$

where the two-mode fermionic coherent state is defined as

$$\left|\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\rangle=\exp(F^{\dagger}\alpha-\bar{\alpha}F)\,|00\rangle\,, \tag{17}$$

$$\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}=(\alpha_1,\alpha_2,\bar{\alpha}_1,\bar{\alpha}_2), \tag{18}$$

$(\alpha,\bar{\alpha})$ is a pair of Grassmann numbers, and

$$\begin{pmatrix}Q & L\\ N & P\end{pmatrix}=\exp(\Gamma\Pi),\ \ \Pi=\begin{pmatrix}0 & I_2\\ I_2 & 0\end{pmatrix}. \tag{19}$$

Based on Eqs. (14) and (19), we have

$$Q=\frac{g\mathrm{e}^{(g+\kappa)t}+\kappa\mathrm{e}^{-(g+\kappa)t}}{g+\kappa}I_2,$$

$$P=\frac{\kappa\mathrm{e}^{(g+\kappa)t}+g\mathrm{e}^{-(g+\kappa)t}}{g+\kappa}I_2,$$

$$L=\frac{g\left(\mathrm{e}^{(g+\kappa)t}-\mathrm{e}^{-(g+\kappa)t}\right)}{g+\kappa}J_2,$$

$$N=\frac{\kappa\left(\mathrm{e}^{(g+\kappa)t}-\mathrm{e}^{-(g+\kappa)t}\right)}{g+\kappa}J_2^{T}. \tag{20}$$

Thus, using the technique of integration within an ordered product of operators to carry out the integration (16), we get the normally ordered expansion of the operator $\exp\mathscr{H}$, i.e.,

$$\exp\mathscr{H}=\sqrt{\det P}\colon\exp\left\{\frac{1}{2}F^{\dagger}\left(LP^{-1}\right)\tilde{F}^{\dagger}+F^{\dagger}\left(\tilde{P}^{-1}-I\right)\tilde{F}+\frac{1}{2}F\left(P^{-1}N\right)\tilde{F}\right\}\colon. \tag{21}$$

Further, using the operator identity $\mathrm{e}^{\lambda f^{\dagger}f}=\colon\exp[(\mathrm{e}^{\lambda}-1)f^{\dagger}f]\colon$, we obtain

$$\exp\mathscr{H}=\sqrt{\det P}\exp\left[\frac{1}{2}F^{\dagger}\left(LP^{-1}\right)\tilde{F}^{\dagger}\right]\exp\left[F^{\dagger}\ln\tilde{P}^{-1}\tilde{F}\right]\exp\left[\frac{1}{2}F\left(P^{-1}N\right)\tilde{F}\right]. \tag{22}$$

Substituting Eqs. (13) and (20) into Eq. (22) leads to the form

$$\begin{aligned}\exp\mathscr{H}&=T_4\exp(T_1f^{\dagger}\tilde{f}^{\dagger})\exp\left[(\tilde{f}^{\dagger}\tilde{f}+f^{\dagger}f)\ln(T_2+1)\right]\exp(T_3\tilde{f}f)\\&=T_4(1+T_1f^{\dagger}\tilde{f}^{\dagger})(1+T_2f^{\dagger}f)(1+T_2\tilde{f}^{\dagger}\tilde{f})(1+T_3\tilde{f}f)\\&=T_4(1+T_2\tilde{f}^{\dagger}\tilde{f}+T_1f^{\dagger}\tilde{f}^{\dagger}+T_2f^{\dagger}f+T_2^2f^{\dagger}f\ \tilde{f}^{\dagger}\tilde{f}+T_3\tilde{f}f+T_1T_3f^{\dagger}\tilde{f}^{\dagger}\tilde{f}f),\end{aligned} \tag{23}$$

where the parameters T_1,T_2,T_3, and T_4 are, respectively

$$T_1=\frac{g\mathrm{e}^{2(g+\kappa)t}-g}{\kappa\mathrm{e}^{2(g+\kappa)t}+g},\quad T_2=\frac{g+\kappa}{g\mathrm{e}^{-(g+\kappa)t}+\kappa\mathrm{e}^{(g+\kappa)t}}-1,\quad T_3=\frac{\kappa\mathrm{e}^{2(g+\kappa)t}-\kappa}{\kappa\mathrm{e}^{2(g+\kappa)t}+g},\quad T_4=\frac{g\mathrm{e}^{-2(g+\kappa)t}+\kappa}{g+\kappa}. \tag{24}$$

4. Kraus operator solutions to Eq. (5) and its matrix representation

Using Eqs. (9) and (23), equation (11) can be written as

$$|\rho\rangle=T_4[\rho_0+T_3f\rho_0f^{\dagger}+T_1f^{\dagger}\rho_0f+T_2\left(\rho_0f^{\dagger}f+f^{\dagger}f\rho_0\right)+\left(T_2^2+T_1T_3\right)f^{\dagger}f\rho_0f^{\dagger}f]\,|I\rangle\,, \tag{25}$$

which leads to the operator-sum representation of ρ

$$\rho=T_4\left(T_3f\rho_0f^{\dagger}+T_1f^{\dagger}\rho_0f+\left(1+T_2f^{\dagger}f\right)\rho_0\left(1+T_2f^{\dagger}f\right)+T_1T_3f^{\dagger}f\rho_0f^{\dagger}f\right)=\sum_{m=1}^{4}M_m\rho_0M_m^{\dagger}, \tag{26}$$

where M_m are identified as the Kraus operators for the fermionic thermal reservoir, i.e.,

$$\begin{aligned}&M_1=\sqrt{T_3T_4}f,\ \ M_2=\sqrt{T_1T_4}f^{\dagger},\\&M_3=\sqrt{T_4}\left(1+T_2f^{\dagger}f\right),\\&M_4=\sqrt{T_1T_3T_4}f^{\dagger}f,\end{aligned} \tag{27}$$

which show that M_1 refers to the decay from $|1\rangle$ to $|0\rangle$ (quantum jump) by emitting a photon and M_2 represents the quantum jump contrary to M_1, but M_3 and M_4 show how the state evolves when no quantum jump occurs.

The Kraus operator solutions in Eq. (27) are obtained from density operators for fermionic systems, but it can be used to study two-level atomic systems as the presence of two singlets. For a two-level atomic system, the ground ($|0\rangle$) and

excited ($|1\rangle$) states can be written for the matrix representation as $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, such that the matrix representations of Fermi operators f and $f^\dagger$ are obtained as, respectively

$$f = |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{28}$$

Therefore, the Kraus operators M_m become

$$\begin{aligned}
M_1 &= \sqrt{T_3T_4}f = \sqrt{\mathscr{P}}\begin{pmatrix} 0 & \sqrt{\mathscr{T}} \\ 0 & 0 \end{pmatrix}, \quad M_2 = \sqrt{T_1T_4}f^\dagger = \sqrt{1-\mathscr{P}}\begin{pmatrix} 0 & 0 \\ \sqrt{\mathscr{T}} & 0 \end{pmatrix}, \\
M_3 &= \sqrt{T_4}\left(1+T_2f^\dagger f\right) = \begin{pmatrix} \sqrt{\mathscr{P}+(1-\mathscr{P})(1-\mathscr{T})} & 0 \\ 0 & \sqrt{\dfrac{1-\mathscr{T}}{\mathscr{P}+(1-\mathscr{P})(1-\mathscr{T})}} \end{pmatrix}, \\
M_4 &= \sqrt{T_1T_3T_4}f^\dagger f = \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{T}\sqrt{\dfrac{\mathscr{P}(1-\mathscr{P})}{\mathscr{P}+(1-\mathscr{P})(1-\mathscr{T})}} \end{pmatrix},
\end{aligned} \tag{29}$$

where $\mathscr{P} = \kappa/(g+\kappa)$ and $\mathscr{T} = 1-\mathrm{e}^{-2(g+\kappa)t}$. Next, we can prove

$$\sum_{i=0}^{4} M_m M_m^\dagger = 1, \tag{30}$$

which indicates the normalization for M_m. The normalization condition leads to the following relation:

$$\mathrm{Tr}\,\rho = \mathrm{Tr}\left(\sum_{m=0}^{4}\rho_0 M_m M_m^\dagger\right) = \mathrm{Tr}\,\rho_0 = 1, \tag{31}$$

so M_m is a trace-preserving quantum operation. By means of performing the following unitary matrix

$$\mathscr{U}_{nm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\dfrac{\mathscr{P}}{\mathscr{P}+(1-\mathscr{P})(1-\mathscr{P})}} & -\sqrt{\dfrac{(1-\mathscr{P})(1-\mathscr{T})}{\mathscr{P}+(1-\mathscr{P})(1-\mathscr{T})}} \\ 0 & 0 & \sqrt{\dfrac{(1-\mathscr{P})(1-\mathscr{T})}{\mathscr{P}+(1-\mathscr{P})(1-\mathscr{T})}} & \sqrt{\dfrac{\mathscr{P}}{\mathscr{P}+(1-\mathscr{P})(1-\mathscr{P})}} \end{pmatrix} \tag{32}$$

to transform M_m into $\mathscr{M}_n = \mathscr{U}_{nm}M_m$, which still holds the operator identities

$$\rho = \sum_{n=1}^{4}\mathscr{M}_n\rho_0\mathscr{M}_n^\dagger \quad \text{and} \quad \sum_{n=1}^{4}\mathscr{M}_n^\dagger\mathscr{M}_n = 1,$$

such that the new Kraus operators $\mathscr{M}_n$ reads

$$\begin{aligned}
\mathscr{M}_1 &= \sqrt{\mathscr{P}}\begin{pmatrix} 0 & \sqrt{\mathscr{T}} \\ 0 & 0 \end{pmatrix}, \quad \mathscr{M}_2 = \sqrt{1-\mathscr{P}}\begin{pmatrix} 0 & 0 \\ \sqrt{\mathscr{T}} & 0 \end{pmatrix}, \\
\mathscr{M}_3 &= \sqrt{\mathscr{P}}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\mathscr{T}} \end{pmatrix}, \\
\mathscr{M}_4 &= \sqrt{1-\mathscr{P}}\begin{pmatrix} \sqrt{1-\mathscr{T}} & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned} \tag{33}$$

which is in accordance with the standard form of the generalized amplitude damping model.[10] Usually, in the literature before, one used the matrix representation (33) to investigate the quantum decoherence of fermionic systems in a thermal bath.[10,11] Here we can see that employing the Kraus operator-sum representation (29) is more convenient and concise; the detailed exposition can be seen as shown below.

Moreover, the fermionic thermal reservoir is general enough to reduce to other decoherence modes for some special values of the parameters related to the thermal bath. For convenience, replacing g and κ with $\kappa\bar{\mathrm{n}}$ and $\kappa(\bar{\mathrm{n}}+1)$, respectively, thus when $\bar{\mathrm{n}} \to 0$ and κ keeps finite, $\mathscr{P} \to 1$, $\mathscr{T} \to 1-\mathrm{e}^{-2kt}$, so equation (29) becomes

$$\begin{aligned}
\mathscr{M}_1 &= \sqrt{T_3T_4}f = \begin{pmatrix} 0 & \sqrt{\mathscr{T}} \\ 0 & 0 \end{pmatrix}, \quad \mathscr{M}_2 = 0, \\
\mathscr{M}_3 &= \sqrt{T_4}\left(1+T_2f^\dagger f\right) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\mathscr{T}} \end{pmatrix}, \quad \mathscr{M}_4 = 0,
\end{aligned} \tag{34}$$

which is the matrix representation of the Kraus operators for the amplitude decay model, as shown in the literature.[11,12] The Kraus operator $\mathscr{M}_1$ shows that the decay only occurs in this process. In the case of $\kappa \to 0$ and $\bar{\mathrm{n}} \to \infty$ (but the product $\kappa\bar{\mathrm{n}}$ being finite), due to the fact that $\mathscr{P} \to 0$, $\mathscr{T} \to$

$1-\mathrm{e}^{-2k\bar{n}t} \equiv \mathfrak{T}$, equation (29) reduces to the matrix representation of the Kraus operator corresponding to the fermionic diffusion process at finite temperature, namely

$$\mathscr{M}_1 = 0, \quad \mathscr{M}_2 = \sqrt{T_1 T_4} f^\dagger = \begin{pmatrix} 0 & 0 \\ \sqrt{\mathfrak{T}} & 0 \end{pmatrix},$$
$$\mathscr{M}_3 = \sqrt{T_4}\left(1 + T_2 f^\dagger f\right) = \begin{pmatrix} \sqrt{1-\mathfrak{T}} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathscr{M}_4 = 0, \quad (35)$$

which indicates that not only the diffusion effect ($\mathscr{M}_3$ means no quantum jump) but also quantum excitation ($\mathscr{M}_2$ refers to the excitation from $|0\rangle$ to $|1\rangle$) occurs since the temperature is not zero in the diffusion evolution process.

As an application of Kraus operator solutions (27) to Eq. (5), we now want to discuss how a two-level atom, denoted with subscript A, evolves via absorbing or emitting a photon in the photon field, denoted with subscript E, for the amplitude decay model. When the atom $\rho_{A,|0\rangle}(0)$ evolves into

$$\begin{aligned} \rho_{A,|0\rangle}(t) &= \mathscr{M}_1 |0\rangle_{AA} \langle 0| \mathscr{M}_1^\dagger + \mathscr{M}_3 |0\rangle_{AA} \langle 0| \mathscr{M}_3^\dagger \\ &= |0\rangle_{AA} \langle 0|, \end{aligned} \quad (36)$$

which shows no decay and no photon appears. By entering the photon field, we have

$$|0\rangle_A |0\rangle_E \to |0\rangle_A |0\rangle_E, \quad (37)$$

which shows that the interaction system remains unchanged for the dephasing environment. When the excited atom state $\rho_{A,|1\rangle}(0)$ evolves into

$$\rho_{A,|1\rangle}(t) = (1-T') |1\rangle_{AA} \langle 1| + T' |0\rangle_{AA} \langle 0|,$$

where $T' = 1 - \mathrm{e}^{-2kt}$. Thus, $\rho_A(t)$ also means the decoherence process of a two-level system in a dissipative environment. So,

$$|1\rangle_A |0\rangle_E \to \sqrt{1-\mathscr{T}'} |1\rangle_A |0\rangle_E + \sqrt{\mathscr{T}'} |0\rangle_A |1\rangle_E. \quad (38)$$

Obviously, T' is the probability that $|1\rangle_A$ has decayed to $|0\rangle_A$ and a photon has been emitted so that the photon field has made a transition from $|0\rangle_E$ to $|1\rangle_E$. When $t \to \infty$, $|1\rangle_A |0\rangle_E$ becomes $|0\rangle_A |1\rangle_E$, which means all atoms finally return to ground state for a long time.

In summary, using the fermionic state representation theory, we have addressed the fermionic master equation for a thermal bath and obtained the Kraus operator-sum representation of the density operator describing the time-evolution of the Fermi system. Also, the Kraus operator-sum form is equivalent to the existing matrix representation of the Kraus operators describing the thermal bath. Especially, the Kraus operator-sum representation of density operators as solutions to master equations for describing the amplitude decay model and the diffusion process at finite temperature are presented.

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