(2+1)-dimensional dissipation nonlinear Schrödinger equation for envelope Rossby solitary waves and chirp effect*

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In the past few decades, the (1+1)-dimensional nonlinear Schrödinger (NLS) equation had been derived for envelope Rossby solitary waves in a line by employing the perturbation expansion method. But, with the development of theory, we note that the (1+1)-dimensional model cannot reflect the evolution of envelope Rossby solitary waves in a plane. In this paper, by constructing a new (2+1)-dimensional multiscale transform, we derive the (2+1)-dimensional dissipation nonlinear Schrödinger equation (DNLS) to describe envelope Rossby solitary waves under the influence of dissipation which propagate in a plane. Especially, the previous researches about envelope Rossby solitary waves were established in the zonal area and could not be applied directly to the spherical earth, while we adopt the plane polar coordinate and overcome the problem. By theoretical analyses, the conservation laws of (2+1)-dimensional envelope Rossby solitary waves as well as their variation under the influence of dissipation are studied. Finally, the one-soliton and two-soliton solutions of the (2+1)-dimensional NLS equation are obtained with the Hirota method. Based on these solutions, by virtue of the chirp concept from fiber soliton communication, the chirp effect of envelope Rossby solitary waves is discussed, and the related impact factors of the chirp effect are given.

Keywords: (2+1)-dimensional dissipation nonlinear Schrödinger equation, envelope Rossby solitary waves, chirp effect, two-soliton solutions

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1. Introduction

It is generally known that the ocean and atmosphere move endlessly in complex patterns and on diverse scales. To discuss the whole dynamics of this enormous spectrum of motions is almost impossible. So our special interest is the so-called large scale motions which are prominently affected by the rotation effect of the earth. Among these large scale motions, the nonlinear Rossby waves have attracted considerable attention from the oceanic and atmospheric field in recent years. Long[1] first found the Rossby solitary waves in the westerly shear flow in the 1960s, then Benney,[2] Redekopp[3] and other investigators pushed the research forward. In the 1970s∼1980s, driven by blocking dynamics, the nonlinear theory of Rossby solitary waves achieved rapid development and formed the solitary waves theory, dipole waves theory, and envelope solitary waves theory. For the Rossby solitary waves, some weakly nonlinear models for the evolution of Rossby solitary waves have been extensively studied in the past. Among these, the Korteweg–de Vries (KdV) equation,[4–6] modified KdV equation,[7,8] and Boussinesq equation[9] were derived to describe the classical Rossby solitary waves; the Benjamin–Ono equation[10,11] was obtained to show the algebraic Rossby solitary waves with non-uniform horizontal shear of zonal flow in infinite depth fluid, while the Intermediate Long Wave equation[12,13] was used to describe the condition in finite depth fluid. For the envelope Rossby solitary waves, the nonlinear Schrödinger (NLS) equation was first obtained by Benney[14] and Yamagata,[15] respectively. Later, Luo[16] also derived the NLS equation to express the envelope Rossby solitary waves and showed that the atmospheric blocking in the mid-high latitudes could be explained better by employing a dipole envelope Rossby solitary wave. With the development of theory, the higher order NLS equation[17] was derived to describe the dynamic behavior of envelope Rossby solitary waves. However, all the above-mentioned models are (1+1)-dimensional and used to describe the propagation of Rossby solitary waves or envelope Rossby solitary waves in a line, which are not enough to reflect the evolution of Rossby solitary waves in the real ocean and atmosphere. It seems that few previous researches involved the problem. The purpose of this paper is to obtain a new (2+1)-dimensional dissipation nonlinear Schrödinger equation (DNLS), which is more suitable

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to describe the evolution of envelope Rossby solitary waves in the real ocean and atmosphere. Especially, due to the dissipation character of the real oceanic and atmospheric motion, we will embed the dissipation effect in the study.

On the other hand, in the fiber soliton communication field, the chirp effect is proposed to express frequency modulation in the process of transmission. The interaction of the nonlinear and dispersive effect can cause the excursion of the center wave and lead to chirp effect. Drawing lessons from the chirp concept in fiber soliton communication, Song[18] investigated the chirp effect of an internal wave, while we will discuss the chirp effect during the propagation of envelope Rossby solitary waves.

Besides the above-mentioned problems, we shall use these methods to solve the soliton equations, such as the first integral method,[19] similarity transformation,[20] Jacobi elliptic function expansion method,[21] Bäcklund transformation method,[22] Darboux transformation method,[23] and so on. In the paper, we will derive the one-soliton and two-soliton solutions of the (2+1)-dimensional NLS equation by employing the Hirota method[24] and analyze the chirp effect based on the analytical solutions.

This paper is organized as follows. In Section 2, from the rotational potential vorticity-conserved equation with dissipation, by constructing a (2+1)-dimensional stretching transformation of time and space, a (2+1)-dimensional DNLS equation is derived, which is a generalization of the (1+1)-dimensional NLS equation obtained by the former. In Section 3, by theoretical analyses, the character of the (2+1)-dimensional envelope Rossby solitary waves under the influence of dissipation is studied. Section 4 will be devoted to investigating the one-soliton and two-soliton solutions of the DNLS equation by using the Hirota method and dissipation effect. Based on these analytical solutions, the chirp effect of envelope Rossby solitary waves will be discussed in Section 5. Some conclusions are given in the last section.

2. Derivation of the (2+1)-dimensional DNLS equation

The basic equation of motion used in previous studies is the following non-dimensional quasi-geostrophic equation in the form of potential-vorticity:

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi,\nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0. \tag{1}$$

But we note that a disadvantage exists for Eq. (1): envelope Rossby solitary waves are established in the zonal area, and cannot be applied directly to the spherical case; we need to consider envelope Rossby solitary waves in rotational fluid. So let us take the plane polar coordinate (r,θ), the rotational potential vorticity-conserved equation including dissipation is, in non-dimensional form, written as

$$\left[\frac{\partial}{\partial t}+\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\right)\right]\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]+\frac{\beta}{r}\frac{\partial\psi}{\partial\theta} = -\lambda_0\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]+Q, \tag{2}$$

where some symbols need to be explained, ψ denotes the non-dimensional stream function, $\beta=\beta_0(L^2/U)$ and $\beta_0=(\omega_0/R_0)\cos\phi_0$, in which R_0 represents the earth's radius, ω_0 is the angular frequency of the earth's rotation, ϕ_0 is the latitude; the first term on the right-hand side of Eq. (2) expresses the vorticity dissipation induced by the Ekman boundary layer, and $\lambda_0>0$ is the dissipative coefficient; the second term on the right-hand side of Eq. (2) denotes the external source and the form will be given in the latter.

Because of the multiple time and space scales feature of Rossby waves, in order to consider the nonlinear effect, we take the following slow scales:

$$T_1=\varepsilon t,\ \ T_2=\varepsilon^2 t,\ \ \Theta_1=\varepsilon\theta,\ \ \Theta_2=\varepsilon^2\theta,\ \ R=\varepsilon r, \tag{3}$$

where ε is a small parameter which weighs the weakness of the nonlinearity.

Based on the WKB method, we can expand the stream function as

$$\begin{aligned}\psi = &\int_0^r(\Omega(s)-c_0)s\,\mathrm{d}s+\varepsilon\psi_1(T_1,T_2,\Theta_1,\Theta_2,R,\theta,r,t)\\ &+\varepsilon^2\psi_2(T_1,T_2,\Theta_1,\Theta_2,R,\theta,r,t)\\ &+\varepsilon^3\psi_3(T_1,T_2,\Theta_1,\Theta_2,R,\theta,r,t)+\cdots,\end{aligned} \tag{4}$$

where the first term on the right-hand side of Eq. (4) means the basic stream which rotates around the polar at angular velocity $\Omega=\Omega(r)$; c_0 is the angular position phase velocity of perturbation and a constant.

Here for the sake of achieving a balance between nonlinearity and turbulent dissipation

$$\lambda_0=\varepsilon^2\lambda, \tag{5}$$

meanwhile restricting ourselves to discuss the dissipation caused by the perturbation stream and eliminate the dissipation caused by the basic stream, so we take the external source as

$$Q=\lambda_0\frac{1}{r}\frac{\partial}{\partial r}[r^2(\Omega-c_0)]. \tag{6}$$

Substituting Eqs. (3)–(6) to Eq. (2), by means of equating the coefficients of like powers of ε, the following set of equations for perturbation stream function ψ can be obtained

$$
\begin{aligned}
O(\varepsilon^1):\ F\psi_1 &= \left[\frac{\partial}{\partial t}+(\Omega(r)-c_0)r\frac{\partial}{\partial\theta}\right]\nabla^2\psi_1+\beta\frac{\partial\psi_1}{\partial\theta}=0,\\
O(\varepsilon^2):\ F\psi_2 &= -\left[\frac{\partial}{\partial T_1}+(\Omega(r)-c_0)r\frac{\partial}{\partial\Theta_1}\right]\nabla^2\psi_1-2\left[\frac{\partial}{\partial t}+(\Omega-c_0)r\frac{\partial}{\partial\theta}\right]\left(\frac{\partial^2\psi_1}{\partial\theta\partial\Theta_1}+\frac{\partial^2\psi_1}{\partial r\partial R}\right)-\beta\frac{\partial\psi_1}{\partial\Theta_1}\\
&\quad-\frac{\partial\psi_1}{\partial\theta}\frac{\partial}{\partial r}\nabla^2\psi_1+\frac{\partial\psi_1}{\partial r}\frac{\partial}{\partial\theta}\nabla^2\psi_1,\\
O(\varepsilon^3):\ F\psi_3 &= -\left[\frac{\partial}{\partial T_2}+(\Omega(r)-c_0)r\frac{\partial}{\partial\Theta_2}+\lambda\right]\nabla^2\psi_1-2\left[\frac{\partial}{\partial t}+(\Omega-c_0)r\frac{\partial}{\partial\theta}\right]\frac{\partial^2\psi_1}{\partial\theta\partial\Theta_2}-2\left[\frac{\partial}{\partial T_1}+(\Omega-c_0)r\frac{\partial}{\partial\Theta_1}\right]\\
&\quad\times\left(\frac{\partial^2\psi_1}{\partial\theta\partial\Theta_1}+\frac{\partial^2\psi_1}{\partial R\partial r}\right)+\frac{\partial^2\psi_1}{\partial r\partial R}-\beta\frac{\partial\psi_1}{\partial\Theta_2}-2\left[\frac{\partial}{\partial t}+(\Omega-c_0)r\frac{\partial}{\partial\theta}\right]\left(\frac{\partial^2\psi_2}{\partial\theta\partial\Theta_1}+\frac{\partial^2\psi_2}{\partial R\partial r}\right)\\
&\quad-\left[\frac{\partial}{\partial t}+(\Omega-c_0)r\frac{\partial}{\partial\theta}\right]\left(\frac{\partial^2\psi_2}{\partial\Theta_1^2}+\frac{\partial^2\psi_2}{\partial R^2}\right)-\left[\frac{\partial}{\partial T_1}+(\Omega-c_0)r\frac{\partial}{\partial\Theta_1}\right]\nabla^2\psi_2-\left(\frac{\partial\psi_2}{\partial\theta}+\frac{\partial\psi_1}{\partial\Theta_1}\right)\frac{\partial}{\partial r}\nabla^2\psi_1\\
&\quad-\frac{\partial\psi_1}{\partial\theta}\left(\frac{\partial}{\partial R}\nabla^2\psi_1+\frac{\partial}{\partial r}\nabla^2\psi_2\right)-2\frac{\partial\psi_1}{\partial\theta}\frac{\partial}{\partial r}\left(\frac{\partial^2\psi_1}{\partial\theta\partial\Theta_1}+\frac{\partial^2\psi_1}{\partial R\partial r}\right)-\beta\frac{\partial\psi_2}{\partial\Theta_1}\\
&\quad-\frac{\partial\psi_1}{\partial r}\left[\frac{\partial\nabla^2\psi_2}{\partial\theta}+\frac{\partial\nabla^2\psi_1}{\partial\Theta_1}+2\frac{\partial}{\partial\theta}\left(\frac{\partial^2\psi_1}{\partial\theta\partial\Theta_1}-\frac{\partial^2\psi_1}{\partial R\partial r}\right)\right]-\left(\frac{\partial\psi_1}{\partial R}+\frac{\partial\psi_2}{\partial r}\right)\frac{\partial\nabla^2\psi_1}{\partial\theta},
\end{aligned}
\tag{7}
$$

where for simplicity, we redefine the Laplace operator ∇^2 in the plane polar coordinate as

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{8}$$

Assume the perturbation does not exist at the boundaries, so we take the boundary conditions as

$$\psi_0=\psi_1=\cdots=0,\qquad r=0,r_1. \tag{9}$$

We look for an amplitude equation, so in the order of ε^1, assuming ψ_1 is periodic in θ and t in the form of

$$\psi_1=\mathrm{Re}\,A(T_1,T_2,\Theta_1,\Theta_2,R)\phi_1(r)\,\mathrm{e}^{\mathrm{i}(k\theta-\omega t)}, \tag{10}$$

where A is the complex wave amplitude; k represents the polar angle direction wave number of Rossby waves; ω denotes the frequency of Rossby waves; *Re* stands for the real part.

Substituting Eq. (10) into the first equation of Eq. (7) yields

$$
\begin{aligned}
&\frac{\mathrm{d}^2\phi_1}{\mathrm{d}r^2}+\left(\frac{\beta k}{k(\Omega(r)-c_0)r-\omega}-k^2\right)\phi_1=0,\\
&\phi_1(0)=\phi_1(r_1)=0.
\end{aligned}
\tag{11}
$$

The polar axis direction structure on the small scale r of ϕ_1 can be determined at the leading order, meanwhile, we can obtain the dispersion relation of linear Rossby waves

$$\omega=(\Omega-c_0)rk-\frac{\beta k}{k^2+m^2} \tag{12}$$

and the corresponding eigenfunction

$$\phi_1(r)=\sqrt{\frac{2}{r_1}}\sin(mr). \tag{13}$$

where the eigenvalue $m=n\pi/r_1$, $n=\pm1,2,3,\ldots$, and ϕ_1 satisfies

$$\int_0^{r_1}\phi_1^2(r)\,\mathrm{d}r=1.$$

In order to obtain the governing equation of complex amplitude A, we need to consider the higher order problem. At the next order, $O(\varepsilon^2)$, by introducing Eq. (10), we can find that to obtain the solution of ψ_2, it is necessary to eliminate these secular terms. So the following equation for nonsecularity must be satisfied

$$\phi_1\left(\frac{\partial A}{\partial T_1}+C_\mathrm{g}\frac{\partial A}{\partial\Theta_1}\right)-\frac{2\mathrm{i}[(\Omega(r)-c_0)rk-\omega]}{k^2+m^2}\frac{\mathrm{d}\phi_1}{\mathrm{d}r}\frac{\partial A}{\partial R}=0, \tag{14}$$

where

$$C_\mathrm{g}=(\Omega(r)-c_0)r-\beta\frac{m^2-k^2}{(m^2+k^2)^2}$$

denotes the group velocity of Rossby waves. By employing Eq. (14), the second equation of Eq. (7) can be rewritten as

$$F(\psi_2)=\left[\frac{\partial}{\partial t}+(\Omega-c_0)r\frac{\partial}{\partial\theta}\right]\nabla^2(\psi_2)+\beta\frac{\partial\psi_2}{\partial\theta}=0. \tag{15}$$

Equation (15) admits solution in the form

$$\psi_2=\bar{\psi}_2(T_1,T_2,\Theta_1,\Theta_2,R,r). \tag{16}$$

But we can find that the solution ψ_2 cannot be obtained in the first two order equations, so in order to obtain the form of ψ_2, it is necessary to consider the next order equation.

To determine ψ_2, we then substitute the solution (16) into the third equation of Eq. (7) to obtain the governing equation

for ψ_2 and corresponding boundary conditions as follows:

$$\begin{cases} \left[\dfrac{\partial}{\partial T_1}+(\Omega(r)-c_0)r\dfrac{\partial}{\partial\Theta_1}\right]\dfrac{\partial^2\bar{\psi}_2}{\partial r^2} \\ \qquad +\beta\dfrac{\partial\bar{\psi}_2}{\partial\Theta_1}=-\dfrac{4pk^2m}{r_1}\sin(2mr)\dfrac{\partial|A|^2}{\partial\Theta_1}, \\ \bar{\psi}_2|_{r=0}=\bar{\psi}_2|_{r=r_1}=0, \end{cases} \tag{17}$$

where p is a constant. Introducing the transformation $\Theta=\Theta_1-C_gT_1$ and using the time scale $T_2=\varepsilon T_1$ in Eq. (3), then equation (17) can be rewritten as

$$\begin{cases} \varepsilon\dfrac{\partial}{\partial T_2}\dfrac{\partial^2\bar{\psi}_2}{\partial r^2}+[(\Omega(r)-c_0)r-C_g]\dfrac{\partial}{\partial\Theta}\dfrac{\partial^2\bar{\psi}_2}{\partial r^2} \\ \qquad +\beta\dfrac{\partial\bar{\psi}_2}{\partial\Theta}=-\dfrac{4pk^2m}{r_1}\sin(2mr)\dfrac{\partial|A|^2}{\partial\Theta}, \\ \bar{\psi}_2|_{r=0}=\bar{\psi}_2|_{r=r_1}=0. \end{cases} \tag{18}$$

Due to $\varepsilon\ll1$, in view of the fact that the first term of the first equation in Eq. (18) is always less than the others, the approximate form of Eq. (18) is

$$\begin{cases} [(\Omega(r)-c_0)r-C_g]\dfrac{\partial}{\partial\Theta}\dfrac{\partial^2\bar{\psi}_2}{\partial r^2} \\ \qquad +\beta\dfrac{\partial\bar{\psi}_2}{\partial\Theta}=-\dfrac{4pk^2m}{r_1}\sin(2mr)\dfrac{\partial|A|^2}{\partial\Theta}, \\ \bar{\psi}_2|_{r=0}=\bar{\psi}_2|_{r=r_1}=0. \end{cases} \tag{19}$$

It is easy to find that the solution of Eq. (19) can be expressed as

$$\psi_2=\bar{\psi}_2=\frac{4pmk^2(k^2+m^2)^2\sin(2mr)|A|^2}{\beta r_1[3m^2(m^2-2k^2)-k^4]}. \tag{20}$$

Substituting Eqs. (10) and (20 into the third equation of Eq. (7) and comparing the coefficients of $\exp\mathrm{i}(k\Theta-\Omega t)$ leads to

$$\begin{aligned} &-(k^2+m^2)\phi_1\left[\frac{\partial A}{\partial T_2}+(\Omega(r)-c_0)r\frac{\partial A}{\partial\Theta_2}+\lambda A\right]-2k[(\Omega(r)-c_0)rk-\Omega]\phi_1\frac{\partial A}{\partial\Theta_2}+\beta\phi_1\frac{\partial A}{\partial\Theta_2} \\ &+2\left[\frac{\partial}{\partial T_1}+(\Omega(r)-c_0)r\frac{\partial}{\partial\Theta_1}\right]\left(\mathrm{i}k\phi_1\frac{\partial A}{\partial\Theta_1}+\frac{d\phi_1}{dr}\frac{\partial A}{\partial R}\right)+\mathrm{i}[(\Omega(r)-c_0)rk-\Omega]\phi_1\left(\frac{\partial^2A}{\partial\Theta_1^2}+\frac{\partial^2A}{\partial R^2}\right) \\ &+8\mathrm{i}pk^3m^2(k^2-3m^2)(k^2+m^2)^2\frac{\cos(2mr)\phi_1A|A|^2}{\beta r_1[3m^2(m^2-2k^2)-k^4]}=0. \end{aligned} \tag{21}$$

Then multiply by ϕ_1 and integrate from 0 to r_1 with respect to r, by virtue of the boundary conditions, it turns out that the nonlinear equation for the evolution of complex amplitude A satisfies the following form

$$\mathrm{i}\left(\frac{\partial A}{\partial T_2}+C_g\frac{\partial A}{\partial\Theta_2}\right)+\alpha\frac{\partial^2A}{\partial\Theta_1^2}+\gamma\frac{\partial^2A}{\partial R^2}+\delta|A^2|A+\mathrm{i}\lambda A=0, \tag{22}$$

where the coefficients are as follows:

$$\begin{aligned} \alpha&=\frac{\beta pk(3m^2-k^2)}{(k^2+m^2)^3}, \\ \gamma&=\frac{\beta pk(5m^2+k^2)}{(k^2+m^2)^3}, \\ \delta&=\frac{4pk^3m^2(3m^2-k^2)(k^2+m^2)}{\beta r_1[3m^2(m^2-2k^2)-k^4]}. \end{aligned}$$

Based on Ref. [24], adopting the following coordinate transformations:

$$\Theta=(\Theta_2-C_gT_2)/\varepsilon=\Theta_1-C_gT_1,\quad T=T_2, \tag{23}$$

then equation (22) can be rewritten as

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha\frac{\partial^2A}{\partial\Theta^2}+\gamma\frac{\partial^2A}{\partial R^2}+\delta|A^2|A+\mathrm{i}\lambda A=0. \tag{24}$$

The above equation depends on either the time variable T or the space variables Θ and R; $\mathrm{i}\lambda A$ stands for the dissipation effect, when $\lambda=0$, i.e., the dissipation effect is absent, the equation (24) reduces to the (2+1)-dimensional NLS equation; so we call Eq. (24) the (2+1)-dimensional DNLS. The equation is prior to (1+1)-dimensional NLS equation, because it is more suitable to describe the evolution of envelope Rossby solitary waves in a plane instead of in a line. Besides, the equation (24) is established by virtue of the plane polar coordinate and can be applied directly to the spherical earth, while the NLS equation which is established in the rectangular coordinate does not have this advantage.

3. The effect of dissipation on envelope Rossby solitary waves

The theoretical model of envelope Rossby solitary waves has been derived in the above section, now we will discuss the evolution of envelope Rossby solitary waves with the dissipation effect. As we know, many physical characteristics of solitary waves remain invariant during the various processes which happen in the physics world; these invariant physics characteristics are called conservation laws. Conservation laws play an important role in the research on the unsteady problem of waves. We will focus on the change of conservation laws of envelope Rossby solitary waves under the influence of dissipation.

In the following, assuming $A,A^*\to0$ as $|R|\to+\infty$ and the values of A,A^* at $\Theta=0$ equal those at $\Theta=2\pi$, where the superscript * stands for the complex conjugate.

The complex conjugate equation of Eq. (24) is given as follows:

$$-\mathrm{i}\frac{\partial A^*}{\partial T}+\alpha\frac{\partial^2 A^*}{\partial \Theta^2}+\gamma\frac{\partial^2 A^*}{\partial R^2}+\delta|A^2|A^*-\mathrm{i}\lambda A^*=0. \quad (25)$$

equation (24) $\times A^*$ – equation (25) $\times A$ leads to

$$\mathrm{i}(A_T A^*+A_T^* A)+\alpha(A_{\Theta\Theta}A^*-A_{\Theta\Theta}^* A)+\gamma(A_{RR}A^*-A_{RR}^* A_R)+2\mathrm{i}\lambda AA^*=0, \quad (26)$$

i.e.,

$$\mathrm{i}\frac{\partial}{\partial T}|A|^2+\alpha\frac{\partial}{\partial\Theta}\left(A^*\frac{\partial A}{\partial\Theta}-A\frac{\partial A^*}{\partial\Theta}\right)+\gamma\frac{\partial}{\partial R}\left(A^*\frac{\partial A}{\partial R}-A\frac{\partial A^*}{\partial R}\right)+2\mathrm{i}\lambda|A|^2=0. \quad (27)$$

Integrating Eq. (27) with respect to Θ from 0 to 2π and with respect to R from $-\infty$ to $+\infty$, gives

$$\left(\frac{\partial}{\partial T}+2\lambda\right)\int_{-\infty}^{+\infty}\int_0^{2\pi}|A|^2\,\mathrm{d}\Theta\,\mathrm{d}R=0. \quad (28)$$

It is easy to find that

$$\int_{-\infty}^{+\infty}\int_0^{2\pi}|A|^2\,\mathrm{d}\Theta\,\mathrm{d}R=\exp(-2\lambda T)\int_{-\infty}^{+\infty}\int_0^{2\pi}|A_0|^2\,\mathrm{d}\Theta\,\mathrm{d}R, \quad (29)$$

where A_0 is the initial value of A. Equation (29) indicates that dissipation arouses the mass of envelope Rossby solitary waves which decrease exponentially, then when $\lambda=0$, i.e., dissipation is absent, the mass of envelope Rossby solitary waves is conserved.

In the following, performing equation (24) $\times A_\Theta^*$ + equation (25) $\times A_\Theta$ leads to

$$\mathrm{i}\left(\frac{\partial A}{\partial T}\frac{\partial A^*}{\partial\Theta}-\frac{\partial A^*}{\partial T}\frac{\partial A}{\partial\Theta}\right)+\alpha\left(\frac{\partial^2 A}{\partial\Theta^2}\frac{\partial A^*}{\partial\Theta}+\frac{\partial^2 A^*}{\partial\Theta^2}\frac{\partial A}{\partial\Theta}\right)+\gamma\left(\frac{\partial^2 A}{\partial R^2}\frac{\partial A^*}{\partial\Theta}+\frac{\partial^2 A^*}{\partial R^2}\frac{\partial A}{\partial\Theta}\right)+\delta|A|^2\frac{\partial|A|^2}{\partial\Theta}+\mathrm{i}\lambda\left(A\frac{\partial A^*}{\partial\Theta}-A^*\frac{\partial A}{\partial\Theta}\right)=0. \quad (30)$$

Equation (24) $\times A_R^*$ + equation (25) $\times A_R$ gives

$$\mathrm{i}\left(\frac{\partial A}{\partial T}\frac{\partial A^*}{\partial R}-\frac{\partial A^*}{\partial T}\frac{\partial A}{\partial R}\right)+\alpha\left(\frac{\partial^2 A}{\partial\Theta^2}\frac{\partial A^*}{\partial R}+\frac{\partial^2 A^*}{\partial\Theta^2}\frac{\partial A}{\partial R}\right)+\gamma\left(\frac{\partial^2 A}{\partial R^2}\frac{\partial A^*}{\partial R}+\frac{\partial^2 A^*}{\partial R^2}\frac{\partial A}{\partial R}\right)+\delta|A|^2\frac{\partial|A|^2}{\partial R}+\mathrm{i}\lambda\left(A\frac{\partial A^*}{\partial R}-A^*\frac{\partial A}{\partial R}\right)=0. \quad (31)$$

Then equation (30) $\times\,\alpha$ + equation (31) $\times\,\gamma$ gives

$$\mathrm{i}\frac{\partial}{\partial T}\left(\alpha A\frac{\partial A^*}{\partial\Theta}+\gamma A\frac{\partial A^*}{\partial R}\right)+\mathrm{i}\lambda\left[\alpha\left(A\frac{\partial A^*}{\partial\Theta}-A^*\frac{\partial A}{\partial\Theta}\right)+\gamma\left(A\frac{\partial A^*}{\partial R}-A^*\frac{\partial A}{\partial R}\right)\right]+\alpha\gamma\left[\frac{\partial}{\partial R}\left(\frac{\partial A}{\partial R}\frac{\partial A^*}{\partial\Theta}-\frac{\partial A}{\partial\Theta}\frac{\partial A^*}{\partial\Theta}\right)+\frac{\partial}{\partial\Theta}\left(\frac{\partial A^*}{\partial R}\frac{\partial A}{\partial\Theta}-\frac{\partial A}{\partial R}\frac{\partial A^*}{\partial R}\right)\right]+\delta|A|^2\left(\alpha\frac{\partial|A|^2}{\partial\Theta}+\gamma\frac{\partial|A|^2}{\partial R}\right)+\alpha^2\frac{\partial}{\partial\Theta}\left(\frac{\partial A}{\partial\Theta}\frac{\partial A^*}{\partial\Theta}\right)+\gamma^2\frac{\partial}{\partial R}\left(\frac{\partial A}{\partial R}\frac{\partial A^*}{\partial R}\right)=0. \quad (32)$$

Integrating Eq. (32) with respect to Θ from 0 to 2π and with respect to R from $-\infty$ to $+\infty$ again, we can obtain

$$\frac{\partial}{\partial T}\int_{-\infty}^{+\infty}\int_0^{2\pi}\left(\alpha A\frac{\partial A^*}{\partial\Theta}+\gamma A\frac{\partial A^*}{\partial R}\right)\mathrm{d}\Theta\,\mathrm{d}R=-\lambda\int_{-\infty}^{+\infty}\int_0^{2\pi}\left[\alpha\left(A\frac{\partial A^*}{\partial\Theta}-A^*\frac{\partial A}{\partial\Theta}\right)+\gamma\left(A\frac{\partial A^*}{\partial R}-A^*\frac{\partial A}{\partial R}\right)\right]\mathrm{d}\Theta\,\mathrm{d}R. \quad (33)$$

In the same way, by a direct calculation, we have

$$\frac{\partial}{\partial T}\int_{-\infty}^{+\infty}\int_0^{2\pi}\left(\alpha A^*\frac{\partial A}{\partial\Theta}+\gamma A^*\frac{\partial A}{\partial R}\right)\mathrm{d}\Theta\,\mathrm{d}R=-\lambda\int_{-\infty}^{+\infty}\int_0^{2\pi}\left[\alpha\left(A^*\frac{\partial A}{\partial\Theta}-A\frac{\partial A^*}{\partial\Theta}\right)+\gamma\left(A^*\frac{\partial A}{\partial R}-A\frac{\partial A^*}{\partial R}\right)\right]\mathrm{d}\Theta\,\mathrm{d}R. \quad (34)$$

Equations (33) and (34) give

$$\left(\frac{\partial}{\partial T}+2\lambda\right)\int_{-\infty}^{+\infty}\int_0^{2\pi}\left[\alpha\left(A\frac{\partial A^*}{\partial\Theta}-A^*\frac{\partial A}{\partial\Theta}\right)+\gamma\left(A\frac{\partial A^*}{\partial R}-A^*\frac{\partial A}{\partial R}\right)\right]\mathrm{d}\Theta\,\mathrm{d}R=0. \quad (35)$$

Based on Eq. (35), by analogy with the KdV equation, it is very obvious that the momentum of envelope Rossby solitary waves is conserved without dissipation; when the dissipation is present, the momentum of envelope Rossby solitary waves decreases exponentially with the dissipation coefficient λ and the evolution of time. Furthermore, we find that the decay rate of the momentum equals that of mass.

Finally, let us continue to consider the third conservation quantity. equation (24) $\times A_T^*$ + equation (25) $\times A_T$, we have

$$\frac{\partial}{\partial T}\left(|A|^4-\left|\frac{\partial A}{\partial\Theta}\right|^2-\left|\frac{\partial A}{\partial R}\right|^2\right)+\frac{\partial}{\partial\Theta}\left(\frac{\partial A}{\partial\Theta}\frac{\partial A^*}{\partial T}+\frac{\partial A^*}{\partial\Theta}\frac{\partial A}{\partial T}\right)+\frac{\partial}{\partial R}\left(\frac{\partial A}{\partial R}\frac{\partial A^*}{\partial T}+\frac{\partial A^*}{\partial R}\frac{\partial A}{\partial T}\right)+\mathrm{i}\lambda\left(A\frac{\partial A^*}{\partial T}-A^*\frac{\partial A}{\partial T}\right)=0. \quad (36)$$

From Eq. (36), we can obtain

$$\frac{\partial}{\partial T}\int_{-\infty}^{+\infty}\int_0^{2\pi}\left(|A|^4-\left|\frac{\partial A}{\partial\Theta}\right|^2-\left|\frac{\partial A}{\partial R}\right|^2\right)\mathrm{d}\Theta\,\mathrm{d}R=-\mathrm{i}\lambda\int_{-\infty}^{+\infty}\int_0^{2\pi}\left(A\frac{\partial A^*}{\partial T}-A^*\frac{\partial A}{\partial T}\right)\mathrm{d}\Theta\,\mathrm{d}R. \quad (37)$$

By Eq. (37), we can find that the energy of envelope Rossby solitary waves is conserved without dissipation; the energy of envelope Rossby solitary waves decreases with dissipation, but their decay relation cannot be derived, which is worth studying in the future.

As mentioned above, three conservation quantities of (2+1)-dimensional envelope Rossby solitary waves have been derived. Furthermore, the conservation laws are destroyed due to dissipation, which shows that the dissipation has an important influence for the evolution of (2+1)-dimensional envelope Rossby solitary waves.

4. One and two-soliton solutions of (2+1)-dimensional NLS equation and chirp effect of envelope Rossby solitary waves

In the above section, we have analyzed the variation of conservation laws under the influence of the dissipation effect and revealed the impact of dissipation on the evolution of envelope Rossby solitary waves. This section will be devoted to discussing the chirp effect of envelope Rossby solitary waves and assume that the dissipation is absent. In order to make the reader understand the content more clearly, it is necessary to explain the concept of chirp. The concept of chirp comes from the optical soliton communication area and is also called the frequency modulation effect. During the propagation of the wave, due to the nonlinear and dispersion effect, the excursion phenomenon of the center wave will occur, i.e., the chirp effect will occur. Here drawing lessons from the chirp concept of optical soliton communication, we will study the frequency modulation effect of (2+1)-dimensional envelope Rossby solitary waves. It is very meaningful for recognizing the character of envelope Rossby solitary waves. In order to accomplish this aim, we must explore the analytical solutions of the (2+1)-dimensional NLS equation. As noted previously, many methods have been proposed to solve soliton equations; here we adopt the Hirota method.

Neglecting the dissipation effect ($\lambda = 0$), equation (24) is rewritten as

$$\mathrm{i}\frac{\partial A}{\partial T} + \alpha\frac{\partial^2 A}{\partial \Theta^2} + \gamma\frac{\partial^2 A}{\partial R^2} + \delta|A|^2 A = 0. \tag{38}$$

For Eq.(38), first let us make the following bilinear transform

$$A = \frac{G}{F}, \tag{39}$$

substituting Eq. (39) into Eq. (38), then equation (38) can be rewritten as the following bilinear representation:

$$\begin{aligned} &(\mathrm{i}D_T + \alpha D_\Theta^2 + \gamma D_R^2)G\cdot F = 0,\\ &(\alpha D_\Theta^2 + \gamma D_R^2)F\cdot F = \delta|G|^2. \end{aligned} \tag{40}$$

Letting

$$G = \xi g_1, \quad F = 1 + \xi^2 f_1, \tag{41}$$

substituting Eq. (41) into Eq. (40), we can obtain the different order equations of ξ as follows:

$$\begin{aligned} &O(\xi):\ (\mathrm{i}D_T + \alpha D_\Theta^2 + \gamma D_R^2)g_1\cdot 1 = 0,\\ &O(\xi^2):\ (2\alpha D_\Theta^2 + 2\gamma D_R^2)f_1\cdot 1 = \delta g_1\cdot g_1^*,\\ &O(\xi^3):\ (\mathrm{i}D_T + \alpha D_\Theta^2 + \gamma D_R^2)g_1\cdot f_1 = 0,\\ &O(\xi^4):\ (\alpha D_\Theta^2 + \gamma D_R^2)f_1\cdot f_1 = 0. \end{aligned} \tag{42}$$

Taking

$$g_1 = \exp\eta_1 = \exp(p_1\Theta + q_1 R + wT), \tag{43}$$

substituting Eq. (43) into the first equation of Eq. (42), we have

$$\mathrm{i}w + \alpha p_1^2 + \gamma q_1^2 = 0, \tag{44}$$

where $p_1, q_1 \in C$. Substituting Eq. (44) into the second equation of Eq. (42), we have

$$f_1 = \frac{\delta}{2[\alpha(p_1+p_1^*)^2 + \gamma(q_1+q_1^*)^2]}\exp(\eta_1+\eta_1^*). \tag{45}$$

Letting $\xi = 1$, then we can get the one-soliton solution of Eq. (38)

$$\begin{aligned} A &= \frac{\exp(p_1\Theta + q_1 R + wT)}{1 + \dfrac{\delta}{2[\alpha(p_1+p_1^*)^2 + \gamma(q_1+q_1^*)^2]}\exp(\eta_1+\eta_1^*)}\\ &= \sqrt{\frac{1}{2\delta}[\alpha(p_1+p_1^*)^2 + \gamma(q_1+q_1^*)^2]}\,\mathrm{sech}(p_1\Theta + q_1 R)\exp[\mathrm{i}(\alpha p_1^2 + \gamma q_1^2)T]. \end{aligned} \tag{46}$$

In fact, we can also derive the two-solitons solution of Eq. (38). Taking

$$\begin{aligned} G &= \xi g_1 + \xi^3 g_2,\\ F &= 1 + \xi^2 f_1 + \xi^4 f_2, \end{aligned} \tag{47}$$

substituting Eq. (47) into Eq. (40), the different order equations of ξ are given as follows:

$$\begin{aligned} &O(\xi):\ (\mathrm{i}D_T + \alpha D_\Theta^2 + \gamma D_R^2)g_1\cdot 1 = 0,\\ &O(\xi^2):\ (2\alpha D_\Theta^2 + 2\gamma D_R^2)f_1\cdot 1 = \delta g_1\cdot g_1^*, \end{aligned}$$

$$
\begin{aligned}
&O(\xi^3):\ (\mathrm{i}D_T+\alpha D_\Theta^2+\gamma D_R^2)(g_1\cdot f_1+g_2\cdot 1)=0,\\
&O(\xi^4):\ (\alpha D_\Theta^2+\gamma D_R^2)f_1\cdot f_1+(2\alpha D_\Theta^2\\
&\qquad +2\gamma D_R^2)f_2\cdot 1=\delta(g_2\cdot g_1^*+g_1\cdot g_2^*),\\
&O(\xi^5):\ (\mathrm{i}D_T+\alpha D_\Theta^2+\gamma D_R^2)(g_2\cdot f_1+g_1\cdot f_2)=0,\\
&O(\xi^6):\ (\alpha D_\Theta^2+\gamma D_R^2)f_1\cdot f_2=\delta g_2\cdot g_2^*,\\
&O(\xi^7):\ (\mathrm{i}D_T+\alpha D_\Theta^2+\gamma D_R^2)g_2\cdot f_2=0,\\
&O(\xi^8):\ (\alpha D_\Theta^2+\gamma D_R^2)f_2\cdot f_2=0. \qquad (48)
\end{aligned}
$$

Then letting

$$
g_1=\exp\eta_1+\exp\eta_2=\exp(p_1\Theta+q_1R+wT)
$$

$$
+\exp(p_2\Theta+q_2R+wT), \qquad (49)
$$

where $p_1,q_1,p_2,q_2\in C$.

After similar procedure and tedious calculation, we have

$$
\begin{aligned}
&\mathrm{i}w+\alpha p_1^2+\gamma q_1^2=0,\\
&\mathrm{i}w+\alpha p_2^2+\gamma q_2^2=0.
\end{aligned}
$$

Then, we can obtain

$$
2\mathrm{i}w+\alpha(p_1^2+p_2^2)+\gamma(q_1^2+q_2^2)=0. \qquad (50)
$$

By virtue of Eq. (50), we have

$$
\begin{aligned}
f_1=&\ m_1\exp(\eta_1+\eta_1^*)+m_2\exp(\eta_2+\eta_1^*)+m_3\exp(\eta_1+\eta_2^*)+m_4m_1\exp(\eta_2+\eta_2^*),\\
g_2=&\ k_1\exp(2\eta_1+\eta_1^*)+k_2\exp(2\eta_1+\eta_2^*)+k_3\exp(\eta_1+\eta_1^*+\eta_2)+k_4\exp(\eta_1+\eta_2+\eta_2^*)\\
&+k_5\exp(\eta_1+\eta_2+\eta_1^*)+k_6\exp(\eta_1+\eta_2+\eta_1^*)+k_7\exp(2\eta_2+\eta_1^*)+k_8\exp(2\eta_2+\eta_2^*),\\
f_2=&\ l_1\exp(2\eta_1+2\eta_1^*)+l_2\exp(2\eta_1+\eta_1^*+\eta_2^*)+l_3\exp(\eta_1+\eta_2+2\eta_1^*)+l_4\exp(\eta_1+\eta_2+\eta_1^*+\eta_2^*)\\
&+l_5\exp(2\eta_2+2\eta_1^*)+l_6\exp(2\eta_2+\eta_1^*+\eta_2^*)+l_7\exp(2\eta_1+2\eta_2^*)+l_8\exp(\eta_1+\eta_2+2\eta_2^*)\\
&+l_9\exp(2\eta_2+2\eta_2^*), \qquad (51)
\end{aligned}
$$

where m_i $(i=1,2,3,4)$, k_i $(i=1,2,\ldots,8)$, l_i $(i=1,2,\ldots,9)$ are all constants; for the detailed form see Appendix A. Taking $\xi=1$ and substituting Eqs. (47) and (51) into Eq. (39), the two-solitons solution of Eq. (38) is given. We note that the two-solitons solution is very meaningful for discussing the interaction between (2+1)-dimensional solitary waves. We will study the problem in the future.

In the following, we will study the chirp effect of the (2+1)-dimensional envelope Rossby solitary waves based on the solution (46). The initial wave shape is as

$$
A=A_0\,\mathrm{sech}(p_1\Theta+q_1R) \qquad (52)
$$

according to Eq. (46), where A_0 denotes the initial amplitude. First, let us consider the dispersion effect and overlook the nonlinear effect, then equation (38) is written as

$$
\frac{\partial A}{\partial T}=\mathrm{i}\left(\alpha\frac{\partial^2A}{\partial\Theta^2}+\gamma\frac{\partial^2A}{\partial R^2}\right). \qquad (53)
$$

Observing the time T from 0 to ΔT, where ΔT is an infinitesimal variable, then the following approximate solution of Eq. (53) is obtained

$$
A(\Delta T,\Theta,R)=A_0\,\mathrm{sech}(p_1\Theta+q_1R)\exp\left[\mathrm{i}\frac{\alpha p_1^2+\gamma q_1^2}{p_1^2(p_1^2+q_1^2)}(1-2\,\mathrm{sech}^2(p_1\Theta+q_1R))\Delta T\right], \qquad (54)
$$

then we can get the phase of the wave as

$$
\varphi_{\mathrm{D}}=\frac{\alpha p_1^2+\gamma q_1^2}{p_1^2(p_1^2+q_1^2)}(1-2\,\mathrm{sech}^2(p_1\Theta+q_1R))\Delta T, \qquad (55)
$$

the chirp caused by dispersion effect is

$$
\triangle\nu_D=-\nabla\varphi_D=4\frac{\alpha p_1^2+\gamma q_1^2}{p_1^2(p_1^2+q_1^2)}(p_1+q_1)\Delta T\,\mathrm{sech}^2(p_1\Theta+q_1R)\tanh(p_1\Theta+q_1R). \qquad (56)
$$

Next, focusing on the chirp caused by nonlinear, then equation (38) can be changed as

$$
\frac{\partial A}{\partial T}=\mathrm{i}\delta|A|^2A. \qquad (57)
$$

In the same way, observing the time from 0 to ΔT, then the approximate solution is

$$
\begin{aligned}
A(\Delta t,x,y)=&\ A_0\,\mathrm{sech}(p_1\Theta+q_1R)\\
&\times\exp[\mathrm{i}\delta A_0^2\mathrm{sech}^2(p_1\Theta+q_1R)\Delta T], \qquad (58)
\end{aligned}
$$

then it is easy to find that the phase of the wave is

$$\varphi_N = \delta A_0^2 \operatorname{sech}^2(p_1\Theta + q_1 R)\Delta T. \tag{59}$$

So the chirp caused by nonlinear is

$$\begin{aligned}\Delta\nu_N = &-\nabla\varphi_N = -2\delta A_0^2(p_1+q_1)\\ &\times \operatorname{sech}^2(p_1\Theta+q_1R)\tanh(p_1\Theta+q_1R)\Delta T.\end{aligned} \tag{60}$$

By using Eqs. (56) and (60), we can obtain the whole chirp as

$$\begin{aligned}\Delta\nu_S = \Delta\nu_D + \nu_N = &\left(4\frac{\alpha p_1^2+\gamma q_1^2}{p_1^2(p_1^2+q_1^2)} - 2\delta A_0^2\right)(p_1+q_1)\\ &\times \operatorname{sech}^2(p_1\Theta+q_1R)\tanh(p_1\Theta+q_1R)\Delta T.\end{aligned} \tag{61}$$

By analyzing Eq. (61), we can find that when the initial amplitude A_0 satisfies

$$A_0 = \sqrt{2\frac{\alpha p_1^2+\gamma q_1^2}{p_1^2(p_1^2+q_1^2)\delta}}, \tag{62}$$

the whole chirp $\Delta\nu_S = 0$, which means that the dispersion balances to the nonlinear; when

$$A_0 < \sqrt{2\frac{\alpha p_1^2+\gamma q_1^2}{p_1^2(p_1^2+q_1^2)\delta}}, \tag{63}$$

i.e., $|\Delta\nu_D| > |\Delta\nu_N|$, then the dispersion effect is more than the nonlinear effect; when

$$A_0 > \sqrt{2\frac{\alpha p_1^2+\gamma q_1^2}{p_1^2(p_1^2+q_1^2)\delta}}, \tag{64}$$

i.e., $|\Delta\nu_D| < |\Delta\nu_N|$, then the dispersion effect is less than the nonlinear effect.

In conclusion, the chirp effect depends on the initial amplitude, the critical value is given in Eq. (62). Based on the critical value, we draw the conclusion that the chirp effect not only relates to the polar angle direction wave number k, the frequency of Rossby waves Ω as well as Coriolis force parameter β, but also the modal number m of (2+1)-dimensional envelope Rossby solitary waves.

5. Conclusion

In the paper, based on the multiscale and perturbation method, by constructing a new (2+1)-dimensional multiscale transform, the (2+1)-dimensional dissipation NLS equation to describe envelope Rossby solitary waves propagating in a plane under the influence of dissipation is derived. By theoretical analysis, the conservation laws of (2+1)-dimensional envelope Rossby solitary waves is obtained. Furthermore, their variation state under the influence of dissipation reveals the impact of dissipation on the evolution of envelope Rossby solitary waves. By employing the Hirota method, after tedious calculation, the one-soliton, two-solitons solutions of the (2+1)-dimensional NLS equation are given. One-soliton solution is used to discuss the chirp effect and two-solitons solution possesses potential application in the inaction of envelope Rossby solitary waves. Finally, with the help of the one-soliton solution, the chirp effect of (2+1)-dimensional envelope Rossby solitary waves is studied. By analysis, we find the critical value of the chirp effect and the related factors with chirp effect, i.e., the polar angle direction wave number k, the frequency of Rossby waves Ω, the Coriolis force parameter β as well as the modal number m of (2+1)-dimensional envelope Rossby solitary waves.

Appendix A

All the constants, m_i $(i = 1,2,3,4)$, k_i $(i = 1,2,\ldots,8)$, l_i $(i = 1,2,\ldots,9)$, that appeared in Eq. (51) are given below.

$$m_1 = \frac{\delta}{8(\lambda p_1^2 + Qq_1^2)},$$

$$m_2 = m_3 = \frac{\delta}{2\lambda(p_1^2+p_2^2)+2Q(q_1+q_2)^2},$$

$$m_4 = \frac{\delta}{8(\lambda p_2^2 + Qq_2^2)};$$

$$k_1 = \frac{(\mathrm{i}w+\lambda p_1^2+Qq_1^2)m_1}{-\mathrm{i}w-9\lambda p_1^2-9Qq_1^2},$$

$$k_2 = \frac{(\mathrm{i}w+\lambda p_2^2+Qq_2^2)m_2}{-\mathrm{i}w-\lambda(2p_1+p_2)^2-Q(2q_1+q_2)^2},$$

$$k_3 = \frac{(\mathrm{i}w+\lambda p_2^2+Qq_2^2)m_3}{-\mathrm{i}w-\lambda(2p_1+p_2)^2-Q(2q_1+q_2)^2},$$

$$k_4 = \frac{[\mathrm{i}w+\lambda(p_1-2p_2)^2+Q(q_1-2q_2)^2]m_4}{-\mathrm{i}w-\lambda(p_1+2p_2)^2-Q(q_1+2q_2)^2},$$

$$k_5 = \frac{[\mathrm{i}w+\lambda(p_2-2p_1)^2+Q(p_2-2p_1^2)]m_1}{-\mathrm{i}w-\lambda(2p_1+p_2)^2-Q(2q_1+q_2)^2},$$

$$k_6 = \frac{(\mathrm{i}w+\lambda p_1^2+Qq_1^2)m_2}{-\mathrm{i}w-\lambda(p_1+2p_2)^2-Q(q_1+2q_2)^2},$$

$$k_7 = \frac{(\mathrm{i}w+\lambda p_1^2+Qq_1^2)m_3}{-\mathrm{i}w-\lambda(p_1+2p_2)^2-Q(q_1+2q_2)^2},$$

$$k_8 = \frac{(\mathrm{i}w+\lambda p_2^2+Qq_2^2)m_4}{-\mathrm{i}w-9\lambda p_2^2-9Qq_2^2};$$

$$l_1 = \frac{k_1}{32\lambda p_1^2+32Qq_1^2},$$

$$l_2 = \frac{k_1+k_2-2\lambda(p_1-p_2)^2m_1m_2-2Q(q_1-q_2)^2m_1m_2}{2\lambda(3p_1+p_2)^2+2Q(3q_1+q_2)^2},$$

$$l_3 = \frac{k_3+k_5-2\lambda(p_1-p_2)^2m_1m_3-2Q(q_1-q_2)^2m_1m_3}{2\lambda(3p_1+p_2)^2+2Q(3q_1+q_2)^2},$$

$$l_4=\frac{k_4+k_6+k_3+k_5-8\lambda(p_1-p_2)^2m_1m_4 8Q(q_1-q_2)^2m_1m_4}{8\lambda(p_1+p_2)^2+8Q(q_1+q_2)^2},\quad l_5=\frac{k_7}{8\lambda(p_1+p_2)^2+8Q(q_1+q_2)^2},$$

$$l_6=\frac{k_7+k_8-2\lambda(p_1-p_2)^2m_3m_4-2Q(q_1-q_2)^2m_3m_4}{2\lambda(p_1+3p_2)^2+2Q(q_1+3q_2)^2},\quad l_7=\frac{k_2}{8\lambda(p_1+p_2)^2+8Q(q_1+q_2)^2},$$

$$l_8=\frac{k_4+k_6-2\lambda(p_1-p_2)^2m_2m_4-2Q(q_1-q_2)^2m_2m_4}{2\lambda(p_1+3p_2)^2+2Q(q_1+3q_2)^2},\quad l_9=\frac{k_8}{32\lambda p_2^2+32Qq_2^2}.$$

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