Successive lag synchronization on dynamical networks with communication delay*

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In this paper, successive lag synchronization (SLS) on a dynamical network with communication delay is investigated. In order to achieve SLS on the dynamical network with communication delay, we design linear feedback control and adaptive control, respectively. By using the Lyapunov function method, we obtain some sufficient conditions for global stability of SLS. To verify these results, some numerical examples are further presented. This work may find potential applications in consensus of multi-agent systems.

Keywords: successive lag synchronization, dynamical network, communication delay, feedback control

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1. Introduction

As an important branch of network science, synchronization of complex networks has been paid great attention and studied deeply in the past two decades. It is well known that synchronization plays an important role in various fields, such as laser systems, communication systems, and parallel image processing.[1–6] In the research of complex dynamical networks, a variety of synchronization patterns have been defined and studied, including complete synchronization,[7] partial synchronization,[8] cluster synchronization,[9,10] projective synchronization,[11] phase synchronization,[12] lag synchronization (LS),[13] generalized synchronization,[14] etc. Among these patterns, the LS between two coupled systems has been studied extensively.[15–17]

Most existing researches of LS are based on two coupled dynamical systems (or networks). The LS is achieved if $y(t) \to x(t-\tau), \tau > 0$ as $t \to +\infty$, which is investigated originally in Ref. [13]. From then on, LS has gradually become a research focus. The LS has been generalized to many other forms, including projective lag synchronization,[18] generalized lag synchronization[19] and successive lag synchronization (SLS).[20] Here, SLS, a new kind of lag synchronization in a multi-coupled dynamical system proposed by Li in Ref. [20], can be considered as a generalized pattern of the traditional LS in two coupled systems. SLS means that a node achieves the LS with its next node, i.e., $x_i(t-\tau) \to x_{i+1}(t)$ for a positive time delay τ as $t \to +\infty$. It may be found potential applications of SLS in multi-agent systems. For example, in military parade, sometimes we can see that the aircrafts (nodes) pass the same spatial position with the same time delay to avoid collision and keep orderly queue, which can be considered as the SLS realization of these aircrafts. In Ref. [20], the authors supposed that there is no communication delay between nodes in the network means that each node in the network can immediately receive the coupling signals from its neighbor nodes. However, because of the finiteness of signal transmission and switching speed, the communication delay is inevitable in real dynamical systems.[21] Communication delay often appears in numerous systems frequently, such as aircraft, seismology, and laser systems.

Motivated by the above discussions, the main aim of this paper is to construct a general dynamical network with communication delay and investigate its SLS. Based on the constructed dynamical network, we apply linear feedback control and adaptive control to achieve its SLS. Depending on topological structure of the considered dynamical network, a linear feedback control is designed to realize its SLS. Moreover, in order to choose the feasible control adaptively, an adaptive linear feedback control is also proposed. By using the Lyapunov function method and Barbalat's Lemma, some sufficient conditions for the global stability of SLS are obtained.

The rest of this paper is organized as follows. In Section 2, we give some Lemmas and some preliminaries about the model formulation of a dynamical network with communication delay. In Section 3, we investigate the global stability conditions of the SLS of the dynamical network under linear feedback control and adaptive linear feedback control, respectively. In Section 4, some numerical examples are performed to verify the theoretical results. Finally, in Section 5, we con-

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clude the paper.

2. Preliminaries

The considered dynamical network with communication delay under control is described as follows:

$$\dot{\boldsymbol{x}}_i(t) = f(\boldsymbol{x}_i(t)) + c\sum_{j=1}^{n} a_{ij}\boldsymbol{x}_j(t-\tau) + \boldsymbol{u}_i(t), \quad i = 1,2,\ldots,n, \tag{1}$$

where $\boldsymbol{x}_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{im}(t))^{\mathrm{T}} \in \mathbb{R}^m$ is the state vector of the i-th node, $\tau > 0$ is the communication delay, $c > 0$ denotes the coupling strength, and $\boldsymbol{u}_i(t)$ denotes the feedback control input into the i-th node. The function $f(\cdot)$ is a continuously differentiable function. The matrix $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$ is irreducible, showing the topological structure of the network. If nodes i and j are connected, then $a_{ij} = a_{ji} = 1$ ($i \neq j$ and $i, j = 1,2,\ldots,n$), otherwise $a_{ij} = a_{ji} = 0$ ($i \neq j$ and $i, j = 1,2,\ldots,n$). $a_{ii} = -\sum_{j=1, j\neq i}^{n} a_{ij} = -k_i$ ($i = 1,2,\ldots,n$), where k_i denotes the degree of the i-th node. So $\sum_{j=1}^{n} a_{ij} = 0$, for all $i \in \{1,2,\ldots,n\}$. The network is required to be connected, i.e., $k_i \neq 0$ for all $i \in \{1,2,\ldots,n\}$. Certainly, a tree structure is always contained in the network.

Definition 1[20] Successive lag synchronization (SLS) of dynamical network is said to be achieved if, for any initial condition $\boldsymbol{x}_i(t) = \boldsymbol{\varphi}_i(t) \in \mathscr{C}([-(n-2)\tau' - \tau, 0], \mathbb{R})$ and every $i \in \{1,2,\ldots,n-1\}$,

$$\lim_{t\to+\infty} \|\boldsymbol{x}_i(t-\tau') - \boldsymbol{x}_{i+1}(t)\| = 0, \tag{2}$$

where $\mathscr{C}([-(n-2)\tau' - \tau, 0], \mathbb{R})$ denotes function space of all continuous functions from $[-(n-2)\tau' - \tau, 0]$ to $\mathbb{R}$, and the time delay parameter $\tau' > 0$.

It should be noticed that we should first number the nodes in a network according to the Definition 1 to prepare for the study of its certain SLS. The schematic figure of SLS is shown in Fig. 1.

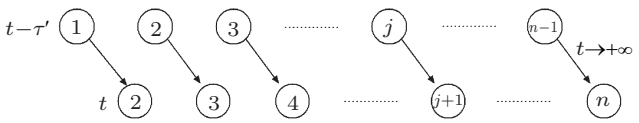


Fig. 1. The schematic figure of successive lag synchronization in a network with size n.

Definition 2[22] Function class QUAD($\boldsymbol{\Delta}, P, \boldsymbol{\omega}$): let a diagonal matrix $\boldsymbol{\Delta} = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_m\}$ and a positive-definite diagonal matrix $\boldsymbol{P} = \mathrm{diag}\{p_1, p_2, \ldots, p_m\}$. QUAD($\boldsymbol{\Delta}, \boldsymbol{P}, \boldsymbol{\omega}$) denotes a class of continuous functions $f(\boldsymbol{x}, t) : \mathbb{R}^m \times [0, +\infty) \to \mathbb{R}^m$ satisfying

$$(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}\boldsymbol{P}\{[f(\boldsymbol{x},t) - f(\boldsymbol{y},t)] - \boldsymbol{\Delta}[\boldsymbol{x}-\boldsymbol{y}]\} \leq -\boldsymbol{\omega}(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{y}), \tag{3}$$

for some $\boldsymbol{\omega} > 0$, all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ and $t \geq 0$.

Lemma 1[23] For a matrix $\boldsymbol{M} = (m_{ij})_{p\times q}$, the inequality

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y} \leq \pi(\boldsymbol{M})(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}), \tag{4}$$

holds for all $\boldsymbol{x} \in \mathbb{R}^p$, $\boldsymbol{y} \in \mathbb{R}^q$, where $\pi(\boldsymbol{M}) = \max\{p, q\} \times \max_{ij}\{|m_{ij}|\}/2$.

From Barbalat's Lemma, Li gives the following result in Ref. [20].

Lemma 2 If $g(t) : \mathbb{R} \to \mathbb{R}^+$ is a uniformly continuous function for $t \geq 0$ and if the integral

$$\int_0^t g(s)\,\mathrm{d}s \tag{5}$$

is bounded in $[0, +\infty)$, then $\lim_{t\to+\infty} g(t) = 0$.

3. Feedback control design and global stability of SLS

In order to eliminate suppressive items in the following error system (10) and accelerate the realization of SLS, we design the following preliminary functions and control. First, for $i = 1,2,\ldots,n-1$, some preliminary functions are defined as

$$\begin{aligned} v_i(t) &= c\left(\sum_{k=1}^{n-1} w_{ik}(t) + a_{(i+1)1}x_1(t-\tau) - a_{in}x_n(t-\tau'-\tau)\right) \\ &\quad - d(x_i(t-\tau') - x_{i+1}(t)), \end{aligned}$$

where $d > 0$ denotes control strength, for $k \neq i$,

$$w_{ik}(t) = \begin{cases} 0, & \text{if } a_{ik} = a_{(i+1)(k+1)}, \\ -a_{ik}x_k(t-\tau'-\tau), & \text{if } a_{ik} > a_{(i+1)(k+1)}, \\ a_{(i+1)(k+1)}x_{k+1}(t-\tau), & \text{if } a_{ik} < a_{(i+1)(k+1)}, \end{cases} \tag{6}$$

and

$$w_{ii}(t) = (a_{(i+1)(i+1)} - a_{ii})x_{i+1}(t-\tau). \tag{7}$$

Then a linear feedback control can be designed as

$$\begin{aligned} u_1(t) &= 0, \\ u_i(t) &= -v_{i-1}(t) - v_{i-2}(t-\tau') - \cdots - v_1(t-(i-2)\tau'), \\ &\quad i = 2,3,\ldots,n. \end{aligned} \tag{8}$$

We can easily have

$$u_i(t-\tau') - u_{i+1}(t) = v_i(t), \quad i = 1,2,\ldots,n-1. \tag{9}$$

According to Definition 1, the SLS errors of dynamical network (1) are defined as $\boldsymbol{e}_i(t) = \boldsymbol{x}_i(t-\tau') - \boldsymbol{x}_{i+1}(t)$ for $i = 1,2,\ldots,n-1$. Then the error system can be described as

$$\begin{aligned} \dot{\boldsymbol{e}}_i(t) &= \dot{\boldsymbol{x}}_i(t-\tau') - \dot{\boldsymbol{x}}_{i+1}(t) \\ &= f(\boldsymbol{x}_i(t-\tau')) + c\sum_{j=1}^{n} a_{ij}\boldsymbol{x}_j(t-\tau'-\tau) + u_i(t-\tau') \end{aligned}$$

$$
\begin{aligned}
&- f(\boldsymbol{x}_{i+1}(t)) - c\sum_{j=1}^{n} a_{(i+1)j}\boldsymbol{x}_j(t-\tau) - u_{i+1}(t)\\
&= f(\boldsymbol{x}_i(t-\tau')) - f(\boldsymbol{x}_{i+1}(t)) + c\sum_{j=1}^{n}[a_{ij}\boldsymbol{x}_j(t-\tau'-\tau)\\
&\quad - a_{(i+1)j}\boldsymbol{x}_j(t-\tau)] + v_i(t)\\
&= f(\boldsymbol{x}_i(t-\tau')) - f(\boldsymbol{x}_{i+1}(t))\\
&\quad + c\sum_{j=1}^{n-1} b_{ij}\boldsymbol{e}_j(t-\tau) - d\boldsymbol{e}_i(t), \qquad (10)
\end{aligned}
$$

where according to control (8) for $j \neq i$,

$$
b_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} = a_{(i+1)(j+1)},\\ 0, & \text{if } a_{ij} \neq a_{(i+1)(j+1)}, \end{cases} \qquad (11)
$$

and

$$
b_{ii} = a_{ii}. \qquad (12)
$$

Obviously, B is symmetric. Let $\boldsymbol{I}_n$ and $\boldsymbol{O}_n$ represent the identity matrix and the zero matrix in $\mathbb{R}^{n\times n}$, respectively.

Theorem 1 Let $\boldsymbol{\Delta} = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_m\}$ be diagonal matrix and $\boldsymbol{P} = \mathrm{diag}\{p_1, p_2, \ldots, p_m\}$ be positive-definite diagonal matrix, such that $f(x) \in \mathrm{QUAD}(\boldsymbol{\Delta}, \boldsymbol{P})$. If there exists $d > 0$ such that

$$
\begin{aligned}
&(2\pi(N) - \omega)\boldsymbol{I}_{(n-1)m} + \boldsymbol{I}_{n-1} \otimes \boldsymbol{P}\boldsymbol{\Delta} - d\boldsymbol{I}_{n-1} \otimes \boldsymbol{P}\\
&< \boldsymbol{O}_{(n-1)m}, \qquad (13)
\end{aligned}
$$

then the SLS of dynamical network (1) under control (8) can be achieved for any initial conditions, where

$$
\begin{aligned}
&N = c\boldsymbol{B} \otimes \boldsymbol{P} = (n_{ij})_{(n-1)m\times(n-1)m},\\
&\pi(N) = \frac{1}{2}(n-1)m \times \max_{ij}\{|n_{ij}|\}.
\end{aligned}
$$

Proof Define the following Lyapunov functional candidate:

$$
V(t) = \frac{1}{2}\sum_{i=1}^{n-1} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_i(t) + \int_{t-\tau}^{t}\sum_{i=1}^{n-1} \boldsymbol{e}_i^{\mathrm{T}}(u)\boldsymbol{Q}\boldsymbol{e}_i(u)\,\mathrm{d}u, \qquad (14)
$$

where $\boldsymbol{Q} = \pi(N)\boldsymbol{I}_m$.

The derivative of $V(t)$ along the trajectories of error system (10) can be obtained as follows:

$$
\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \sum_{i=1}^{n-1} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\Big(f(\boldsymbol{x}_i(t-\tau')) - f(\boldsymbol{x}_{i+1}(t))\\
&\quad + c\sum_{j=1}^{n-1}[a_{ij}\boldsymbol{x}_j(t-\tau'-\tau) - a_{(i+1)j}\boldsymbol{x}_j(t-\tau)]\\
&\quad + u_i(t-\tau') - u_{i+1}(t)\Big) + \sum_{i=1}^{n-1}\Big(\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}_i(t)\\
&\quad - \boldsymbol{e}_i^{\mathrm{T}}(t-\tau)\boldsymbol{Q}\boldsymbol{e}_i(t-\tau)\Big)\\
&= \sum_{i=1}^{n-1} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\Big(f(x_i(t-\tau')) - f(\boldsymbol{x}_{i+1}(t))\\
&\quad + c\sum_{j=1}^{n-1} b_{ij}\boldsymbol{e}_j(t-\tau) - d\boldsymbol{e}_i(t)\Big)\\
&\quad + \sum_{i=1}^{n-1}\Big(\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}_i(t) - \boldsymbol{e}_i^{\mathrm{T}}(t-\tau)\boldsymbol{Q}\boldsymbol{e}_i(t-\tau)\Big)\\
&\le \sum_{i=1}^{n-1}\Big(-\omega\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) + \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{\Delta}\boldsymbol{e}_i(t)\\
&\quad + c\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\sum_{j=1}^{n-1} b_{ij}\boldsymbol{e}_j(t-\tau)\\
&\quad - d\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_i(t) + \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}_i(t)\\
&\quad - \boldsymbol{e}_i^{\mathrm{T}}(t-\tau)\boldsymbol{Q}\boldsymbol{e}_i(t-\tau)\Big). \qquad (15)
\end{aligned}
$$

Letting $\boldsymbol{e}(t) = (\boldsymbol{e}_1^{\mathrm{T}}(t), \boldsymbol{e}_2^{\mathrm{T}}(t), \ldots, \boldsymbol{e}_{n-1}^{\mathrm{T}}(t))^{\mathrm{T}}$, from Eq. (15), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\le -\omega\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{I}_{(n-1)m}\boldsymbol{e}(t) + \boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_{n-1} \otimes \boldsymbol{P}\boldsymbol{\Delta})\boldsymbol{e}(t)\\
&\quad + c\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{B} \otimes \boldsymbol{P})\boldsymbol{e}(t-\tau) - d\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_{n-1} \otimes \boldsymbol{P})\boldsymbol{e}(t)\\
&\quad + \boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_{n-1} \otimes \boldsymbol{Q})\boldsymbol{e}(t)\\
&\quad - \boldsymbol{e}^{\mathrm{T}}(t-\tau)(\boldsymbol{I}_{n-1} \otimes \boldsymbol{Q})\boldsymbol{e}(t-\tau)\\
&= \boldsymbol{e}^{\mathrm{T}}(t)(-\omega\boldsymbol{I}_{(n-1)m} + I_{n-1} \otimes \boldsymbol{P}\boldsymbol{\Delta} - d\boldsymbol{I}_{n-1} \otimes \boldsymbol{P}\\
&\quad + \boldsymbol{I}_{n-1} \otimes \boldsymbol{Q})\boldsymbol{e}(t) - \boldsymbol{e}^{\mathrm{T}}(t-\tau)(\boldsymbol{I}_{n-1} \otimes \boldsymbol{Q})\boldsymbol{e}(t-\tau)\\
&\quad + \boldsymbol{e}^{\mathrm{T}}(t)c\boldsymbol{B} \otimes \boldsymbol{P}\boldsymbol{e}(t-\tau). \qquad (16)
\end{aligned}
$$

From condition (13), there exists a constant $\varepsilon > 0$ such that $(2\pi(N) - \omega)\boldsymbol{I}_{(n-1)m} + \boldsymbol{I}_{n-1} \otimes \boldsymbol{P}\boldsymbol{\Delta} - d\boldsymbol{I}_{n-1} \otimes \boldsymbol{P} \le -\varepsilon\boldsymbol{I}_{(n-1)m}$. From Lemma 1, letting $N = c\boldsymbol{B} \otimes \boldsymbol{P}$, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\le \boldsymbol{e}^{\mathrm{T}}(t)(-\omega\boldsymbol{I}_{(n-1)m} + \boldsymbol{I}_{n-1} \otimes \boldsymbol{P}\boldsymbol{\Delta} - d\boldsymbol{I}_{n-1} \otimes \boldsymbol{P}\\
&\quad + \boldsymbol{I}_{n-1} \otimes \boldsymbol{Q})\boldsymbol{e}(t) - \boldsymbol{e}^{\mathrm{T}}(t-\tau)(\boldsymbol{I}_{n-1} \otimes \boldsymbol{Q})\boldsymbol{e}(t-\tau)\\
&\quad + \boldsymbol{e}^{\mathrm{T}}(t-\tau)\pi(N)\boldsymbol{I}_{(n-1)m}\boldsymbol{e}(t-\tau)\\
&\quad + \boldsymbol{e}^{\mathrm{T}}(t)\pi(N)\boldsymbol{I}_{(n-1)m}\boldsymbol{e}(t)\\
&= \boldsymbol{e}^{\mathrm{T}}(t)\Big(-\omega\boldsymbol{I}_{(n-1)m} + \boldsymbol{I}_{n-1} \otimes \boldsymbol{P}\boldsymbol{\Delta} - d\boldsymbol{I}_{n-1} \otimes \boldsymbol{P}\\
&\quad + 2\pi(N)\boldsymbol{I}_{(n-1)m}\Big)\boldsymbol{e}(t)\\
&\le -\varepsilon\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)\\
&= -\varepsilon\sum_{i=1}^{n-1} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t). \qquad (17)
\end{aligned}
$$

Integrating the above equation from 0 to t yields

$$
\begin{aligned}
\varepsilon\int_0^t\sum_{i=1}^{n-1} \boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{e}_i(s)\,\mathrm{d}s &\le V(0) - V(t)\\
&\le V(0). \qquad (18)
\end{aligned}
$$

Thus, the integral $\int_0^t\sum_{i=1}^{n-1} \boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{e}_i(s)\,\mathrm{d}s$ is bounded. From Lemma 2, we obtain $\lim_{t\to+\infty}\|\boldsymbol{e}_i(t)\| = 0$ for every $i \in \{1, 2, \ldots, n-1\}$, which means that $\lim_{t\to+\infty}\|\boldsymbol{x}_i(t-\tau') - \boldsymbol{x}_{i+1}(t)\| = 0$. This completes the proof.

In practice, the control strength is not allowed to be arbitrarily large, so we adopt an adaptive control method[22] to reduce the control strength.

For $i = 1,2,\ldots,n-1$, some preliminary functions are similarly defined as

$$v_i(t) = c\left(\sum_{k=1}^{n-1} w_{ik}(t) + a_{(i+1)1}x_1(t-\tau) - a_{in}x_n(t-\tau'-\tau)\right) - d_i(t)(x_i(t-\tau') - x_{i+1}(t)),$$

where $d_i(t) \geq 0$ denotes the time-varying control strength, the definition of $w_{ik}(t)$ is the same as Eqs. (6) and (7).

Then an adaptive control is designed as

$$\begin{aligned} u_1(t) &= 0, \\ u_i(t) &= -v_{i-1}(t) - v_{i-2}(t-\tau') - \cdots - v_1(t-(i-2)\tau'), \\ \dot{d}_j(t) &= \boldsymbol{e}_j^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_j(t),\ i = 2,\ldots,n,\ j = 1,\ldots,n-1. \end{aligned} \tag{19}$$

Suppose that $\boldsymbol{\Delta} = \mathrm{diag}\{\delta_1,\delta_2,\ldots,\delta_m\}$ is a diagonal matrix and $\boldsymbol{P} = \mathrm{diag}\{p_1,p_2,\ldots,p_m\}$ is a positive-definite diagonal matrix. Then we have the following result under adaptive control (19).

Theorem 2 If $f(\boldsymbol{x}) \in \mathrm{QUAD}(\boldsymbol{\Delta},\boldsymbol{P})$, then the SLS of dynamical network (1) can be achieved for any initial conditions under adaptive control (19).

Proof Define the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2}\sum_{i=1}^{n-1}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_i(t) + \int_{t-\tau}^{t}\sum_{i=1}^{n-1}\boldsymbol{e}_i^{\mathrm{T}}(u)\boldsymbol{Q}\boldsymbol{e}_i(u)\,\mathrm{d}u + \sum_{i=1}^{n-1}\alpha(d_i(t)-d_i^*)^2,$$

where positive constants α and d_i^* will be determined later.

The derivative of $V(t)$ along trajectories of error system (10) is calculated as follows:

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \sum_{i=1}^{n-1}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\Big(f(\boldsymbol{x}_i(t-\tau')) - f(\boldsymbol{x}_{i+1}(t)) \\ &\quad + c\sum_{j=1}^{n-1}(a_{ij}x_j(t-\tau'-\tau) - a_{(i+1)j}\boldsymbol{x}_j(t-\tau)) \\ &\quad + u_i(t-\tau') - u_{i+1}(t)\Big) + \sum_{i=1}^{n-1}\Big(\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}_i(t) \\ &\quad - \boldsymbol{e}_i^{\mathrm{T}}(t-\tau)\boldsymbol{Q}\boldsymbol{e}_i(t-\tau)\Big) \\ &\quad + 2\alpha\sum_{i=1}^{n-1}(d_i(t)-d_i^*)\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_i(t) \\ &= \sum_{i=1}^{n-1}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\Big(f(\boldsymbol{x}_i(t-\tau')) - f(\boldsymbol{x}_{i+1}(t)) \\ &\quad + c\sum_{j=1}^{n-1}b_{ij}\boldsymbol{e}_j(t-\tau) - d_i(t)\boldsymbol{e}_i(t)\Big) \\ &\quad + \sum_{i=1}^{n-1}\Big(\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}_i(t) - \boldsymbol{e}_i^{\mathrm{T}}(t-\tau)\boldsymbol{Q}\boldsymbol{e}_i(t-\tau)\Big) \\ &\quad + 2\alpha\sum_{i=1}^{n-1}(d_i(t)-d_i^*)\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_i(t). \end{aligned} \tag{20}$$

Letting $\boldsymbol{D}(t) = \mathrm{diag}\{d_1(t),d_2(t),\ldots,d_{n-1}(t)\}$ and $\boldsymbol{D}^* = \mathrm{diag}\{d_1^*,d_2^*,\ldots,d_{n-1}^*\}$, from Eq. (20), we have

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leq \boldsymbol{e}^{\mathrm{T}}(t)\Big(-\omega\boldsymbol{I}_{(n-1)m} + \boldsymbol{I}_{n-1}\otimes\boldsymbol{P}\boldsymbol{\Delta} \\ &\quad + 2\pi(N)\boldsymbol{I}_{(n-1)m} - \boldsymbol{D}(t)\otimes\boldsymbol{P}\Big)\boldsymbol{e}(t) \\ &\quad + 2\alpha\boldsymbol{e}^{\mathrm{T}}(t)\Big(\boldsymbol{D}(t)\otimes\boldsymbol{P} - \boldsymbol{D}^*\otimes\boldsymbol{P}\Big)\boldsymbol{e}(t) \\ &= \boldsymbol{e}^{\mathrm{T}}(t)\Big(-\omega\boldsymbol{I}_{(n-1)m} + \boldsymbol{I}_{n-1}\otimes\boldsymbol{P}\boldsymbol{\Delta} \\ &\quad + 2\pi(N)\boldsymbol{I}_{(n-1)m} - 2\alpha\boldsymbol{D}^*\otimes\boldsymbol{P}\Big)\boldsymbol{e}(t) \\ &\quad - (1-2\alpha)\boldsymbol{e}^{\mathrm{T}}(t)\Big(\boldsymbol{D}(t)\otimes\boldsymbol{P}\Big)\boldsymbol{e}(t). \end{aligned} \tag{21}$$

We can choose appropriate α such that $1-2\alpha > 0$ and select appropriate $\boldsymbol{D}^*$ such that

$$\begin{aligned} &(2\pi(N)-\omega)\boldsymbol{I}_{(n-1)m} + \boldsymbol{I}_{n-1}\otimes\boldsymbol{P}\boldsymbol{\Delta} - 2\alpha\boldsymbol{D}^*\otimes\boldsymbol{P} \\ &\leq -\tilde{\varepsilon}\boldsymbol{I}_{(n-1)m}, \end{aligned} \tag{22}$$

for a positive constant $\tilde{\varepsilon}$. Combining Eqs. (21) and (22), we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -\tilde{\varepsilon}\sum_{i=1}^{n-1}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t). \tag{23}$$

Thus, the integral $\int_0^t\sum_{i=1}^{n-1}\boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{e}_i(s)\,\mathrm{d}s$ is bounded. From Lemma 2, we obtain $\lim_{t\to+\infty}\|\boldsymbol{e}_i(t)\| = 0$ for every $i \in \{1,2,\ldots,n-1\}$, which means that $\lim_{t\to+\infty}\|\boldsymbol{x}_i(t-\tau') - \boldsymbol{x}_{i+1}(t)\| = 0$. This theorem is proven.

4. Numerical simulations

In this section, some numerical examples are given to verify the theorems in Section 3. The size of dynamical network (1) is set as $n = 4$ and its topological structure is shown in Fig. 2 with two cases. The local dynamics of all nodes in this network is characterized by a 3D neural network,[9] which is described by

$$\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = -\boldsymbol{D}\boldsymbol{y} + \boldsymbol{T}\boldsymbol{g}(\boldsymbol{y}), \tag{24}$$

with $\boldsymbol{y} = (y_1,y_2,y_3)^{\mathrm{T}} \in \mathbb{R}^3$,

$$\boldsymbol{T} = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{pmatrix}, \tag{25}$$

$\boldsymbol{D} = \boldsymbol{I}_3$, and $\boldsymbol{g}(\boldsymbol{y}) = (\boldsymbol{g}(y_1), \boldsymbol{g}(y_2), \boldsymbol{g}(y_3))^{\mathrm{T}}$ where $\boldsymbol{g}(u) = (|\boldsymbol{u}+1| - |\boldsymbol{u}-1|)/2$. For this system, we have $m = 3$ and choose $\boldsymbol{P}_1 = \boldsymbol{I}_3$, $\boldsymbol{\Delta}_1 = 10\boldsymbol{I}_3$, and $\omega = 0.6218$ to achieve inequality (3).

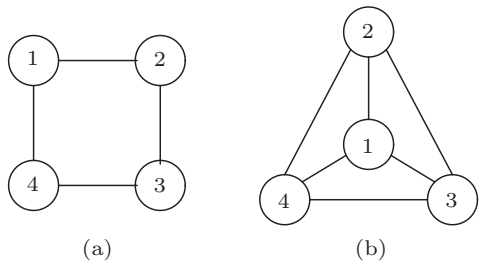


Fig. 2. Two kinds of topological structure for dynamical network (1) with size $n = 4$. (a) Loop-shaped network; (b) fully connected network.

4.1. Linear feedback control

First, we use linear feedback control (8) to realize SLS on a dynamical network, whose topological structure is shown in Fig. 2(a). The coupling matrix of this network structure is

$$\boldsymbol{A} = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}. \tag{26}$$

From Eqs. (11) and (12), we obtain

$$\boldsymbol{B} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}. \tag{27}$$

According to control (8), we have

$$\begin{aligned}
\boldsymbol{u}_1(t) &= 0, \\
\boldsymbol{u}_2(t) &= -c(\boldsymbol{x}_1(t-\tau) - \boldsymbol{x}_4(t-\tau'-\tau)) \\
&\quad + d(\boldsymbol{x}_1(t-\tau') - \boldsymbol{x}_2(t)), \\
\boldsymbol{u}_3(t) &= -c(\boldsymbol{x}_1(t-\tau'-\tau) - \boldsymbol{x}_4(t-\tau'-\tau)) \\
&\quad + d(\boldsymbol{x}_1(t-2\tau') - \boldsymbol{x}_3(t)), \\
\boldsymbol{u}_4(t) &= -c(x_1(t-\tau) + \boldsymbol{x}_1(t-2\tau'-\tau) \\
&\quad - \boldsymbol{x}_4(t-\tau'-\tau) - \boldsymbol{x}_4(t-3\tau'-\tau)) \\
&\quad + d(\boldsymbol{x}_1(t-3\tau') - \boldsymbol{x}_4(t)).
\end{aligned}$$

We can obtain $\pi(N) = (n-1)m \times \max_{ij}\{|n_{ij}|\}/2 = 9c$ from the matrix $\boldsymbol{B}$. In order to achieve (13), we choose $c = 0.1$ and search the lower bound of the control strength d by solving the following inequality:

$$(1.8 - \omega + 10 - d)\boldsymbol{I}_9 \leq 0. \tag{28}$$

We can easily get the lower bound of the control strength $d_l = 11.18$. So, if $d > d_l$, the conditions of Theorem 1 are satisfied, which means that the SLS of dynamical network (1) under control (8) can be achieved for any initial conditions. Figure 3 gives a realization under control strength $d = 12$ and time delay $\tau = 0.5$, and figure 5 shows its SLS errors on the same condition. From these simulations, we can see that the SLS is achieved under linear feedback control (8). In order to explore the influence of communication delay on the SLS, we change τ to 1 and obtain Fig. 4. This realization is under control strength $d = 12$, which shows that SLS is still achieved. Hence, the stability condition is independent on communication delay.

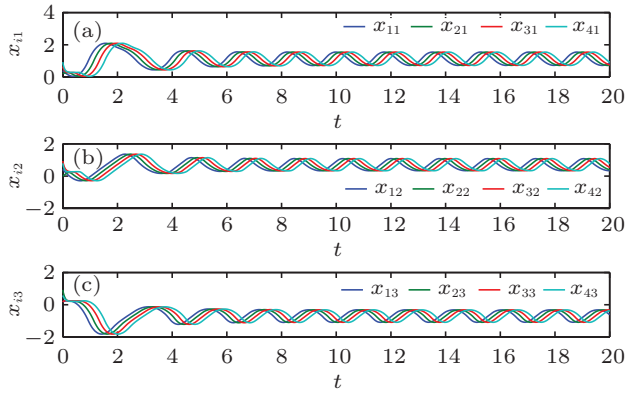


Fig. 3. (color online) Trajectories of (a) x_{i1}, (b) x_{i2}, and (c) x_{i3} of dynamical network (1) under control (8), where $i = 1,2,3,4$, $c = 0.1$, $d = 12$, $\tau' = 0.2$, and $\tau = 0.5$.

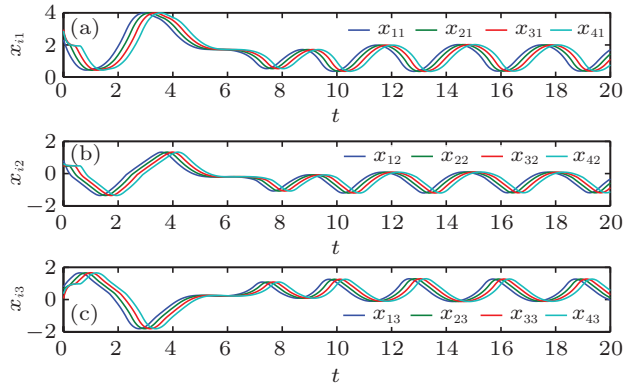


Fig. 4. (color online) Trajectories of (a) x_{i1}, (b) x_{i2}, and (c) x_{i3} of dynamical network (1) under control (8), where $i = 1,2,3,4$, $c = 0.1$, $d = 12$, $\tau' = 0.2$, and $\tau = 1$.

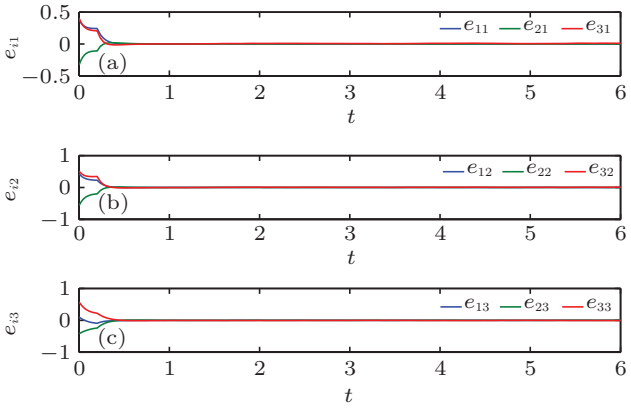


Fig. 5. (color online) Trajectories of SLS errors (a) e_{i1}, (b) e_{i2}, and (c) e_{i3} of dynamical network (1) under control (8), where $i = 1,2,3$, $c = 0.1$, $d = 12$, $\tau' = 0.2$, and $\tau = 0.5$.

4.2. Adaptive linear feedback control

Now, we use adaptive control (19) to realize SLS on another dynamical network, whose topological structure is shown in Fig. 2(b). The coupling matrix of this network is

$$\boldsymbol{A} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}. \tag{29}$$

According to adaptive linear feedback control (19), we have

$$\begin{aligned}
\boldsymbol{u}_1(t) &= 0, \\
\boldsymbol{u}_2(t) &= -c(\boldsymbol{x}_1(t-\tau)-\boldsymbol{x}_4(t-\tau'-\tau)+d_1(t)\boldsymbol{e}_1(t), \\
\boldsymbol{u}_3(t) &= -c(\boldsymbol{x}_1(t-\tau)+\boldsymbol{x}_1(t-\tau'-\tau) \\
&\quad -\boldsymbol{x}_4(t-\tau'-\tau)-\boldsymbol{x}_4(t-2\tau'-\tau) \\
&\quad d_1(t-\tau')\boldsymbol{e}_1(t-\tau')+d_2(t)\boldsymbol{e}_2(t), \\
\boldsymbol{u}_4(t) &= -c(\boldsymbol{x}_1(t-\tau)+\boldsymbol{x}_1(t-\tau'-\tau)+\boldsymbol{x}_1(t-2\tau'-\tau) \\
&\quad -\boldsymbol{x}_4(t-\tau'-\tau) \\
&\quad -\boldsymbol{x}_4(t-2\tau'-\tau)-\boldsymbol{x}_4(t-3\tau'-\tau)) \\
&\quad +d_1(t-2\tau')\boldsymbol{e}_1(t-2\tau') \\
&\quad +d_2(t-\tau')\boldsymbol{e}_2(t-\tau')+d_3(t)\boldsymbol{e}_3(t).
\end{aligned}$$

Figure 6 presents a realization under adaptive linear feedback control (19) with coupling strength $c = 0.1$ and $\tau' = 0.2$ and $\tau = 0.5$. Figure 7 gives the errors. As shown in Figs. 6 and 7, we can see that the SLS of dynamical network (1) is achieved under adaptive linear feedback control (19).

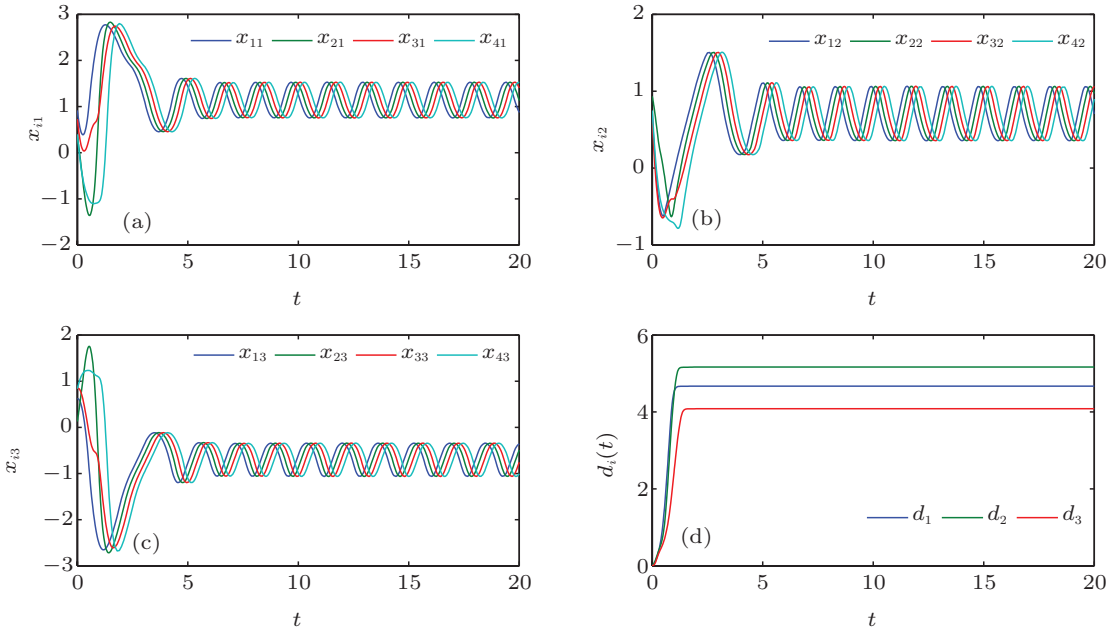


Fig. 6. (color online) Trajectories of all state variables (a) x_{i1}, (b) x_{i2}, (c) x_{i3}, and (d) $d_i(t)$ of dynamical network (1) under control (19), where $i = 1,2,3,4$, $c = 0.1$, $\tau' = 0.2$, and $\tau = 0.5$.

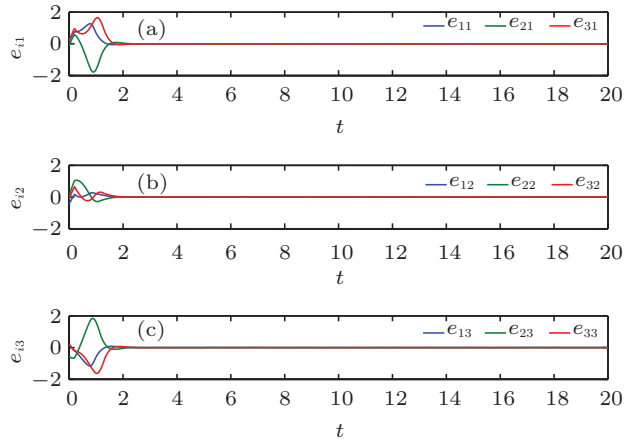


Fig. 7. (color online) Trajectories of all SLS errors (a) e_{i1}, (b) e_{i2}, and (c) e_{i3} of dynamical network (1) under control (19), where $i = 1,2,3$, $c = 0.1$, $\tau' = 0.2$, and $\tau = 0.5$.

Similar to that in subsection 4.1, by solving the following inequality:

$$(2.7-\omega+10-d)\boldsymbol{I}_9 \leq 0, \tag{30}$$

we gain the lower bound of the control strength $d_l = 12.08$. As shown in Fig. 6, we can observe that all control strengths are obviously smaller than d_l, which confirms that the adaptive linear feedback control can decrease the control strength required for the realization of SLS.

5. Conclusions

In this paper, we address the successive lag synchronization (SLS) on dynamical networks with communication delay via linear feedback control and adaptive linear feedback con-

trol. We construct a general dynamical network with communication delay to make the SLS more realistic and significant for real multi-coupled systems. By using the linear feedback control, the SLS of the network has been realized. In order to choose the feasible control gain adaptively, we design an adaptive linear feedback control. By theoretical analysis, we obtained some sufficient conditions for the global stability of SLS under the controls.

In many real dynamical networks, their topological structures may be generally directed. Therefore, it is worth to consider SLS of dynamical networks with directed topological structures in the future.

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