Stochastic response of van der Pol oscillator with two kinds of fractional derivatives under Gaussian white noise excitation*

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This paper aims to investigate the stochastic response of the van der Pol (VDP) oscillator with two kinds of fractional derivatives under Gaussian white noise excitation. First, the fractional VDP oscillator is replaced by an equivalent VDP oscillator without fractional derivative terms by using the generalized harmonic balance technique. Then, the stochastic averaging method is applied to the equivalent VDP oscillator to obtain the analytical solution. Finally, the analytical solutions are validated by numerical results from the Monte Carlo simulation of the original fractional VDP oscillator. The numerical results not only demonstrate the accuracy of the proposed approach but also show that the fractional order, the fractional coefficient and the intensity of Gaussian white noise play important roles in the responses of the fractional VDP oscillator. An interesting phenomenon we found is that the effects of the fractional order of two kinds of fractional derivative items on the fractional stochastic systems are totally contrary.

Keywords: stochastic averaging method, fractional derivative, van der Pol, equivalent stochastic system

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1. Introduction

Fractional calculus[1–5] can be traced back to the Leibniz's letter to L'Hospital in 1695. Unfortunately, for quite a long time, fractional calculus remained in an abstract mathematical field and developed very slowly. In the past few decades, fractional calculus has attracted considerable attention, partly due to the fact that the fractional-order mathematical models can be used to make a more accurate description than the integer-order models. Thus, fractional calculus can be considered as an old and yet novel branch of mathematics.[6,7]

Because of the extensive existence of the random factor,[8,9] it is of practical significance to study the stochastic fractional systems. In order to obtain the approximate analytical solution of fractional systems, a lot of approaches have been used by many authors. The stochastic averaging method has been used by many researchers to obtain the analytical solutions. Specifically, Huang and Jin[10] investigated the stationary response and stability of a single-degree-of-freedom (SDOF) strongly nonlinear stochastic system with light damping modeled by a fractional derivative. Chen *et al.*[11] and Chen and Zhu[12] studied the stationary responses. Chen and Zhu[13] investigated the stochastic jump and bifurcation. Chen *et al.*[14] studied first passage failure; Chen *et al.*[15,16] fractional control of stochastic fractional systems. Yang *et al.*[17,18] studied the stationary response of a nonlinear system with Caputo-type fractional derivative under Gaussian white noise excitation by the stochastic averaging method. Zhang *et al.*[19] discussed the response of a Duffing–Rayleigh system with fractional derivative under Gaussian white noise excitation. There are several other methods to deal with stochastic systems besides the stochastic averaging method. Spanos and Zeldin[20] put forward a frequency-domain method for systems with fractional derivatives. Agrawal[21] proposed an analytical approach for stochastic dynamic systems endowed with a fractional derivative by the eigenvector expansion method and Laplace transforms. Xu *et al.*[22,23] studied the response of the stochastic oscillator with fractional derivative by using a new technique based on the Lindstedt–Poincaré (LP) method and multiple-scale method. Liu *et al.*[24] investigated the principal resonance responses of SDOF systems with small fractional derivative damping subjected to narrow-band random parametric excitation by the multiple scale method.

However, the systems mentioned in the above literature only contain one kind of fractional derivative, little work is related to the study of deterministic or stochastic systems containing two kinds of fractional derivative to the best of the authors' knowledge. Huang *et al.*[25] discussed the statistical behaviors of the response of stochastic systems endowed with two kinds of fractional derivatives by using the Laplace transform in conjunction with the weighted generalized Mittag–Leffler function. Shen *et al.*[26] investigated the primary resonance of the Duffing oscillator with two kinds of fractional-order derivatives. Liao[27] studied the Duffing oscillator with two kinds of fractional-order derivative by us-

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ing the improved version of the constrained optimization harmonic balance method.

Inspired by the above discussion, in this paper we are to investigate the stationary response of the VDP oscillator with two kinds of fractional derivatives excited by Gaussian white noise.

The rest of this paper is organized as follows. The fractional VDP oscillator is reduced to the equivalent stochastic VDP oscillator without fractional derivative terms by using the generalized harmonic function technique in Section 2. The approximate analytical solutions of the equivalent VDP oscillator are obtained by the stochastic averaging method in Section 3. The predictor–corrector algorithm for the solution of the initial value problem with fractional derivative is briefly introduced in Section 4. Numerical simulations are performed to verify the theoretical results in Section 5. Finally, the conclusions are given in Section 6.

2. Equivalent VDP oscillator

Consider a fractional VDP oscillator[28–30] with two kinds of fractional derivatives subjected to Gaussian white noise. The motion of the system is governed by the following equation:

$$\ddot{x}+\beta_1 D^{\alpha_1}x(t)+\beta_2 D^{\alpha_2}x(t)+(-b_1+b_2x^2)\dot{x}+\omega^2 x=W(t),$$
$$0<\alpha_1<1, 1<\alpha_2<2, \tag{1}$$

where a dot over a variable denotes differentiation with respect to time t; β_1, β_2, and ω are constant coefficients; $W(t)$ is a stationary Gaussian white noise with correlation function $E[W(t)W(t+\tau)]=2D\delta(\tau)$. There are many definitions of the fractional derivative. In the present paper, the Caputo-type fractional derivative is adopted as follows:

$$D^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x^{(n)}(u)}{(t-u)^{\alpha-n+1}}\,\mathrm{d}u, \tag{2}$$

in which $n-1<\alpha<n$, $n\in N$, and $\Gamma(\cdot)$ is a Gamma function.

According to Refs. [11], [14], and [26], the term associated with the fractional derivative not only serves as the classical damping force, but also contributes to the restoring force. Based on the generalized harmonic function technique,[11,14] this term can be replaced by a linear restoring force and a linear damping force:

$$\beta_1 D^{\alpha_1}x(t)=C_1(\alpha_1)\dot{x}(t)+K_1(\alpha_1)x(t), \tag{3}$$
$$\beta_2 D^{\alpha_2}x(t)=C_2(\alpha_2)\dot{x}(t)+K_2(\alpha_2)x(t). \tag{4}$$

In order to obtain the values of $C_1(\alpha_1)$, $C_2(\alpha_2)$, $K_1(\alpha_1)$, and $K_2(\alpha_2)$, we first introduce two important formulae:

$$\lim_{T\to\infty}\int_0^T \frac{\sin(\omega t)}{t^{\alpha}}\,\mathrm{d}t=\omega^{\alpha-1}\Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}, \tag{5}$$
$$\lim_{T\to\infty}\int_0^T \frac{\cos(\omega t)}{t^{\alpha}}\,\mathrm{d}t=\omega^{\alpha-1}\Gamma(1-\alpha)\sin\frac{\alpha\pi}{2}. \tag{6}$$

We can assume that the solution of the fractional VDP oscillator (1) has the following form:

$$x(t)=A(t)\cos\Phi'(t),$$
$$\dot{x}(t)=-A(t)\omega\sin\Phi'(t),$$
$$\Phi'(t)=\omega t+\Theta'(t). \tag{7}$$

First, we present the detailed derivation procedure for $C_1(\alpha_1)$.

$$\begin{aligned}C_1(\alpha_1) &= -\frac{\beta_1}{\pi A\omega}\int_0^{2\pi} D^{\alpha_1}(A\cos\Phi')\sin\Phi'\,\mathrm{d}\Phi'\\ &= \frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\sin(\omega u+\Phi')}{(t-u)^{\alpha_1}}\,\mathrm{d}u\right]\sin(\omega t+\Phi')\right\}\mathrm{d}t.\end{aligned} \tag{8}$$

Using the transformations of coordinates $s=t-u$ and $\mathrm{d}s=-\mathrm{d}u$, one can obtain

$$\begin{aligned}C_1(\alpha_1) &= \frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\sin(\omega t+\Phi'-\omega s)}{s^{\alpha_1}}\,\mathrm{d}s\right]\sin(\omega t+\Phi')\right\}\mathrm{d}t\\ &= \frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\cos(\omega s)}{s^{\alpha_1}}\,\mathrm{d}s\right]\sin^2(\omega t+\Phi')\right\}\mathrm{d}t\\ &\quad -\frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\sin(\omega s)}{s^{\alpha_1}}\,\mathrm{d}s\right]\sin(\omega t+\Phi')\cos(\omega t+\Phi')\right\}\mathrm{d}t=C_{11}+C_{12}.\end{aligned} \tag{9}$$

Define the first part of Eq. (9) as C_{11} and the second part as C_{12}, then

$$\begin{aligned}C_{11} &= \frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\cos(\omega s)}{s^{\alpha_1}}\,\mathrm{d}s\right]\sin^2(\omega t+\Phi')\right\}\mathrm{d}t\\ &= \frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\int_0^t\frac{\cos(\omega s)}{s^{\alpha_1}}\,\mathrm{d}s\right)\mathrm{d}\left(\frac{2\omega t-\sin(2\omega t+2\Phi')}{4\omega}\right)\\ &= \frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\left(\frac{2\omega t-\sin(2\omega t+2\Phi')}{4\omega T}\right)\left(\int_0^t\frac{\cos(\omega s)}{s^{\alpha_1}}\,\mathrm{d}s\right)\Bigg|_0^T\end{aligned}$$

$$-\frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\frac{2\omega t-\sin(2\omega t+2\Phi')}{4\omega}\right)\left(\frac{\cos(\omega t)}{t^{\alpha_1}}\right)\mathrm{d}t=C_{111}+C_{112}. \tag{10}$$

Define the first part of Eq. (10) as C_{111} and the second part as C_{112}, then

$$\begin{aligned}C_{111}&=\frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\left(\frac{2\omega t-\sin(2\omega t+2\Phi')}{4\omega T}\right)\left(\int_0^t\frac{\cos(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right)\Bigg|_0^T\\&=\frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\left(\frac{2\omega T-\sin(2\omega T+2\Phi')}{4\omega T}\right)\left(\int_0^T\frac{\cos(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right)=\beta_1\omega^{\alpha_1-1}\sin\frac{\alpha_1\pi}{2},\end{aligned} \tag{11}$$

$$C_{112}=0, \tag{12}$$

$$C_{12}=-\frac{2\beta_1}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\sin(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right]\sin(\omega t+\Phi')\cos(\omega t+\Phi')\right\}\mathrm{d}t=0. \tag{13}$$

According to Eqs. (11)–(13), one can obtain

$$C_1(\alpha_1)=C_{11}+C_{12}=(C_{111}+C_{112})+C_{12}=C_{111}=\beta_1\omega^{\alpha_1-1}\sin\frac{\alpha_1\pi}{2}. \tag{14}$$

Second, we present the detailed derivation procedure for $K_1(\alpha_1)$.

$$\begin{aligned}K_1(\alpha_1)&=\frac{\beta_1}{\pi A}\int_0^{2\pi}D^{\alpha_1}(A\cos\Phi')\cos\Phi'\,\mathrm{d}\Phi'\\&=-\frac{2\beta_1\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\sin(\omega u+\Phi')}{(t-u)^{\alpha_1}}\mathrm{d}u\right]\cos(\omega t+\Phi')\right\}\mathrm{d}t.\end{aligned} \tag{15}$$

Using the transformations of coordinates $s=t-u$ and $\mathrm{d}s=-\mathrm{d}u$, one can obtain

$$\begin{aligned}K_1(\alpha_1)&=-\frac{2\beta_1\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\sin(\omega t+\Phi'-\omega s)}{s^{\alpha_1}}\mathrm{d}s\right]\cos(\omega t+\Phi')\right\}\mathrm{d}t\\&=-\frac{2\beta_1\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\cos(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right]\sin(\omega t+\Phi')\cos(\omega t+\Phi')\right\}\mathrm{d}t\\&\quad+\frac{2\beta_1\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\sin(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right]\cos^2(\omega t+\Phi')\right\}\mathrm{d}t=K_{11}+K_{12}.\end{aligned} \tag{16}$$

Define the first part of Eq. (16) as K_{11} and the second part as K_{12}, then

$$\begin{aligned}K_{11}&=-\frac{2\beta_1\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\cos(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right]\right.\\&\quad\left.\times\sin(\omega t+\Phi')\cos(\omega t+\Phi')\right\}\mathrm{d}t=0,\end{aligned} \tag{17}$$

$$\begin{aligned}K_{12}&=\frac{2\beta_1\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left[\int_0^t\frac{\sin(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right]\right.\\&\quad\left.\times\cos^2(\omega t+\Phi')\right\}\mathrm{d}t\\&=\frac{2\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\int_0^t\frac{\sin(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right)\\&\quad\times\mathrm{d}\left(\frac{2\omega t+\sin(2\omega t+2\Phi')}{4\omega}\right)\\&=\frac{2\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\left(\frac{2\omega t+\sin(2\omega t+2\Phi')}{4\omega T}\right)\\&\quad\times\left(\int_0^t\frac{\sin(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right)\Bigg|_0^T\\&\quad-\frac{2\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\frac{2\omega t+\sin(2\omega t+2\Phi')}{4\omega}\right)\\&\quad\times\left(\frac{\sin(\omega t)}{t^{\alpha_1}}\right)\mathrm{d}t=K_{121}+K_{122}.\end{aligned} \tag{18}$$

Define the first part of Eq. (16) as K_{121} and the second part as K_{122}, then

$$\begin{aligned}K_{121}&=\frac{2\beta_1\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\left(\frac{2\omega t+\sin(2\omega t+2\Phi')}{4\omega T}\right)\\&\quad\times\left(\int_0^t\frac{\sin(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right)\Bigg|_0^T\\&=\frac{2\beta_1\omega}{\Gamma(1-\alpha_1)}\lim_{T\to\infty}\left(\frac{2\omega T+\sin(2\omega T+2\Phi')}{4\omega T}\right)\\&\quad\times\left(\int_0^T\frac{\sin(\omega s)}{s^{\alpha_1}}\mathrm{d}s\right)=\beta_1\omega^{\alpha_1}\cos\frac{\alpha_1\pi}{2},\end{aligned} \tag{19}$$

$$K_{122}=0. \tag{20}$$

According to Eqs. (17), (19), and Eq. (20), one can obtain

$$\begin{aligned}K_1(\alpha_1)&=K_{11}+K_{12}=K_{11}+(K_{121}+K_{122})\\&=K_{121}=\beta_1\omega^{\alpha_1}\cos\frac{\alpha_1\pi}{2}.\end{aligned} \tag{21}$$

According to Eqs. (14) and (21), one can obtain

$$\beta_1 D^{\alpha_1}x(t)$$

$$= C_1(\alpha_1)\dot{x}(t) + K_1(\alpha_1)x(t)$$
$$= \beta_1\omega^{\alpha_1-1}\sin\frac{\alpha_1\pi}{2}\dot{x}(t) + \beta_1\omega^{\alpha_1}\cos\frac{\alpha_1\pi}{2}x(t). \quad (22)$$

Similarly, one could also obtain $C_2(\alpha_2)$, $K_2(\alpha_2)$:

$$C_2(\alpha_2) = \beta_2\omega^{\alpha_2-1}\cos\frac{(\alpha_2-1)\pi}{2}, \quad (23)$$
$$K_2(\alpha_2) = \beta_2\omega^{\alpha_2}\sin\frac{(\alpha_2-1)\pi}{2}. \quad (24)$$

Thus

$$\beta_2 D^{\alpha_2}x(t) = \left[\beta_2\omega^{\alpha_2-1}\cos\frac{(\alpha_2-1)\pi}{2}\right]\dot{x}(t) - \left[\beta_2\omega^{\alpha_2}\sin\frac{(\alpha_2-1)\pi}{2}\right]x(t). \quad (25)$$

Therefore, the equivalent VDP oscillator associated with system (1) is given by

$$\ddot{x} + \varepsilon r\dot{x} + \omega_0^2 x = W(t), \quad (26)$$

in which

$$\varepsilon r = \beta_1\omega^{\alpha_1-1}\sin\frac{\alpha_1\pi}{2} + \beta_2\omega^{\alpha_2-1}\cos\frac{(\alpha_2-1)\pi}{2} - b_1 + b_2x^2, \quad (27)$$
$$\omega_0^2 = \beta_1\omega^{\alpha_1}\cos\frac{\alpha_1\pi}{2} - \beta_2\omega^{\alpha_2}\sin\frac{(\alpha_2-1)\pi}{2} + \omega^2. \quad (28)$$

3. Stochastic averaging

In this section, the stochastic averaging method is applied to the VDP oscillator (26).

According to Ref. [20], we can assume that the solution of system (26) has the following form:

$$X = x(t) = A(t)\cos\Phi(t),$$
$$Y = \dot{x}(t) = -A(t)\omega_0\sin\Phi(t),$$
$$\Phi(t) = \omega_0 t + \Theta,$$
$$U(x) = \int_0^x \omega_0^2 u\,\mathrm{d}u = \frac{1}{2}\omega_0^2x^2, \quad (29)$$

where A, Φ, and Θ are random processes. One can obtain the following equations for the amplitude A and the phase angle Θ:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \varepsilon F_1(A,\Theta) + \varepsilon^{1/2}G_1(A,\Theta)W(t),$$
$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \varepsilon F_2(A,\Theta) + \varepsilon^{1/2}G_2(A,\Theta)W(t), \quad (30)$$

where

$$\varepsilon F_1 = -\varepsilon rA\sin^2\Phi = -A\sin^2\Phi\left[\beta_1\omega^{\alpha_1-1}\sin\frac{\alpha_1\pi}{2} + \beta_2\omega^{\alpha_2-1}\times\cos\frac{(\alpha_2-1)\pi}{2} - b_1 + b_2A^2\cos^2\Phi\right],$$
$$\varepsilon F_2 = -\varepsilon r\sin\Phi\cos\Phi = -\sin\Phi\cos\Phi\left[\beta_1\omega^{\alpha_1-1}\sin\frac{\alpha_1\pi}{2} + \beta_2\omega^{\alpha_2-1}\times\cos\frac{(\alpha_2-1)\pi}{2} - b_1 + b_2A^2\cos^2\Phi\right],$$
$$\varepsilon^{1/2}G_1 = -\frac{\sin\Phi}{\omega_0},$$
$$\varepsilon^{1/2}G_2 = -\frac{\cos\Phi}{A\omega_0}. \quad (31)$$

According to the Stratonovich–Khasminskii theorem, $A(t)$ and $\Phi(t)$ converge weakly into a two-dimensional diffusion Markov process as $\varepsilon\to 0$, in a time interval $0\le t\le T$, where $T\sim O(\varepsilon^{-1})$. Equation (30) shows that the averaged Itô equation for $A(t)$ is independent of $\Theta(t)$, the limiting process $A(t)$ is a one-dimensional diffusion process governed by

$$\mathrm{d}A = m(A)\,\mathrm{d}t + \sigma(A)\,\mathrm{d}B(t), \quad (32)$$

where the drift and diffusion coefficients, respectively, are

$$m(A) = \varepsilon\left\langle F_1^{(1)} + F_1^{(2)} + D\frac{\partial G_1}{\partial A}G_1 + D\frac{\partial G_1}{\partial\Theta}G_2\right\rangle_{\Phi},$$
$$\sigma^2(A) = \varepsilon\langle 2DG_1^2\rangle_{\Phi}. \quad (33)$$

Then substituting Eq. (31) into Eq. (33), the explicit expression for the averaged drift and diffusion coefficients can be obtain as follows:

$$m(A) = -\frac{A}{2}\left[\beta_1\omega^{\alpha_1-1}\sin\frac{\alpha_1\pi}{2} + \beta_2\omega^{\alpha_2-1}\cos\frac{(\alpha_2-1)\pi}{2}\right] + \frac{b_1A}{2} - \frac{b_2A^3}{8} + \frac{D}{2A\omega_0^2}, \quad (34)$$
$$\sigma^2(A) = \frac{D}{\omega_0^2}, \quad (35)$$

in which

$$\omega_0^2 = \beta_1\omega^{\alpha_1}\cos\frac{\alpha_1\pi}{2} - \beta_2\omega^{\alpha_2}\sin\frac{(\alpha_2-1)\pi}{2} + \omega^2. \quad (36)$$

The corresponding FPK equation associated with Eq. (29) has the following form:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial A}[m(A)p] + \frac{1}{2}\frac{\partial^2}{\partial A^2}[\sigma^2(A)p]. \quad (37)$$

The boundary conditions are $p = c$ in which $c\in(-\infty,+\infty)$ at $A = 0$ and $p\to 0$, $\partial p/\partial A\to 0$ as $A\to\infty$. Thus, based on these boundary conditions, the stationary solution of Eq. (37) is

$$p(A) = \frac{C}{\sigma^2(A)}\exp\left[\int_0^A\frac{2m(u)}{\sigma^2(u)}\,\mathrm{d}u\right], \quad (38)$$

where C is the normalization constant.

The stationary probability density of the system Hamiltonian $H = U(A) = \frac{1}{2}\omega_0^2A^2$ can be obtained from Eq. (38) as follows:

$$p(H) = p(A)\left|\frac{\mathrm{d}A}{\mathrm{d}H}\right| = \left.\frac{p(A)}{\omega_0^2A}\right|_{A=U^{-1}(H)}. \quad (39)$$

Then the joint stationary probability density of the displacement and velocity can be obtained as follows:

$$p(x,y) = \frac{p(H)}{T(H)}\bigg|_{H=\omega_0^2 x^2/2+y^2/2}, \tag{40}$$

where

$$T(H) = 2\pi/\omega_0. \tag{41}$$

4. Numerical simulation

In the paper, the following algorithm[31] is used to deal with the numerical solution of differential equations of fractional order, equipped with suitable initial conditions, i.e.,

$$\begin{cases} D^{\alpha}x(t) = f(t,x(t)), \\ x(0) = x_0. \end{cases} \tag{42}$$

Based on the predictor–corrector algorithm, the following iterative formula is adopted:

$$\begin{aligned} x_h(t_{n+1}) &= x_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{n+1}, x_h^p(t_{n+1})) \\ &\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} f(t_i, x_h(t_i)), \end{aligned} \tag{43}$$

in which the weight of the corrector $a_{i,n+1}$ is in the following form:

$$a_{i,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & i = 0, \\ (n-i+2)^{\alpha+1} + (n-i)^{\alpha+1} & \\ \quad -2(n-i+1)^{\alpha+1}, & 1 \le i \le n, \\ 1, & i = n+1, \end{cases} \tag{44}$$

where $x_h^p(t_{n+1})$ denotes the predicted value, which is governed by

$$x_h^p(t_{n+1}) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{n} b_{i,n+1} f(t_i, x_h(t_i)); \tag{45}$$

$b_{i,n+1}$ represents the weight of the predictor and is expressed as

$$b_{i,n+1} = \frac{h^{\alpha}}{\alpha} [(n-i+1)^{\alpha} - (n-i)^{\alpha}]. \tag{46}$$

The error of this approach is estimated to be $O(h^{\min(2,1+\alpha)})$. The pseudo-code is available in the appendix of Ref. [31].

5. Numerical results

In order to demonstrate the efficiency of the proposed method, some numerical results are obtained for the fractional VDP oscillator (1) with different parameter values. It is found that the analytical results and the Monte Carlo simulations of the original VDP oscillator are in satisfactory agreement with each other. We also discuss the effects of fractional order α_1 and α_1, the fractional coefficients β_1 and β_2, and the intensity of Gaussian white noise on the fractional VDP oscillator.

In this section, the time step is 0.01 and the total computation time is 10^4 s in the implementation of the Monte Carlo simulation. The results from the analytical solutions and the Monte Carlo simulations are shown in the following figures by solid line (—) and star ($*$), respectively.

5.1. Effect of fractional order $\alpha_1 \in (0,1)$

We examine the effect of the order of Caputo-type fractional derivative α_1 on the fractional VDP oscillator (1). The stationary probability density of amplitude $p(A)$, displacement $p(x)$, and velocity $p(y)$ from an analytical solution for different values of the fractional derivative order are shown in Figs. 1 and 2. The results from Monte Carlo simulation of the original system (1) are shown in Figs. 1 and 2. Examining those figures, it is found that the analytical results and the Monte Carlo simulations are in very satisfactory agreement with each other. We can conclude that the smaller the fractional derivative order α_1, the greater the probability of the oscillator with large amplitude (displacement or velocity) is.

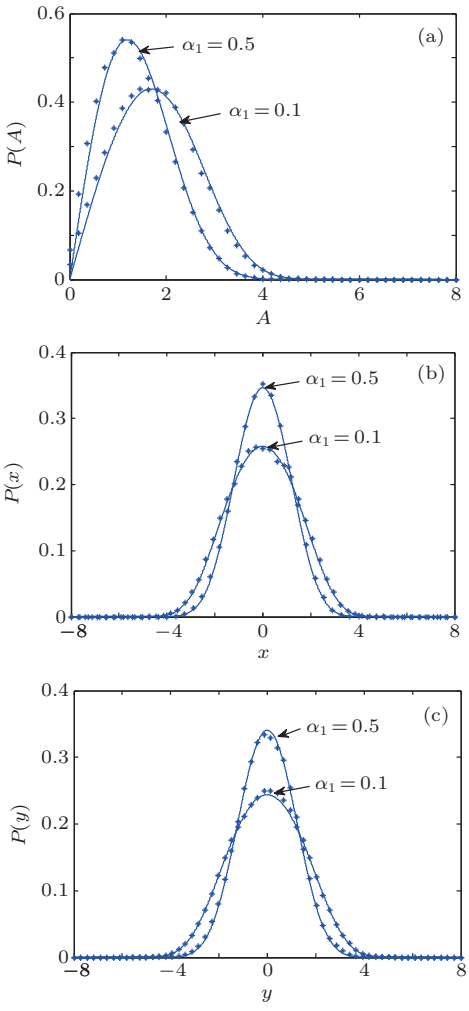


Fig. 1. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_2 = 1.7$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $b_1 = b_2 = 0.05$, $\omega = 1.0$, and $D = 0.4$.

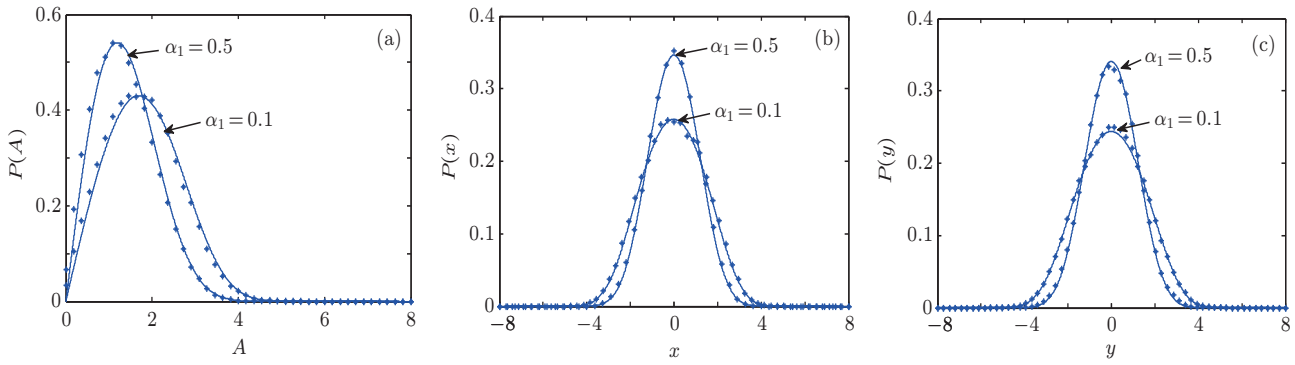


Fig. 2. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_2 = 1.8$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $b_1 = b_2 = 0.05$, $\omega = 1.0$, and $D = 0.4$.

5.2. Effect of fractional order $\alpha_2 \in (1,2)$

We examine the effect of the order of Caputo-type fractional derivative α_2 on the fractional VDP oscillator (1). The results from the analytical solution and the Monte Carlo simulation are shown in Figs. 3 and 4. Examining those figures, it is found that the analytical results and the Monte Carlo simulations are in good agreement with each other. We can obtain the following interesting conclusion, i.e., the smaller the fractional derivative order α_2, the smaller the probability of the oscillator with large amplitude (displacement or velocity) is, which is totally contrary to the conclusion in Subsection 5.1.

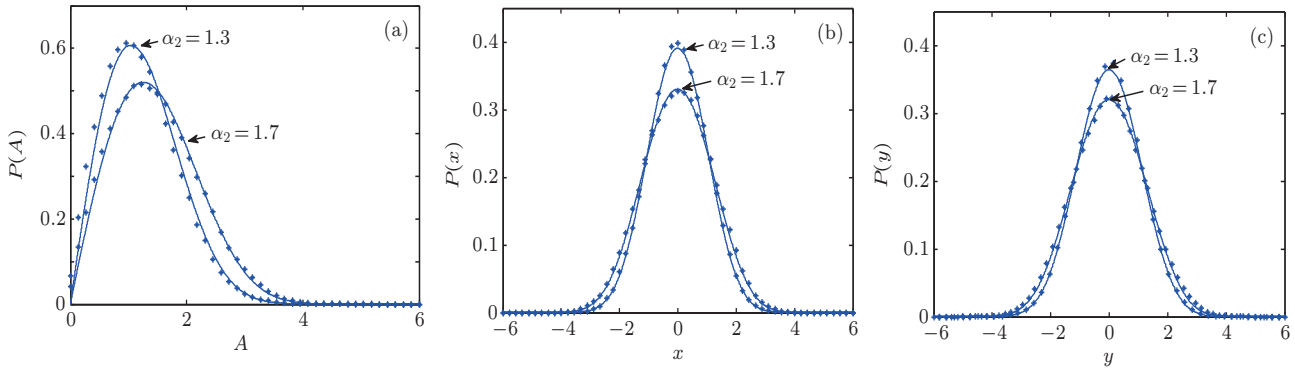


Fig. 3. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_1 = 0.4$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $b_1 = b_2 = 0.05$, $\omega = 1.0$, and $D = 0.4$.

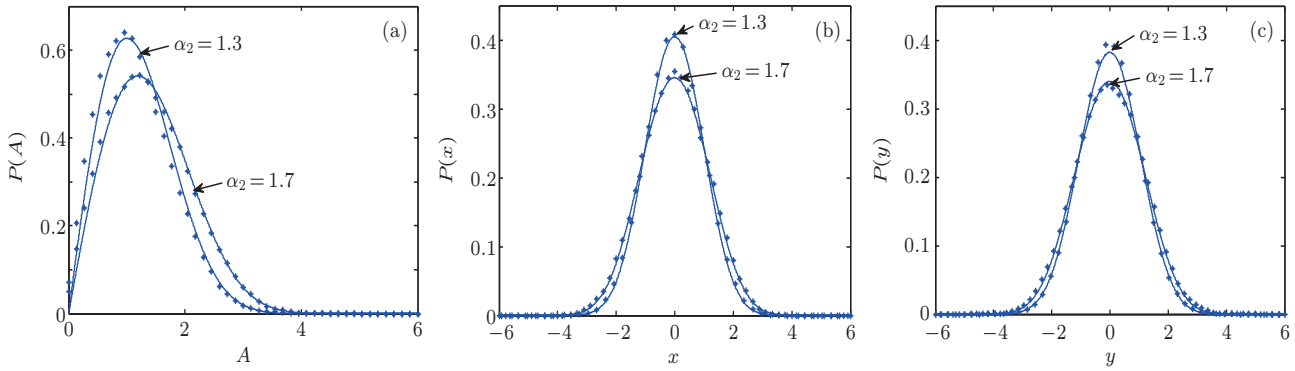


Fig. 4. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_1 = 0.5$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $b_1 = b_2 = 0.05$, $\omega = 1.0$, and $D = 0.4$.

5.3. Effect of fractional coefficient β_1

We investigate the effect of the coefficient of Caputo-type fractional derivative β_1 on the fractional VDP oscillator (1). The results from the analytical solution and the Monte Carlo simulation are shown in Figs. 5 and 6. Those figures show very satisfactory agreement between the analytical results and the Monte Carlo simulations. We can obtain the following conclusions: the smaller the fractional coefficient β_1, the larger the probability of the oscillator with large amplitude (displacement or velocity) is.

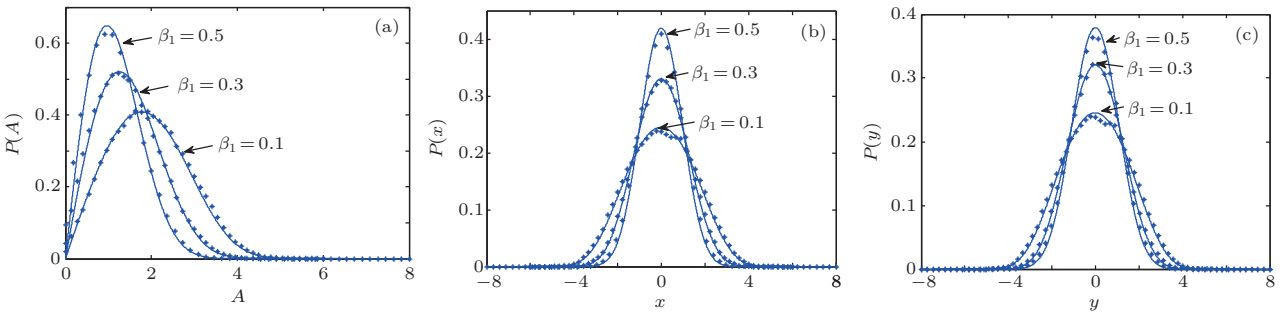


Fig. 5. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_1 = 0.4$, $\alpha_1 = 1.7$, $\beta_2 = 0.2$, $b_1 = b_2 = 0.05$, $\omega = 1.0$, and $D = 0.4$.

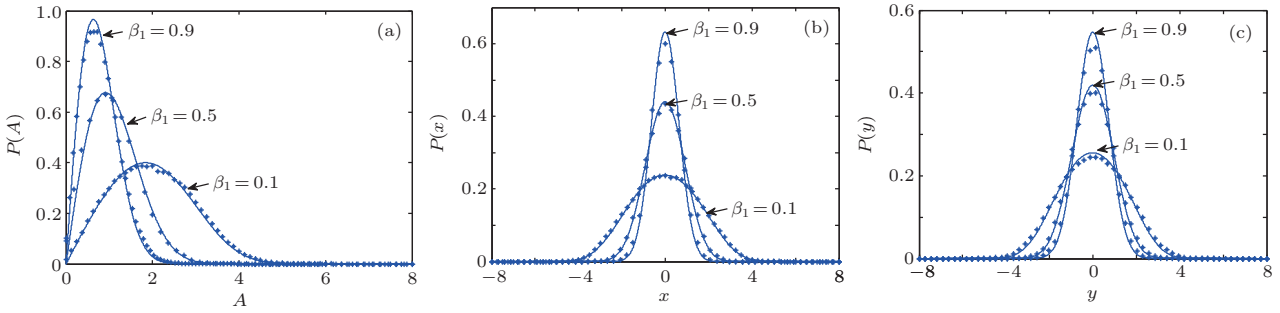


Fig. 6. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_1 = 0.6$, $\alpha_1 = 1.8$, $\beta_2 = 0.2$, $b_1 = b_2 = 0.05$, $\omega = 1.0$, and $D = 0.4$.

5.4. Effect of fractional coefficient β_2

Now we come to examine the effect of the coefficient of Caputo-type fractional derivative β_2 on the fractional VDP oscillator (1). The results from the analytical solution and the Monte Carlo simulation are shown in Figs. 7–9. Examining those figures shows very satisfactory agreement between the analytical results and the Monte Carlo simulations. We can obtain the following conclusion: the smaller the fractional coefficient β_2, the larger the probability of the oscillator with large amplitude (displacement or velocity) is, which is the same as the conclusion in Subsection 5.3.

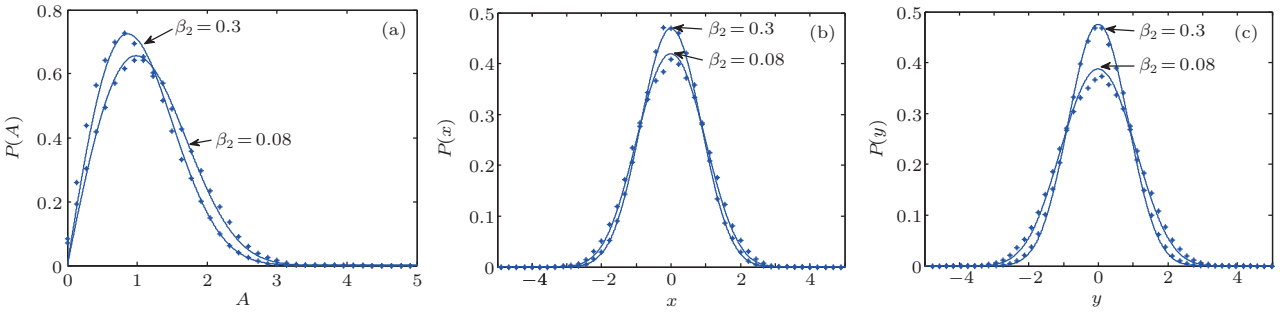


Fig. 7. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_1 = 0.4$, $\alpha_1 = 1.7$, $\beta_1 = 0.3$, $b_1 = b_2 = 0.05$, $\omega = 1.0$, and $D = 0.2$.

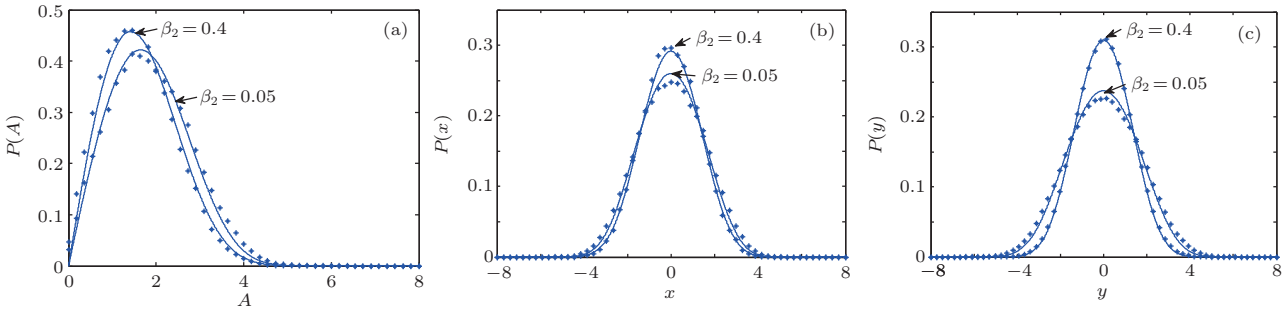


Fig. 8. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_1 = 0.4$, $\alpha_1 = 1.7$, $\beta_1 = 0.3$, $b_1 = b_2 = 0.05$, $\omega = 1.0$, and $D = 0.6$.

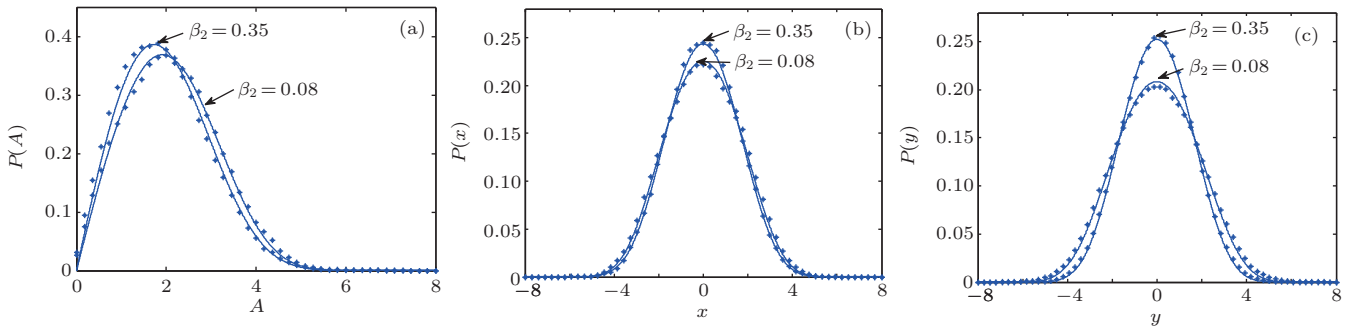


Fig. 9. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_1 = 0.4$, $\alpha_1 = 1.7$, $\beta_1 = 0.3$, $b_1 = b_2 = 0.05$, $\omega = 1.0$, and $D = 0.9$.

5.5. Effect of the intensity *D*

Finally, we consider the effect of the intensity of Gaussian white noise D on the fractional VDP oscillator (1). The results from the analytical solution and the Monte Carlo simulation are shown in Figs. 10 and 11. Examining those figures indicates very satisfactory agreement between the analytical results and the Monte Carlo simulations. We can obtain the following conclusions: the smaller the intensity of Gaussian white noise D, the smaller the probability of the oscillator with large amplitude (displacement or velocity) is.

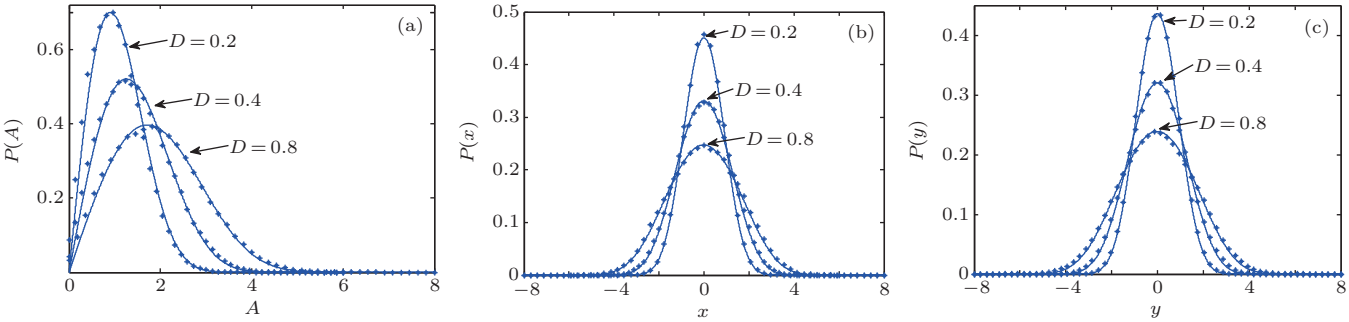


Fig. 10. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_1 = 0.4$, $\alpha_1 = 1.7$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $b_1 = b_2 = 0.05$, and $\omega = 1.0$.

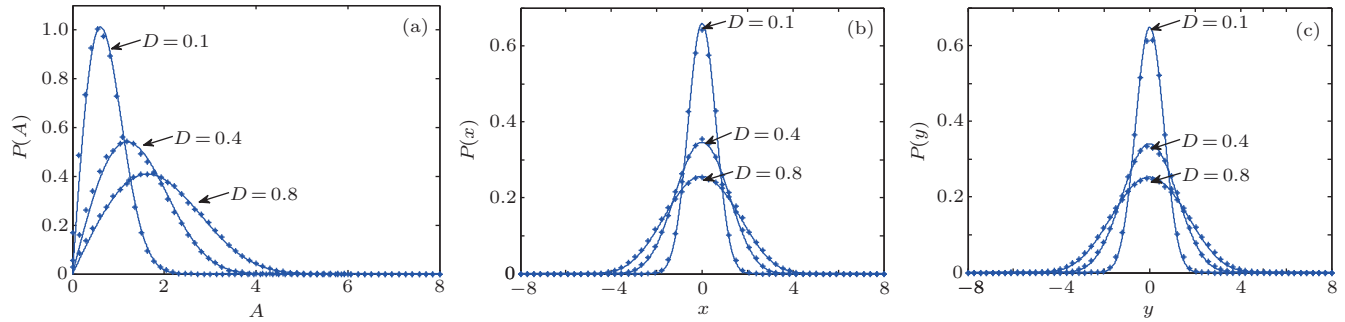


Fig. 11. (color online) Stationary probability densities of amplitude A, displacement x, and velocity y of oscillator (1) with $\alpha_1 = 0.5$, $\alpha_1 = 1.7$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $b_1 = b_2 = 0.05$, and $\omega = 1.0$.

6. Conclusions

In this paper, we investigate the stochastic response of VDP oscillator with two kinds of fractional derivatives under Gaussian white noise excitation. The equivalent VDP oscillator without fractional derivative terms for the fractional VDP oscillator is obtained by using the generalized harmonic balance technique. Then, the analytical solution of the equivalent VDP oscillator can be derived by the stochastic averaging method. Finally, the analytical results are validated by those from the Monte Carlo simulation of the original fractional VDP oscillator.

The numerical results not only demonstrate the accuracy of the proposed approach but also show that the fractional orders α_1 and α_2, the fractional coefficients β_1 and β_2 and the intensity D of Gaussian white noise play important roles in the responses of the fractional VDP oscillator.

More specifically, the smaller the fractional derivative order α_1, the greater the probability of the oscillator with large amplitude (displacement or velocity) is; the effect of fractional order α_2 is contrary to that of fractional order α_1, i.e., the

smaller the fractional derivative order α_2, the smaller the probability of the oscillator with large amplitude (displacement or velocity) is. The smaller the fractional coefficient β_1 or β_2, the larger the probability of the oscillator with large amplitude (displacement or velocity) is. The smaller the intensity D of Gaussian white noise, the smaller the probability of the oscillator with large amplitude (displacement or velocity) is.

It is noted that the approach is only suitable for the fractional VDP oscillator with two kinds of fractional derivatives subjected to weakly Gaussian white noise and light damping.

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