Landau damping in a dipolar Bose–Fermi mixture in the Bose–Einstein condensation (BEC) limit

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By using a mean-field approximation which describes the coupled oscillations of condensate and noncondensate atoms in the collisionless regime, Landau damping in a dilute dipolar Bose–Fermi mixture in the BEC limit where Fermi superfluid is treated as tightly bounded molecules, is investigated. In the case of a uniform quasi-two-dimensional (2D) case, the results for the Landau damping due to the Bose–Fermi interaction are obtained at low and high temperatures. It is shown that at low temperatures, the Landau damping rate is exponentially suppressed. By increasing the strength of dipolar interaction, and the energy of boson quasiparticles, Landau damping is suppressed over a broader temperature range.

Keywords: Landau damping, mixtures of Bose and Fermi superfluids, dipole–dipole interaction

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1. Introduction

In the last few years, ultracold atoms have appeared as an inimitable tool to investigate quantum many-body systems. This new research area, with the unique possibility of tuning the system features, allows researchers to study the new conditions in condensed matter or low temperature physics. There are a large number of parameters such as temperature, number of atoms, trap potentials, interactions between atoms, which can be controlled in an ultracold system.

Significant progress in understanding the phenomena of BEC for bosons and the BCS-BEC crossover for fermions are among the key successes of the field.[1,2] On the other hand, many-body Bose–Fermi mixtures of particles with different quantum statistics, are believed to exhibit very different behavior from pure Bose and Fermi systems. In addition, preparing and studying the mixtures of bosons and fermions experimentally in the quantum degenerate regime, lead to direct experimental tests of theoretical predictions for these mixtures.

Low-lying condensate oscillations of a trapped Bose–Fermi mixture have been studied at finite temperature in the collisionless regime by Liu and Hu.[3] They studied the behaviors of monopole and quadrupole excitations as a function of the coupling of condensate, noncondensate, and degenerate Fermi gas. In a Bose–Fermi superfluid mixture the quasiparticles possess a special property. Recently, a mixture of Bose and Fermi superfluids preparing bosonic ^{7}Li atoms and two spin components of fermionic ^{6}Li atoms, has been realized by ENS group.[4] In the ENS experiment, the Fermi superfluid was controlled from the BCS to the BEC region via Feshbach resonance. In a Bose–Fermi superfluid mixture, there are one gapped fermionic mode of Cooper pair breaking in the Fermi superfluid and two gapless bosonic modes. The two bosonic modes are Goldstone modes of Bose and Fermi superfluids. The ENS experiment leads to many theoretical studies on this new superfluid mixture.[5–11]

At low temperatures and low densities in these systems, where the mean free path of the elementary excitations becomes comparable to the size of the system, collisions play a minor role and the appropriate regime is the collisionless regime. In this regime, the interactions between quasiparticles can be characterized by the familiar Bogoliubov dispersion and determine many features of these systems. Landau–Beliaev damping is one of the widely known effects in these systems. In a damping mechanism, called Beliaev damping, an excitation of the system decays into two states with smaller energies and momenta.[12,13] Beliaev damping can happen even at $T = 0$. At finite temperatures, a mechanism of damping (known as Landau damping) occurs by the process of a collective mode absorbed by a quasiparticle that is then turned into another quasiparticle. Landau damping becomes the dominant damping mechanism in higher temperatures.[14,15]

The effect of quasiparticle interactions in a Bose–Fermi superfluid mixture has been investigated by Shen and Zheng.[10] They have shown that the Landau damping of the Bose superfluid due to its interaction with the Fermi superfluid is exponentially suppressed and on the BCS and the BEC side of the Fermi superfluid, the damping rate has a distinct threshold manner and momentum dependence. The effect of quasiparticle interaction in a Bose–Fermi superfluid mixture, has been investigated by Zheng and Zhai.[11] They found that at low temperatures, the Landau damping is a constant near the threshold momentum on the BCS side, while it increases rapidly on the BEC side. In fact, the fermion quasiparticles are

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located near the Fermi surface on the BCS side, while such a restriction does not exist on the BEC side. This restriction causes different damping rate behavior on the BCS and the BEC side of the Fermi superfluid.

The development of experimentally realized dipolar quantum gases such as the heteronuclear ^{40}K^{87}Rb molecules with electric dipoles[16–18] and spinor ^{87}Rb condensates with magnetic dipoles,[19] has aroused the interest in cold-atom mixtures involving dipolar condensates. These experimental studies have now expanded from alkali elements to the transition element, ^{52}Cr,[20] ^{164}Dy,[21] and ^{168}Er[22] with large magnetic dipoles. Mixtures of two different fermionic species further allow the creation of polar molecules, each of which has a long-range dipole–dipole interaction.[17,23] The mixture ^{6}Li$-^{40}$K is a prime candidate for these studies. ^{6}Li$-^{40}$K are the only stable fermionic alkali isotopes and thus belong to the experimentally best-mastered class of atoms. Moreover, both species have bosonic isotopes which can also be used to create boson-fermion gases. Furthermore, the difference in the electronic structure between the two species is large, leading to a large electric dipole moment for heteronuclear diatomic ^{6}Li^{40}K molecule.[24] Thus, it is conceivable that the dipolar Bose–Fermi and Fermi–Fermi mixtures can be realized in cold-atom experiments not too far down the road.

Landau damping of excitations in a quasi-2D dipolar Bose gas has been studied in conformity with the time-dependent mean-field formalism by Natu and Wilson.[25] They calculated the Landau damping rates for phonons at low and high temperatures. The effect of long-range $1/r$ interaction on the Landau damping in a Bose–Fermi superfluid mixture has also been studied.[26]

To the best of our knowledge, the effect of dipole–dipole interaction between particles on the Landau damping of a Bose–Fermi superfluid mixture has not been studied. In this paper, we study the Landau damping in a mixture of dipolar Bose–Fermi superfluid gas. We consider a case in which the Fermi superfluid is in the strongly interacting regime and on the BEC side, whilst the Bose superfluid is in the weakly interacting regime. On the BEC side we can consider the Fermi superfluid as tightly bounded molecules and treat it as a molecular BEC. Using the time-dependent mean-field approach according to the Popov approximation,[27] the damping coefficients are calculated. Finally, for a 2D dipolar Bose–Fermi mixture, the temperature dependence of Landau damping due to boson–fermion interaction is calculated and the results are plotted for different values of dipole–dipole interaction. We think our results are useful for theoretical and experimental research on the effects of dipole–dipole interaction on the Landau damping of a dipolar Bose–Fermi mixture.

2. Theoretical method

The Hamiltonian of a trapped mixture of bosons and molecules in the BEC limit where Fermi superfluid is treated as tightly bounded molecules, reads

$$\hat{\mathscr{H}} = \hat{\mathscr{H}}_{\rm b} + \hat{\mathscr{H}}_{\rm m} + \hat{\mathscr{H}}_{\rm bm}, \tag{1}$$

where the bosonic ($\hat{\mathscr{H}}_{\rm b}$), molecular ($\hat{\mathscr{H}}_{\rm m}$), and bosons–molecule interaction ($\hat{\mathscr{H}}_{\rm bm}$) parts of the Hamiltonian in the second quantized form respectively are

$$\hat{\mathscr{H}}_{\rm b} = \int {\rm d}\boldsymbol{r}\,\hat{\Psi}_{\rm b}^{\dagger}(\boldsymbol{r},t)[-\frac{1}{2m_{\rm b}}\nabla^2 - \mu_{\rm b}(\boldsymbol{r})]\hat{\Psi}_{\rm b}(\boldsymbol{r},t) + \frac{1}{2}\int {\rm d}\boldsymbol{r}\int {\rm d}\boldsymbol{r}'\,V_{\rm b}(\boldsymbol{r}-\boldsymbol{r}')\hat{\Psi}_{\rm b}^{\dagger}(\boldsymbol{r},t)\hat{\Psi}_{\rm b}^{\dagger}(\boldsymbol{r}',t)\hat{\Psi}_{\rm b}(\boldsymbol{r}',t)\hat{\Psi}_{\rm b}(\boldsymbol{r},t), \tag{2}$$

$$\hat{\mathscr{H}}_{\rm m} = \int {\rm d}\boldsymbol{r}\hat{\Psi}_{\rm m}^{\dagger}(\boldsymbol{r},t)\left[-\frac{1}{2m_{\rm m}}\nabla^2 - \mu_{\rm m}(\boldsymbol{r})\right]\hat{\Psi}_{\rm m}(\boldsymbol{r},t) + \frac{1}{2}\int {\rm d}\boldsymbol{r}\int {\rm d}\boldsymbol{r}'V_{\rm m}(\boldsymbol{r}-\boldsymbol{r}')\hat{\Psi}_{\rm m}^{\dagger}(\boldsymbol{r},t)\hat{\Psi}_{\rm m}^{\dagger}(\boldsymbol{r}',t)\hat{\Psi}_{\rm m}(\boldsymbol{r}',t)\hat{\Psi}_{\rm m}(\boldsymbol{r},t), \tag{3}$$

$$\hat{\mathscr{H}}_{\rm bm} = \int {\rm d}\boldsymbol{r}\int {\rm d}\boldsymbol{r}'\,V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}')\hat{\Psi}_{\rm b}^{\dagger}(\boldsymbol{r},t)\hat{\Psi}_{\rm m}^{\dagger}(\boldsymbol{r}',t)\hat{\Psi}_{\rm m}(\boldsymbol{r}',t)\hat{\Psi}_{\rm b}(\boldsymbol{r},t), \tag{4}$$

where $m_{\rm b}$ ($m_{\rm m}$) is the atomic mass of boson (molecule), $\mu_{\rm b(m)}$ $(\boldsymbol{r})$ is the spatially varying bosonic (molecular) chemical potential, and $\hat{\Psi}_{\rm b}(\boldsymbol{r},t)$ ($\hat{\Psi}_{\rm m}$ $(\boldsymbol{r},t)$) and $\hat{\Psi}_{\rm b}{}^{\dagger}$ $(\boldsymbol{r},t)$ ($\hat{\Psi}_{\rm m}{}^{\dagger}$ $(\boldsymbol{r},t)$) are the creation and annihilation bosonic (molecular) field operators. In the presence of the bosonic (molecular) trapping potentials $U_{\rm b(m)}$ $(\boldsymbol{r})$, $\mu_{\rm b(m)}$ $(\boldsymbol{r}) = \mu_{0\rm b(m)} - U_{\rm b(m)}(\boldsymbol{r})$, where $\mu_{0\rm b(m)}$ is the bosonic (molecular) chemical potential in the center of trap. The two-body interactions in Eqs. (2)–(4) include both contact and the dipole–dipole potentials as

$$V_{\rm b}(\boldsymbol{r}-\boldsymbol{r}') = g_{\rm b}\delta(\boldsymbol{r}-\boldsymbol{r}') + g_{\rm db}\frac{1-3\cos^2\theta}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \tag{5}$$

$$V_{\rm m}(\boldsymbol{r}-\boldsymbol{r}') = g_{\rm m}\delta(\boldsymbol{r}-\boldsymbol{r}') + g_{\rm dm}\frac{1-3\cos^2\theta}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \tag{6}$$

$$V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}') = g_{\rm bm}\delta(\boldsymbol{r}-\boldsymbol{r}') + g_{\rm dbm}\frac{1-3\cos^2\theta}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \tag{7}$$

where the contact potential for boson (molecule) is parameterized by coupling constant $g_{\rm b(m)} = 4\pi a_{\rm b(m)}/m_{\rm b(m)}$ to the lowest order in the s-wave scattering length $a_{\rm b}$. The contact potential between boson and molecule are also described by the coupling constant $g_{\rm bm} = 2\pi a_{\rm bm}/m_r$ to the lowest order in the s-wave scattering length $a_{\rm bm}$ with $m_r = m_{\rm b}m_{\rm m}/(m_{\rm b}+m_{\rm m})$ being the reduced mass. The coefficient $g_{\rm db(m)}$ is the strength

of dipole–dipole interaction of the bosons (molecules), and θ is the angle between the vector $\boldsymbol{r}-\boldsymbol{r}'$ and the z axis.

The Heisenberg equation of motion for the bosonic field operator reads

$$\mathrm{i}\partial_t\hat{\Psi}_{\mathrm{b}}(\boldsymbol{r},t)=\left(-\frac{1}{2m_{\mathrm{b}}}\nabla^2-\mu_{\mathrm{b}}(\boldsymbol{r})+\int \mathrm{d}r'\left[V_{\mathrm{b}}(\boldsymbol{r}-\boldsymbol{r}')|\hat{\Psi}_{\mathrm{b}}(r',t)|^2+V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')|\hat{\Psi}_{\mathrm{m}}(r',t)|^2\right]\right)\hat{\Psi}_{\mathrm{b}}(\boldsymbol{r},t), \tag{8}$$

$$\mathrm{i}\partial_t\hat{\Psi}_{\mathrm{m}}(\boldsymbol{r},t)=\left(-\frac{1}{2m_{\mathrm{m}}}\nabla^2-\mu_{\mathrm{m}}(\boldsymbol{r})+\int \mathrm{d}r'\left[V_{\mathrm{m}}(\boldsymbol{r}-\boldsymbol{r}')|\hat{\Psi}_{\mathrm{m}}(r',t)|^2+V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')|\hat{\Psi}_{\mathrm{b}}(r',t)|^2\right]\right)\hat{\Psi}_{\mathrm{m}}(\boldsymbol{r},t). \tag{9}$$

By considering the Fermi superfluid on the BEC side as molecular condensate, in the usual treatment for Bose systems with broken gauge symmetry, the bosonic (molecular) operators can be written as:

$$\hat{\Psi}_{\mathrm{b}}(\boldsymbol{r},t)=\phi_{\mathrm{b}}(\boldsymbol{r},t)+\hat{\psi}_{\mathrm{b}}(\boldsymbol{r},t), \tag{10}$$

$$\hat{\Psi}_{\mathrm{m}}(\boldsymbol{r},t)=\phi_{\mathrm{m}}(\boldsymbol{r},t)+\hat{\psi}_{\mathrm{m}}(\boldsymbol{r},t), \tag{11}$$

where $\phi_{\mathrm{b\,(m)}}(\boldsymbol{r},t)=\langle\hat{\Psi}_{b\,(m)}(\boldsymbol{r},t)\rangle$ is a time-dependent condensate wave function, and the operator $\hat{\psi}_{\mathrm{b\,(m)}}(\boldsymbol{r},t)$ acts on the noncondensed particles, which by definition have the property $\langle\hat{\psi}_{\mathrm{b\,(m)}}(\boldsymbol{r},t)\rangle=0$. Nonequilibrium average, which describes the fluctuations of the condensed and noncondensed atoms, is shown by $\langle\cdots\rangle$. Equilibrium average will be denoted by $\langle\cdots\rangle_0$. By substituting the decompositions (10) and (11) to Eqs. (8) and (9), the equations of motion for the condensate field for boson and molecule respectively become,

$$\begin{aligned}\mathrm{i}\partial_t\phi_{\mathrm{b}}(\boldsymbol{r},t)=&\left(-\frac{1}{2m_{\mathrm{b}}}\nabla^2-\mu_{\mathrm{b}}(\boldsymbol{r})\right)\phi_{\mathrm{b}}(\boldsymbol{r},t)\\&+\int \mathrm{d}r'V_{\mathrm{b}}(\boldsymbol{r}-\boldsymbol{r}')\left[|\phi_{\mathrm{b}}(r',t)|^2\phi_{\mathrm{b}}(\boldsymbol{r},t)+\phi_{\mathrm{b}}(r',t)n_{\mathrm{b}}(r',\boldsymbol{r},t)+\phi^*_{\mathrm{b}}(r',t)m_{\mathrm{b}}(r',\boldsymbol{r},t)+n_{\mathrm{b}}(r',t)\phi_{\mathrm{b}}(\boldsymbol{r},t)\right]\\&+\int \mathrm{d}r'V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')\left[|\phi_{\mathrm{m}}(r',t)|^2\phi_{\mathrm{b}}(\boldsymbol{r},t)+n_{\mathrm{m}}(r',t)\phi_{\mathrm{b}}(\boldsymbol{r},t)\right],\end{aligned} \tag{12}$$

$$\begin{aligned}\mathrm{i}\partial_t\phi_{\mathrm{m}}(\boldsymbol{r},t)=&\left(-\frac{1}{2m_{\mathrm{m}}}\nabla^2-\mu_{\mathrm{m}}(\boldsymbol{r})\right)\phi_{\mathrm{m}}(\boldsymbol{r},t)\\&+\int \mathrm{d}r'V_{\mathrm{m}}(\boldsymbol{r}-\boldsymbol{r}')\left[|\phi_{\mathrm{m}}(r',t)|^2\phi_{\mathrm{m}}(\boldsymbol{r},t)+\phi_{\mathrm{m}}(r',t)n_{\mathrm{m}}(r',\boldsymbol{r},t)+\phi^*_{\mathrm{m}}(r',t)m_{\mathrm{m}}(r',\boldsymbol{r},t)+n_{\mathrm{m}}(r',t)\phi_{\mathrm{m}}(\boldsymbol{r},t)\right]\\&+\int \mathrm{d}r'V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')\left[|\phi_{\mathrm{b}}(r',t)|^2\phi_{\mathrm{m}}(\boldsymbol{r},t)+n_{\mathrm{b}}(r',t)\phi_{\mathrm{m}}(\boldsymbol{r},t)\right],\end{aligned} \tag{13}$$

where the normal and anomalous densities are defined as

$$n_{\mathrm{b\,(m)}}(r',\boldsymbol{r},t)=\langle\hat{\psi}^{\dagger}_{\mathrm{b\,(m)}}(r',t)\hat{\psi}_{\mathrm{b\,(m)}}(\boldsymbol{r},t)\rangle, \tag{14}$$

$$n_{\mathrm{b\,(m)}}(\boldsymbol{r},t)=\langle\hat{\psi}^{\dagger}_{\mathrm{b\,(m)}}(\boldsymbol{r},t)\hat{\psi}_{\mathrm{b\,(m)}}(\boldsymbol{r},t)\rangle, \tag{15}$$

$$m_{\mathrm{b\,(m)}}(r',\boldsymbol{r},t)=\langle\hat{\psi}_{\mathrm{b\,(m)}}(r',t)\hat{\psi}_{\mathrm{b\,(m)}}(\boldsymbol{r},t)\rangle. \tag{16}$$

We also assume

$$0=\langle\hat{\psi}^{\dagger}_{\mathrm{b\,(m)}}(r',t)\hat{\psi}_{\mathrm{m\ (b)}}(\boldsymbol{r},t)\rangle, \tag{17}$$

$$0=\langle\hat{\psi}_{\mathrm{b\,(m)}}(r',t)\hat{\psi}_{\mathrm{m\ (b)}}(\boldsymbol{r},t)\rangle. \tag{18}$$

In the small amplitude regime and small fluctuations of the condensate, normal and anomalous densities around their equilibrium values can be written respectively as

$$\phi_{\mathrm{b\,(m)}}(\boldsymbol{r},t)=\phi_{0\mathrm{b\,(m)}}(\boldsymbol{r})+\delta\phi_{0\mathrm{b\,(m)}}(\boldsymbol{r},t), \tag{19}$$

$$n_{\mathrm{b\,(m)}}(r',\boldsymbol{r},t)=n_{\mathrm{eqb\,(m)}}(r',\boldsymbol{r})+\delta n_{\mathrm{b\,(m)}}(r',\boldsymbol{r},t), \tag{20}$$

$$m_{\mathrm{b\,(m)}}(r',\boldsymbol{r},t)=m_{\mathrm{eqb\,(m)}}(r',\boldsymbol{r})+\delta m_{\mathrm{b\,(m)}}(r',\boldsymbol{r},t), \tag{21}$$

where we use the shorthand notation

$$n_{0\mathrm{b\,(m)}}(\boldsymbol{r})=\left|\phi_{0\mathrm{b\,(m)}}(\boldsymbol{r})\right|^2, \tag{22}$$

$$n_{\mathrm{eqb\,(m)}}(r',\boldsymbol{r})=\left\langle\hat{\psi}^{\dagger}_{\mathrm{b\,(m)}}(r')\hat{\psi}_{\mathrm{b\,(m)}}(\boldsymbol{r})\right\rangle_0, \tag{23}$$

$$m_{\mathrm{eqb\,(m)}}(r',\boldsymbol{r})=\left\langle\hat{\psi}_{\mathrm{b\,(m)}}(r')\hat{\psi}_{\mathrm{b\,(m)}}(\boldsymbol{r})\right\rangle_0. \tag{24}$$

Based on the Popov approximation,[13] we ignore the anomalous density of the thermal cloud. By linearizing Eq. (12), the time-dependent equation for $\delta\phi_{\mathrm{b\,(m)}}(\boldsymbol{r},t)$ is obtained as

$$\begin{aligned}\mathrm{i}\partial_t\delta\phi_{\mathrm{b}}(\boldsymbol{r},t)=&\mathscr{L}_{0\mathrm{b}}\delta\phi_{\mathrm{b}}(\boldsymbol{r},t)+\int \mathrm{d}r'V_{\mathrm{b}}(\boldsymbol{r}-\boldsymbol{r}')\big(\phi_{0\mathrm{b}}(\boldsymbol{r})\phi_{0\mathrm{b}}(r')[\delta\phi^*_{\mathrm{b}}(r',t)+\delta\phi_{\mathrm{b}}(r',t)]\\&+n_{\mathrm{eqb}}(r',\boldsymbol{r})\delta\phi_{\mathrm{b}}(r',t)+\phi_{0\mathrm{b}}(r')[\delta m_{\mathrm{b}}(r',\boldsymbol{r},t)+\delta n_{\mathrm{b}}(r',\boldsymbol{r},t)]+\phi_{0\mathrm{b}}(\boldsymbol{r})\delta n_{\mathrm{b}}(r',t)\big)\\&+\int \mathrm{d}r'V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')\big(\phi_{0\mathrm{b}}(\boldsymbol{r})\phi_{0\mathrm{m}}(r')[\delta\phi^*_{\mathrm{m}}(r',t)+\delta\phi_{\mathrm{m}}(r',t)]+\phi_{0\mathrm{b}}(\boldsymbol{r})\delta n_{\mathrm{m}}(r',t)\big),\end{aligned} \tag{25}$$

$$\begin{aligned}\mathrm{i}\partial_t\delta\phi_m(\boldsymbol{r},t) = {}& \hat{\mathscr{L}}_{0\mathrm{m}}\delta\phi_{\mathrm{m}}(\boldsymbol{r},t) + \int \mathrm{d}r' V_{\mathrm{m}}(\boldsymbol{r}-\boldsymbol{r}')\big(\phi_{0\mathrm{m}}(\boldsymbol{r})\phi_{0\mathrm{m}}(r')[\delta\phi^*_{\mathrm{m}}(r',t)+\delta\phi_{\mathrm{m}}(r',t)]\\ &+ n_{\mathrm{eqm}}(r',\boldsymbol{r})\delta\phi_{\mathrm{m}}(r',t)+\phi_{0\mathrm{m}}(r')[\delta m_{\mathrm{m}}(r',\boldsymbol{r},t)+\delta n_{\mathrm{m}}(r',\boldsymbol{r},t)]+\phi_{0\mathrm{m}}(\boldsymbol{r})\delta n_{\mathrm{m}}(r',t)\big)\\ &+\int \mathrm{d}r' V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')\big(\phi_{0\mathrm{m}}(\boldsymbol{r})\phi_{0\mathrm{b}}(r')[\delta\phi^*_{\mathrm{b}}(r',t)+\delta\phi_{\mathrm{b}}(r',t)]+\phi_{0\mathrm{m}}(\boldsymbol{r})\delta n_{\mathrm{b}}(r',t)\big),\end{aligned} \tag{26}$$

where

$$\hat{\mathscr{L}}_{0\mathrm{b}(\mathrm{m})} = -\frac{1}{2m_{0\mathrm{b}(\mathrm{m})}}\nabla^2 - \mu_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r}) + \int \mathrm{d}r'\, V_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r}-\boldsymbol{r}')n_{\mathrm{tot\,b}(\mathrm{m})}(r') + \int \mathrm{d}r'\, V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')n_{\mathrm{tot\,m}(\mathrm{b})}(r') \tag{27}$$

and $n_{\mathrm{tot\,b}(\mathrm{m})}(\boldsymbol{r}) = n_{0b(m)}(\boldsymbol{r}) + n_{\mathrm{eq\,b}(\mathrm{m})}(\boldsymbol{r})$ is the total density of bosons (molecules).

Substituting Eqs. (10) and (19) into Hamiltonian (1), and ignoring the cubic products of the noncondensate operators, the corresponding linearized interaction terms of the Hamiltonian can be decomposed into two parts.

$$\hat{\mathscr{H}}_{\mathrm{int}} = \hat{\mathscr{H}}^0_{\mathrm{int}} + \hat{\mathscr{H}}^{\delta}_{\mathrm{int}}, \tag{28}$$

where

$$\begin{aligned}\hat{\mathscr{H}}^0_{\mathrm{int\,b}(\mathrm{m})} = {}& \int \mathrm{d}\boldsymbol{r}\,\big\{\hat{\psi}^{\dagger}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\hat{\mathscr{L}}_{0\mathrm{b}(\mathrm{m})}\hat{\psi}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\\ &+\frac{1}{2}\int \mathrm{d}r' V_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r}-\boldsymbol{r}')\big(n_{\mathrm{tot\,b}(\mathrm{m})}(r',\boldsymbol{r})[\hat{\psi}^{\dagger}_{\mathrm{b}(\mathrm{m})}(r',t)\hat{\psi}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)+\hat{\psi}^{\dagger}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\hat{\psi}_{\mathrm{b}(\mathrm{m})}(r',t)]\\ &+\phi_{0\mathrm{b}(\mathrm{m})}(\boldsymbol{r})\phi_{0\mathrm{b}(\mathrm{m})}(r')[\hat{\psi}^{\dagger}_{\mathrm{b}(\mathrm{m})}(r',t)\hat{\psi}^{\dagger}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)+\hat{\psi}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\hat{\psi}_{\mathrm{b}(\mathrm{m})}(r',t)]\big)\big\},\end{aligned} \tag{29}$$

$$\begin{aligned}\hat{\mathscr{H}}^0_{\mathrm{int\,bm}} = {}& \int \mathrm{d}\boldsymbol{r}\int \mathrm{d}r' V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')\big(\phi_{0\mathrm{m}}(\boldsymbol{r}')\psi^{\dagger}_{\mathrm{m}}(\boldsymbol{r}',t)\hat{\psi}^{\dagger}_{\mathrm{b}}(r,t)\hat{\psi}_{\mathrm{b}}(\boldsymbol{r},t)+\phi_{0\mathrm{b}}(\boldsymbol{r})\psi_{\mathrm{b}}{}^{\dagger}(r,t)\hat{\psi}^{\dagger}_{\mathrm{m}}(\boldsymbol{r}',t)\hat{\psi}_{\mathrm{m}}(\boldsymbol{r}',t)\\ &+\phi^*_{0\mathrm{m}}(\boldsymbol{r}')\psi_{\mathrm{m}}(\boldsymbol{r}',t)\hat{\psi}^{\dagger}_{\mathrm{b}}(r,t)\hat{\psi}_{\mathrm{b}}(\boldsymbol{r},t)+\phi^*_{0\mathrm{b}}(\boldsymbol{r})\psi_{\mathrm{b}}(\boldsymbol{r},t)\hat{\psi}^{\dagger}_{\mathrm{m}}(\boldsymbol{r}',t)\hat{\psi}_{\mathrm{m}}(\boldsymbol{r}',t)\big).\end{aligned} \tag{30}$$

In general, we define $n_{\mathrm{tot\,b}(\mathrm{m})}(\boldsymbol{r},\boldsymbol{r}') = \phi_{0\mathrm{b}(\mathrm{m})}(\boldsymbol{r})\phi_{0\mathrm{b}(\mathrm{m})}(\boldsymbol{r}') + n_{\mathrm{eq\,b}(\mathrm{m})}(\boldsymbol{r},\boldsymbol{r}')$. The Hamiltonian $\hat{\mathscr{H}}^0_{\mathrm{int}}$ includes the equilibrium condensate and densities. The quartic terms of the Hamiltonians (2)–(4) in the mean-field approximation can be written as

$$\hat{\psi}^{\dagger}\hat{\psi}'^{\dagger}\hat{\psi}'\hat{\psi} = \langle\hat{\psi}'^{\dagger}\hat{\psi}'\rangle\hat{\psi}^{\dagger}\hat{\psi} + \langle\hat{\psi}^{\dagger}\hat{\psi}'\rangle\hat{\psi}'^{\dagger}\hat{\psi} + \langle\hat{\psi}'^{\dagger}\hat{\psi}\rangle\hat{\psi}^{\dagger}\hat{\psi}' + \langle\hat{\psi}^{\dagger}\hat{\psi}\rangle\hat{\psi}'^{\dagger}\hat{\psi}'. \tag{31}$$

The second part of Hamiltonian (28) which is linear in terms of $\delta\phi_{\mathrm{b}(\mathrm{m})}$ reads

$$\begin{aligned}\hat{\mathscr{H}}^{\delta}_{\mathrm{int\,b}(\mathrm{m})} = {}& \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}r' V_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r}-\boldsymbol{r}')[\phi_{0b(m)}(\boldsymbol{r})\{[\delta\phi_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)+\delta\phi^*_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)]\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(r',t)\psi_{\mathrm{b}(\mathrm{m})}(r',t)\\ &+\delta\phi_{\mathrm{b}(\mathrm{m})}(r',t)\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(r',t)\psi_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)+\delta\phi^*_{\mathrm{b}(\mathrm{m})}(r',t)\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\psi_{\mathrm{b}(\mathrm{m})}(r',t)\\ &+\delta\phi_{\mathrm{b}(\mathrm{m})}(r',t)\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(r',t)+\delta\phi^*_{\mathrm{b}(\mathrm{m})}(r',t)\psi_{\mathrm{b}(\mathrm{m})}(r',t)\psi_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\}\\ &+\phi_{0\mathrm{b}(\mathrm{m})}(r')\{[\delta\phi_{\mathrm{b}(\mathrm{m})}(r',t)+\delta\phi^*_{\mathrm{b}(\mathrm{m})}(r',t)]\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\psi_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\\ &+\delta\phi_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\psi_{\mathrm{b}(\mathrm{m})}(r',t)+\delta\phi^*_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(r',t)\psi_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\\ &+\delta\phi_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\psi^{\dagger}_{\mathrm{b}(\mathrm{m})}(r',t)+\delta\phi^*_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\psi_{\mathrm{b}(\mathrm{m})}(r',t)\psi_{\mathrm{b}(\mathrm{m})}(\boldsymbol{r},t)\}]\\ &+\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}r' V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')[\phi_{0\mathrm{b}(\mathrm{m})}(r')\{[\delta\phi_{\mathrm{b}(\mathrm{m})}(r',t)+\delta\phi^*_{\mathrm{b}(\mathrm{m})}(r',t)]\hat{\psi}^{\dagger}_{\mathrm{m}(\mathrm{b})}(r,t)\hat{\psi}_{\mathrm{m}(\mathrm{b})}(r,t)\}.\end{aligned} \tag{32}$$

It is convenient to diagonalize the Hamiltonian $\hat{\mathscr{H}}^{\delta}_{\mathrm{int}}$ by the linear transformations

$$\hat{\psi}_{\mathrm{b}}(\boldsymbol{r},t) = \sum_i [u^{\mathrm{b}}_i(\boldsymbol{r})\hat{a}_i(t) + v^{\mathrm{b}*}_i(\boldsymbol{r})\hat{a}^{\dagger}_i(t)], \tag{33}$$

$$\hat{\psi}_{\mathrm{m}}(\boldsymbol{r},t) = \sum_i [u^{\mathrm{m}}_i(\boldsymbol{r})\hat{d}_i(t) + v^{\mathrm{m}*}_i(\boldsymbol{r})\hat{d}^{\dagger}_i(t)], \tag{34}$$

where $\hat{a}_i(t)$ ($\hat{d}_i(t)$), and $\hat{a}^{\dagger}_i(t)$ ($\hat{d}^{\dagger}_i(t)$) are quasiparticle annihilation and creation operators for bosons (molecules) at time t, which satisfy the usual Bose commutation relations, and the functions $u^{\mathrm{b}(\mathrm{m})}_i(\boldsymbol{r})$ and $v^{\mathrm{b}(\mathrm{m})}_i(\boldsymbol{r})$ obey the following normalization condition:[28]

$$\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\left[u^{\mathrm{b}(\mathrm{m})*}_i(\boldsymbol{r})\,u^{\mathrm{b}(\mathrm{m})}_j(\boldsymbol{r}') - v^{\mathrm{b}(\mathrm{m})*}_i(\boldsymbol{r})\,v^{\mathrm{b}(\mathrm{m})}_j(\boldsymbol{r}')\right] = \delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{35}$$

The resulting Hamiltonians describe a noninteracting gas of Bogoliubov quasiparticles

$$\mathscr{H}_{\rm int}^{\rm 0b} = \sum_i E_i^{\rm b} \hat{a}_i^\dagger \hat{a}_i, \tag{36}$$

$$\mathscr{H}_{\rm int}^{\rm 0m} = \sum_i E_i^{\rm m} \hat{d}_i^\dagger \hat{d}_i, \tag{37}$$

where the quasiparticle energy $E_i^{\rm b(m)}$ is obtained from the solutions of the Bogoliubov equations

$$E_i^{\rm b(m)} u_i^{\rm b(m)}(\boldsymbol{r}) = \hat{\mathscr{L}}_{\rm b(m)}[u_i^{\rm b(m)}(\boldsymbol{r})] + \hat{\mathscr{M}}_{\rm b(m)}[v_i^{\rm b(m)}(\boldsymbol{r})], \tag{38}$$

$$-E_i^{\rm b(m)} v_i^{\rm b(m)}(\boldsymbol{r}) = \hat{\mathscr{L}}_{\rm b(m)}[v_i^{\rm b(m)}(\boldsymbol{r})] + \hat{\mathscr{M}}_{\rm b(m)}[u_i^{\rm b(m)}(\boldsymbol{r})]. \tag{39}$$

Here, we have defined

$$\hat{\mathscr{L}}_{\rm b(m)}[f(\boldsymbol{r})] = \hat{\mathscr{L}}_{\rm 0b(m)} f(\boldsymbol{r}) + \int {\rm d}r' V_{\rm b(m)}(\boldsymbol{r}-\boldsymbol{r}') n_{\rm tot\,b(m)}(\boldsymbol{r}',\boldsymbol{r}) f(\boldsymbol{r}'), \tag{40}$$

$$\hat{\mathscr{M}}_{\rm b(m)}[f(\boldsymbol{r})] = \int {\rm d}r' V_{\rm b(m)}(\boldsymbol{r}-\boldsymbol{r}') \phi_{\rm 0b(m)}(\boldsymbol{r}) \phi_{\rm 0b(m)}(\boldsymbol{r}') f(\boldsymbol{r}'). \tag{41}$$

In the presence of dipole–dipole interactions, the problem of solving nonlocal Bogoliubov equations becomes computationally intensive.[29–31] The normal and anomalous densities in terms of the Bogoliubov operators $a_i(t)$ ($d_i(t)$) and $a_i^\dagger(t)$ ($d_i^\dagger(t)$) can be written respectively as

$$\begin{aligned} n_{\rm b(m)}(\boldsymbol{r}',\boldsymbol{r},t) = & \sum_{ij} u_i^{\rm b(m)*}(\boldsymbol{r}') u_j^{\rm b(m)}(\boldsymbol{r}) f_{ij}^{\rm b(m)} + v_i^{\rm b(m)}(\boldsymbol{r}') v_j^{\rm b(m)*}(\boldsymbol{r}) f_{ji}^{\rm b(m)} + u_i^{\rm b*}(\boldsymbol{r}') v_j^{\rm b*}(\boldsymbol{r}) g_{ij}^{\rm b(m)*} \\ & + v_i^{\rm b(m)}(\boldsymbol{r}') u_j^{\rm b(m)}(\boldsymbol{r}) g_{ij}^{\rm b(m)} + v_i^{\rm b(m)}(\boldsymbol{r}') v_j^{\rm b(m)*}(\boldsymbol{r}), \end{aligned} \tag{42}$$

$$\begin{aligned} m_{\rm b(m)}(\boldsymbol{r}',\boldsymbol{r},t) = & \sum_{ij} u_i^{\rm b(m)}(\boldsymbol{r}') v_j^{\rm b(m)*}(\boldsymbol{r}) f_{ji}^{\rm b(m)} + v_i^{\rm b(m)*}(\boldsymbol{r}') u_j^{\rm b(m)}(\boldsymbol{r}) f_{ij}^{\rm b(m)} + u_i^{\rm b(m)}(\boldsymbol{r}') u_j^{\rm b(m)}(\boldsymbol{r}) g_{ij}^{\rm b(m)} \\ & + v_i^{\rm b(m)*}(\boldsymbol{r}') v_j^{\rm b(m)*}(\boldsymbol{r}) g_{ij}^{\rm b(m)*} + u_i^{\rm b(m)}(\boldsymbol{r}') v_j^{\rm b(m)*}(\boldsymbol{r}), \end{aligned} \tag{43}$$

where it is convenient to define the functions

$$f_{ij}^{\rm b}(t) \equiv \left\langle \hat{a}_i^\dagger(t) \hat{a}_j(t) \right\rangle, \qquad f_{ij}^{\rm m}(t) \equiv \left\langle \hat{d}_i^\dagger(t) \hat{d}_j(t) \right\rangle, \tag{44}$$

$$g_{ij}^{\rm b}(t) \equiv \left\langle \hat{a}_i(t) \hat{a}_j(t) \right\rangle, \qquad g_{ij}^{\rm m}(t) \equiv \left\langle \hat{d}_i(t) \hat{d}_j(t) \right\rangle. \tag{45}$$

Physically, $f_{ij}^{\rm b(m)}(t)$ corresponds to the creation and annihilation of a quasiparticle with energies $E_i^{\rm b(m)}$ and $E_j^{\rm b(m)}$, and contributes to Landau damping, while $g_{ij}^{\rm b(m)}(t)$ $\left(g_{ij}^{\rm b(m)*}(t)\right)$ corresponds to the annihilation (creation) of two quasiparticles with energies $E_i^{\rm b(m)}$ and $E_j^{\rm b(m)}$, and contributes to Beliaev damping.[32]

In terms of $f_{ij}^{\rm b(m)}(t)$ and $g_{ij}^{\rm b(m)}(t)$, the time dependence of the fluctuation of the condensate (12) can be written as

$$\begin{aligned} {\rm i}\partial_t \delta\phi_{\rm b}(\boldsymbol{r},t) = & \hat{\mathscr{L}}_{\rm b}[\delta\phi_{\rm b}(\boldsymbol{r},t)] + \hat{\mathscr{M}}_{\rm b}[\delta\phi_{\rm b}^*(\boldsymbol{r},t)] \\ & + \int {\rm d}r' V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}') \phi_{\rm 0b}(\boldsymbol{r}) \phi_{0m}(\boldsymbol{r}') [\delta\phi_{\rm m}^*(\boldsymbol{r}',t) + \delta\phi_{\rm m}(\boldsymbol{r}',t)] \\ & \times \sum_{ij} f_{ij}^{\rm b}(t) \int {\rm d}r' V_{\rm b}(\boldsymbol{r}-\boldsymbol{r}') \{\phi_{\rm 0b}(\boldsymbol{r}') [u_i^{\rm b*}(\boldsymbol{r}') u_j^{\rm b}(\boldsymbol{r}) + v_j^{\rm b}(\boldsymbol{r}') v_i^{\rm b*}(\boldsymbol{r}) \\ & + v_i^{\rm b*}(\boldsymbol{r}) u_j^{\rm b}(\boldsymbol{r}') + v_i^{\rm b*}(\boldsymbol{r}') u_j^{\rm b}(\boldsymbol{r})] + \phi_{\rm 0b}(\boldsymbol{r}) [u_i^{\rm b*}(\boldsymbol{r}') u_j^{\rm b}(\boldsymbol{r}') + v_i^{\rm b}(\boldsymbol{r}') v_j^{\rm b*}(\boldsymbol{r}')]\} \\ & + \sum_{ij} f_{ij}^{\rm m}(t) \int {\rm d}r' V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}') \phi_{\rm 0b}(\boldsymbol{r}) [u_i^{\rm m*}(\boldsymbol{r}') u_j^{\rm m}(\boldsymbol{r}') + v_i^{\rm m}(\boldsymbol{r}') v_j^{\rm m*}(\boldsymbol{r}')], \end{aligned} \tag{46}$$

$$\begin{aligned} {\rm i}\partial_t \delta\phi_{\rm m}(\boldsymbol{r},t) = & \hat{\mathscr{L}}_{\rm m}[\delta\phi_{\rm m}(\boldsymbol{r},t)] + \hat{\mathscr{M}}_{\rm m}[\delta\phi_{\rm m}^*(\boldsymbol{r},t)] \\ & + \int {\rm d}r' V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}') \phi_{\rm 0m}(\boldsymbol{r}) \phi_{\rm 0b}(\boldsymbol{r}') [\delta\phi_{\rm b}^*(\boldsymbol{r}',t) + \delta\phi_{\rm b}(\boldsymbol{r}',t)] \\ & \times \sum_{ij} f_{ij}^{\rm m}(t) \int {\rm d}r' V_{\rm m}(\boldsymbol{r}-\boldsymbol{r}') \{\phi_{0m}(\boldsymbol{r}') [u_i^{\rm m*}(\boldsymbol{r}') u_j^{\rm m}(\boldsymbol{r}) + v_j^{\rm m}(\boldsymbol{r}') v_i^{\rm m*}(\boldsymbol{r}) \\ & + v_i^{\rm m*}(\boldsymbol{r}) u_j^{\rm m}(\boldsymbol{r}') + v_i^{\rm m*}(\boldsymbol{r}') u_j^{\rm m}(\boldsymbol{r})] + \phi_{\rm 0m}(\boldsymbol{r}) [u_i^{\rm m*}(\boldsymbol{r}') u_j^{\rm m}(\boldsymbol{r}') + v_i^{\rm m}(\boldsymbol{r}') v_j^{\rm m*}(\boldsymbol{r}')]\} \\ & + \sum_{ij} f_{ij}^{\rm b}(t) \int {\rm d}r' V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}') \phi_{\rm 0m}(\boldsymbol{r}) [u_i^{\rm b*}(\boldsymbol{r}') u_j^{\rm b}(\boldsymbol{r}') + v_i^{\rm b}(\boldsymbol{r}') v_j^{\rm b*}(\boldsymbol{r}')]. \end{aligned} \tag{47}$$

Since we are interested in studying Landau damping ($f_{ij}^{\rm b}$ and $f_{ij}^{\rm m}$) in Eqs. (46) and (47), $g_{ij}^{\rm b}$ and $g_{ij}^{\rm m}$ which describe Beliaev damping are not considered. The Heisenberg equations of motion for $f_{ij}^{\rm b}(t)$ and $f_{ij}^{\rm m}(t)$ lead to

$${\rm i}\partial_t f_{ij}^{\rm b}(t) = \left\langle \left[a_i^\dagger(t) a_j(t), \mathscr{H}_{\rm int} \right] \right\rangle$$

$$
\begin{aligned}
&= (\varepsilon_j^{\rm b} - \varepsilon_i^{\rm b}) f_{ij}^{\rm b}(t) + \frac{(n_i^{\rm b} - n_j^{\rm b})}{2} \int {\rm d}\boldsymbol{r}\, {\rm d}\boldsymbol{r}'\, V_{\rm b}(\boldsymbol{r}-\boldsymbol{r}')[2\phi_{0{\rm b}}(\boldsymbol{r})\,[\delta\phi_{\rm b}^*(\boldsymbol{r},t) + \delta\phi_{\rm b}(\boldsymbol{r},t)]\{u_j^{\rm b*}(\boldsymbol{r}')u_i^{\rm b}(\boldsymbol{r}') + v_j^{\rm b*}(\boldsymbol{r}')v_i^{\rm b}(\boldsymbol{r}')\} \\
&\quad + 2[\phi_{0{\rm b}}(\boldsymbol{r})\,\delta\phi_{\rm b}(\boldsymbol{r}',t) + \phi_{0{\rm b}}(\boldsymbol{r}')\,\delta\phi_{\rm b}^*(\boldsymbol{r},t)]\{u_j^{\rm b*}(\boldsymbol{r}')u_i^{\rm b}(\boldsymbol{r}) + v_j^{\rm b*}(\boldsymbol{r})v_i^{\rm b}(\boldsymbol{r}')\} \\
&\quad + [\phi_{0{\rm b}}(\boldsymbol{r})\,\delta\phi_{\rm b}(\boldsymbol{r}',t) + \phi_{0{\rm b}}(\boldsymbol{r}')\,\delta\phi_{\rm b}(\boldsymbol{r},t)]\{u_j^{\rm b*}(\boldsymbol{r})v_i^{\rm b}(\boldsymbol{r}') + u_j^{\rm b*}(\boldsymbol{r}')v_i^{\rm b}(\boldsymbol{r})\} \\
&\quad + [\phi_{0{\rm b}}(\boldsymbol{r})\,\delta\phi_{\rm b}^*(\boldsymbol{r}',t) + \phi_{0{\rm b}}(\boldsymbol{r}')\,\delta\phi_{\rm b}^*(\boldsymbol{r},t)]\{u_i^{\rm b}(\boldsymbol{r})v_j^{\rm b*}(\boldsymbol{r}') + u_i^{\rm b}(\boldsymbol{r}')v_j^{\rm b*}(\boldsymbol{r})\}] \\
&\quad + \frac{(n_i^{\rm b} - n_j^{\rm b})}{2} \int {\rm d}\boldsymbol{r}\, {\rm d}\boldsymbol{r}'\, V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}')[2\phi_{0m}(\boldsymbol{r})\,[\delta\phi_{\rm m}^*(\boldsymbol{r},t) + \delta\phi_{\rm m}(\boldsymbol{r},t)]\{u_j^{\rm b*}(\boldsymbol{r}')u_i^{\rm b}(\boldsymbol{r}') + v_j^{\rm b*}(\boldsymbol{r}')v_i^{\rm b}(\boldsymbol{r}')\}, \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
{\rm i}\partial_t f_{ij}^{\rm m}(t) &= \left\langle \left[d_i^\dagger(t) d_j(t), \mathscr{H}_{\rm int} \right] \right\rangle \\
&= (\varepsilon_j^{\rm m} - \varepsilon_i^{\rm m}) f_{ij}^{\rm m}(t) + \frac{(n_i^{\rm m} - n_j^{\rm m})}{2} \int {\rm d}\boldsymbol{r}\, {\rm d}\boldsymbol{r}'\, V_{\rm m}(\boldsymbol{r}-\boldsymbol{r}')[2\phi_{0{\rm m}}(\boldsymbol{r})\,[\delta\phi_{\rm m}^*(\boldsymbol{r},t) + \delta\phi_{\rm m}(\boldsymbol{r},t)]\{u_j^{\rm m*}(\boldsymbol{r}')u_i^{\rm m}(\boldsymbol{r}') + v_j^{\rm m*}(\boldsymbol{r}')v_i^{\rm m}(\boldsymbol{r}')\} \\
&\quad + 2[\phi_{0{\rm m}}(\boldsymbol{r})\,\delta\phi_{\rm m}(\boldsymbol{r}',t) + \phi_{0{\rm m}}(\boldsymbol{r}')\,\delta\phi_{\rm m}^*(\boldsymbol{r},t)]\{u_j^{\rm m*}(\boldsymbol{r}')u_i^{\rm m}(\boldsymbol{r}) + v_j^{\rm m*}(\boldsymbol{r})v_i^{\rm m}(\boldsymbol{r}')\} \\
&\quad + [\phi_{0{\rm m}}(\boldsymbol{r})\,\delta\phi_{\rm m}(\boldsymbol{r}',t) + \phi_{0{\rm m}}(\boldsymbol{r}')\,\delta\phi_{\rm m}(\boldsymbol{r},t)]\{u_j^{\rm m*}(\boldsymbol{r})v_i^{\rm m}(\boldsymbol{r}') + u_j^{\rm m*}(\boldsymbol{r}')v_i^{\rm m}(\boldsymbol{r})\} \\
&\quad + [\phi_{0{\rm m}}(\boldsymbol{r})\,\delta\phi_{\rm m}^*(\boldsymbol{r}',t) + \phi_{0{\rm m}}(\boldsymbol{r}')\,\delta\phi_{\rm m}^*(\boldsymbol{r},t)]\{u_i^{\rm m}(\boldsymbol{r})v_j^{\rm m*}(\boldsymbol{r}') + u_i^{\rm m}(\boldsymbol{r}')v_j^{\rm m*}(\boldsymbol{r})\}] \\
&\quad + \frac{(n_i^{\rm m} - n_j^{\rm m})}{2} \int {\rm d}\boldsymbol{r}\, {\rm d}\boldsymbol{r}'\, V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}')[2\phi_{0{\rm b}}(\boldsymbol{r})\,[\delta\phi_{\rm b}^*(\boldsymbol{r},t) + \delta\phi_{\rm b}(\boldsymbol{r},t)]\{u_j^{\rm m*}(\boldsymbol{r}')u_i^{\rm m}(\boldsymbol{r}') + v_j^{\rm m*}(\boldsymbol{r}')v_i^{\rm m}(\boldsymbol{r}')\}, \qquad (49)
\end{aligned}
$$

where $n_i^{\rm b}$ and $n_i^{\rm m}$ are the Bose–Einstein distribution functions. In order to solve the coupled system of Eqs. (46)–(48), we suppose that the condensate oscillates with frequency $\omega_{\rm b\,(m)}$

$$
\delta\phi_{\rm b\,(m)}(\boldsymbol{r},t) = \delta\phi_{1b\,(m)}(\boldsymbol{r})\,{\rm e}^{-{\rm i}\omega_{\rm b\,(m)}t}, \qquad (50)
$$

$$
\delta\phi_{\rm b\,(m)}^*(\boldsymbol{r},t) = \delta\phi_{2b\,(m)}(\boldsymbol{r})\,{\rm e}^{-{\rm i}\omega_{\rm b\,(m)}t}. \qquad (51)
$$

The Fourier transforms of Eqs. (48) and (49) are

$$
\begin{aligned}
f_{ij}^{\rm b}(\omega_{\rm b}) &= \frac{(n_i^{\rm b} - n_j^{\rm b})}{2[\omega_{\rm b} - (\varepsilon_j^{\rm b} - \varepsilon_i^{\rm b})]} \int {\rm d}\boldsymbol{r}\, {\rm d}\boldsymbol{r}'\, V_{\rm b}(\boldsymbol{r}-\boldsymbol{r}')[2\phi_{0{\rm b}}(\boldsymbol{r})\,[\delta\phi_{1b}(\boldsymbol{r}) + \delta\phi_{2b}(\boldsymbol{r})]\{u_j^{\rm b*}(\boldsymbol{r}')u_i^{\rm b}(\boldsymbol{r}') + v_j^{\rm b*}(\boldsymbol{r}')v_i^{\rm b}(\boldsymbol{r}')\} \\
&\quad + 2[\phi_{0{\rm b}}(\boldsymbol{r})\,\delta\phi_{1{\rm b}}(\boldsymbol{r}') + \phi_{0{\rm b}}(\boldsymbol{r}')\,\delta\phi_{2{\rm b}}(\boldsymbol{r})]\{u_j^{\rm b*}(\boldsymbol{r}')u_i^{\rm b}(\boldsymbol{r}) + v_j^{\rm b*}(\boldsymbol{r})v_i^{\rm b}(\boldsymbol{r}')\} \\
&\quad + [\phi_{0{\rm b}}(\boldsymbol{r})\,\delta\phi_{1{\rm b}}(\boldsymbol{r}') + \phi_{0{\rm b}}(\boldsymbol{r}')\,\delta\phi_{1{\rm b}}(\boldsymbol{r})]\{u_j^{\rm b*}(\boldsymbol{r})v_i^{\rm b}(\boldsymbol{r}') + u_j^{\rm b*}(\boldsymbol{r}')v_i^{\rm b}(\boldsymbol{r})\} \\
&\quad + [\phi_{0{\rm b}}(\boldsymbol{r})\,\delta\phi_{2{\rm b}}(\boldsymbol{r}') + \phi_{0{\rm b}}(\boldsymbol{r}')\,\delta\phi_{2{\rm b}}(\boldsymbol{r})]\{u_i^{\rm b}(\boldsymbol{r})v_j^{\rm b*}(\boldsymbol{r}') + u_i^{\rm b}(\boldsymbol{r}')v_j^{\rm b*}(\boldsymbol{r})\}] \\
&\quad + \frac{(n_i^{\rm b} - n_j^{\rm b})}{2[\omega_{\rm b} - (\varepsilon_j^{\rm b} - \varepsilon_i^{\rm b})]} \int {\rm d}\boldsymbol{r}\, {\rm d}\boldsymbol{r}'\, V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}')[2\phi_{0{\rm m}}(\boldsymbol{r})\,[\delta\phi_{1{\rm m}}(\boldsymbol{r}) + \delta\phi_{2{\rm m}}(\boldsymbol{r})]\{u_j^{\rm b*}(\boldsymbol{r}')u_i^{\rm b}(\boldsymbol{r}') + v_j^{\rm b*}(\boldsymbol{r}')v_i^{\rm b}(\boldsymbol{r}')\}, \qquad (52)
\end{aligned}
$$

$$
\begin{aligned}
f_{ij}^{\rm m}(\omega_{\rm m}) &= \frac{(n_i^{\rm m} - n_j^{\rm m})}{2[\omega_{\rm m} - (\varepsilon_j^{\rm m} - \varepsilon_i^{\rm m})]} \int {\rm d}\boldsymbol{r}\, {\rm d}\boldsymbol{r}'\, V_{\rm m}(\boldsymbol{r}-\boldsymbol{r}')[2\phi_{0{\rm m}}(\boldsymbol{r})\,[\delta\phi_{1{\rm m}}(\boldsymbol{r}) + \delta\phi_{2{\rm m}}(\boldsymbol{r})]\{u_j^{\rm m*}(\boldsymbol{r}')u_i^{\rm m}(\boldsymbol{r}') + v_j^{\rm m*}(\boldsymbol{r}')v_i^{\rm m}(\boldsymbol{r}')\} \\
&\quad + 2[\phi_{0{\rm m}}(\boldsymbol{r})\,\delta\phi_{1{\rm m}}(\boldsymbol{r}') + \phi_{0{\rm m}}(\boldsymbol{r}')\,\delta\phi_{2{\rm m}}(\boldsymbol{r})]\{u_j^{\rm m*}(\boldsymbol{r}')u_i^{\rm m}(\boldsymbol{r}) + v_j^{\rm m*}(\boldsymbol{r})v_i^{\rm m}(\boldsymbol{r}')\} \\
&\quad + [\phi_{0{\rm m}}(\boldsymbol{r})\,\delta\phi_{1{\rm m}}(\boldsymbol{r}') + \phi_{0{\rm m}}(\boldsymbol{r}')\,\delta\phi_{1{\rm m}}(\boldsymbol{r})]\{u_j^{\rm m*}(\boldsymbol{r})v_i^{\rm m}(\boldsymbol{r}') + u_j^{\rm m*}(\boldsymbol{r}')v_i^{\rm m}(\boldsymbol{r})\} \\
&\quad + [\phi_{0{\rm m}}(\boldsymbol{r})\,\delta\phi_{2{\rm m}}(\boldsymbol{r}') + \phi_{0{\rm m}}(\boldsymbol{r}')\,\delta\phi_{2{\rm m}}(\boldsymbol{r})]\{u_i^{\rm m}(\boldsymbol{r})v_j^{\rm m*}(\boldsymbol{r}') + u_i^{\rm m}(\boldsymbol{r}')v_j^{\rm m*}(\boldsymbol{r})\}] \\
&\quad + \frac{(n_i^{\rm m} - n_j^{\rm m})}{2[\omega_{\rm m} - (\varepsilon_j^{\rm m} - \varepsilon_i^{\rm m})]} \int {\rm d}\boldsymbol{r}\, {\rm d}\boldsymbol{r}'\, V_{\rm bm}(\boldsymbol{r}-\boldsymbol{r}')[2\phi_{0{\rm b}}(\boldsymbol{r})\,[\delta\phi_{1{\rm b}}(\boldsymbol{r}) + \delta\phi_{2{\rm b}}(\boldsymbol{r})]\{u_j^{\rm m*}(\boldsymbol{r}')u_i^{\rm m}(\boldsymbol{r}') + v_j^{\rm m*}(\boldsymbol{r}')v_i^{\rm m}(\boldsymbol{r}')\}. \qquad (53)
\end{aligned}
$$

By disregarding the coupling terms between the condensate and noncondensate atoms and molecules (bosons) in Eq. (46) (Eq. (47)), the Bogoliubov equation for the fluctuation of the condensate with real frequency ω_0 is

$$
\omega_{0{\rm b\,(m)}} \begin{pmatrix} \delta\phi_{1{\rm b\,(m)}}^0(\boldsymbol{r}) \\ \delta\phi_{2{\rm b\,(m)}}^0(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} \hat{\mathscr{L}}_{\rm b\,(m)} & \mathscr{M}_{\rm b\,(m)} \\ -\mathscr{M}_{\rm b\,(m)}^* & -\hat{\mathscr{L}}_{\rm b\,(m)}^* \end{pmatrix} \begin{pmatrix} \delta\phi_{1{\rm b\,(m)}}^0(\boldsymbol{r}) \\ \delta\phi_{2{\rm b\,(m)}}^0(\boldsymbol{r}) \end{pmatrix} \qquad (54)
$$

with the normalization condition

$$\int \mathrm{d}\boldsymbol{r}\left(\left|\delta\phi^0_{1\mathrm{b}(\mathrm{m})}(\boldsymbol{r})\right|^2-\left|\delta\phi^0_{2\mathrm{b}(\mathrm{m})}(\boldsymbol{r})\right|^2\right)=1. \tag{55}$$

For small non-condensate density, the coupling terms may be considered as small perturbations and we can write

$$\delta\phi_{1\mathrm{b}(\mathrm{m})}=\delta\phi^0_{1\mathrm{b}(\mathrm{m})}+\delta\phi'_{1\mathrm{b}(\mathrm{m})}, \tag{56}$$

$$\delta\phi_{2\mathrm{b}(\mathrm{m})}=\delta\phi^0_{2\mathrm{b}(\mathrm{m})}+\delta\phi'_{2\mathrm{b}(\mathrm{m})}, \tag{57}$$

where the oscillation amplitude and the unperturbative oscillation amplitude are corrected by the following orthogonal relation

$$\int \mathrm{d}\boldsymbol{r}\left(\delta\phi^0_{1\mathrm{b}(\mathrm{m})}(\boldsymbol{r})\delta\phi'^*_{1\mathrm{b}(\mathrm{m})}(\boldsymbol{r})-\delta\phi^0_{2\mathrm{b}(\mathrm{m})}(\boldsymbol{r})\delta\phi'^*_{2\mathrm{b}(\mathrm{m})}(\boldsymbol{r})\right)=1. \tag{58}$$

Furthermore, the perturbed eigenfrequency can be written as

$$\omega_{\mathrm{b}(\mathrm{m})}=\omega_{0\mathrm{b}(\mathrm{m})}+\delta\omega_{\mathrm{b}(\mathrm{m})}+\mathrm{i}\gamma_{\mathrm{b}(\mathrm{m})}, \tag{59}$$

where $\delta\omega_{\mathrm{b}(\mathrm{m})}$ is the correction to the real part of the normal mode eigenfrequency and $\gamma_{\mathrm{b}(\mathrm{m})}$ denotes the damping coefficient. Owing to the coupling existing between the condensed and noncondensed atoms, the lowest-order condensate fluctuations for bosons [Eq. (46)] and molecules [Eq. (47)] respectively read

$$\omega_{\mathrm{b}}\delta\phi_{1\mathrm{b}}(\boldsymbol{r})=\hat{\mathscr{L}}_{\mathrm{b}}[\delta\phi^0_{1\mathrm{b}}(\boldsymbol{r})]+\hat{\mathscr{M}}_{\mathrm{b}}[\delta\phi^0_{2\mathrm{b}}(\boldsymbol{r})]+\sum_{ij}f^{\mathrm{b}}_{ij}(\omega_{0\mathrm{b}})A_{ij}(\boldsymbol{r})+\sum_{ij}f^{\mathrm{m}}_{ij}(\omega_{0\mathrm{b}})C_{ij}(\boldsymbol{r}), \tag{60}$$

$$\omega_{\mathrm{b}}\delta\phi_{2\mathrm{b}}(\boldsymbol{r})=-\hat{\mathscr{L}}^*_{\mathrm{b}}[\delta\phi^0_{2\mathrm{b}}(\boldsymbol{r})]-\hat{\mathscr{M}}^*_{\mathrm{b}}[\delta\phi^0_{1\mathrm{b}}(\boldsymbol{r})]-\sum_{ij}f^{\mathrm{b}}_{ji}(\omega_{0\mathrm{b}})A^*_{ij}(\boldsymbol{r})-\sum_{ij}f^{\mathrm{m}}_{ji}(\omega_{0\mathrm{b}})C^*_{ij}(\boldsymbol{r}), \tag{61}$$

$$\omega_{\mathrm{m}}\delta\phi_{1\mathrm{m}}(\boldsymbol{r})=\hat{\mathscr{L}}_{\mathrm{m}}[\delta\phi^0_{1\mathrm{m}}(\boldsymbol{r})]+\hat{\mathscr{M}}_{\mathrm{m}}[\delta\phi^0_{2\mathrm{m}}(\boldsymbol{r})]+\sum_{ij}f^{\mathrm{m}}_{ij}(\omega_{0\mathrm{m}})B_{ij}(\boldsymbol{r})+\sum_{ij}f^{\mathrm{b}}_{ij}(\omega_{0\mathrm{m}})D_{ij}(\boldsymbol{r}), \tag{62}$$

$$\omega_{\mathrm{m}}\delta\phi_{2\mathrm{m}}(\boldsymbol{r})=-\hat{\mathscr{L}}^*_{\mathrm{m}}[\delta\phi^0_{2\mathrm{m}}(\boldsymbol{r})]-\hat{\mathscr{M}}^*_{\mathrm{m}}[\delta\phi^0_{1\mathrm{m}}(\boldsymbol{r})]-\sum_{ij}f^{\mathrm{m}}_{ji}(\omega_{0\mathrm{m}})B^*_{ij}(\boldsymbol{r})-\sum_{ij}f^{\mathrm{b}}_{ji}(\omega_{0\mathrm{m}})D^*_{ij}(\boldsymbol{r}), \tag{63}$$

where we have replaced all the $\delta\phi_{\mathrm{b}(\mathrm{m})}$ and $\delta\phi^*_{\mathrm{b}(\mathrm{m})}$ terms with $\delta\phi^0_{1\mathrm{b}(\mathrm{m})}$ and $\delta\phi^0_{2\mathrm{b}(\mathrm{m})}$, and $\omega^{b(m)}$ with $\omega_0^{\mathrm{b}(\mathrm{m})}$ on the right-hand side of Eq. (60) (Eq. (61)). We also introduce the shorthand

$$\begin{aligned}A_{ij}(\boldsymbol{r})=&\int \mathrm{d}\boldsymbol{r}'V_{\mathrm{b}}(\boldsymbol{r}-\boldsymbol{r}')\phi_{0\mathrm{b}}(\boldsymbol{r}')[u^{\mathrm{b}*}_i(\boldsymbol{r}')u^{\mathrm{b}}_j(\boldsymbol{r})+v^{\mathrm{b}}_j(\boldsymbol{r}')v^{\mathrm{b}*}_i(\boldsymbol{r})+v^{\mathrm{b}*}_i(\boldsymbol{r})u^{\mathrm{b}}_j(\boldsymbol{r}')+v^{\mathrm{b}*}_i(\boldsymbol{r}')u^{\mathrm{b}}_j(\boldsymbol{r})]\\&+\phi_{0\mathrm{b}}(\boldsymbol{r})[u^{\mathrm{b}*}_i(\boldsymbol{r}')u^{\mathrm{b}}_j(\boldsymbol{r}')+v^{\mathrm{b}}_i(\boldsymbol{r}')v^{\mathrm{b}*}_j(\boldsymbol{r}')],\end{aligned} \tag{64}$$

$$\begin{aligned}B_{ij}(\boldsymbol{r})=&\int \mathrm{d}\boldsymbol{r}'V_{\mathrm{m}}(\boldsymbol{r}-\boldsymbol{r}')\phi_{0\mathrm{m}}(\boldsymbol{r}')[u^{\mathrm{m}*}_i(\boldsymbol{r}')u^{\mathrm{m}}_j(\boldsymbol{r})+v^{\mathrm{m}}_j(\boldsymbol{r}')v^{\mathrm{m}*}_i(\boldsymbol{r})+v^{\mathrm{m}*}_i(\boldsymbol{r})u^{\mathrm{m}}_j(\boldsymbol{r}')+v^{\mathrm{m}*}_i(\boldsymbol{r}')u^{\mathrm{m}}_j(\boldsymbol{r})]\\&+\phi_{0\mathrm{m}}(\boldsymbol{r})[u^{\mathrm{m}*}_i(\boldsymbol{r}')u^{\mathrm{m}}_j(\boldsymbol{r}')+v^{\mathrm{m}}_i(\boldsymbol{r}')v^{\mathrm{m}*}_j(\boldsymbol{r}')],\end{aligned} \tag{65}$$

$$C_{ij}(\boldsymbol{r})=\int \mathrm{d}\boldsymbol{r}'\,V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')\phi_{0\mathrm{b}}(\boldsymbol{r})[u^{\mathrm{m}*}_i(\boldsymbol{r}')u^{\mathrm{m}}_j(\boldsymbol{r}')+v^{\mathrm{m}}_i(\boldsymbol{r}')v^{\mathrm{m}*}_j(\boldsymbol{r}')], \tag{66}$$

$$D_{ij}(\boldsymbol{r})=\int \mathrm{d}\boldsymbol{r}'\,V_{\mathrm{bm}}(\boldsymbol{r}-\boldsymbol{r}')\phi_{0\mathrm{m}}(\boldsymbol{r})[u^{\mathrm{b}*}_i(\boldsymbol{r}')u^{\mathrm{b}}_j(\boldsymbol{r}')+v^{\mathrm{b}}_i(\boldsymbol{r}')v^{\mathrm{b}*}_j(\boldsymbol{r}')]. \tag{67}$$

Multiplying Eqs. (60) and (61) ((62) and (63)) respectively by $\delta\phi^*_{1\mathrm{b}}$ ($\delta\phi^*_{1\mathrm{m}}$) and $\delta\phi^*_{2\mathrm{b}}$ ($\delta\phi^*_{2\mathrm{m}}$) and subtracting the two equations after integrating over space we find

$$\omega_{\mathrm{b}}=\omega_{0\mathrm{b}}+\sum_{ij}f^{\mathrm{b}}_{ij}(\omega_{0\mathrm{b}})\int \mathrm{d}\boldsymbol{r}\,[A_{ij}(\boldsymbol{r})\delta\phi^*_{1\mathrm{b}}+A^*_{ji}(\boldsymbol{r})\delta\phi^*_{2\mathrm{b}}]+\sum_{ij}f^{\mathrm{m}}_{ij}(\omega_{0\mathrm{b}})\int \mathrm{d}\boldsymbol{r}\,[C_{ij}(\boldsymbol{r})\delta\phi^*_{1\mathrm{b}}+C^*_{ji}(\boldsymbol{r})\delta\phi^*_{2\mathrm{b}}], \tag{68}$$

$$\omega_{\mathrm{m}}=\omega_{0\mathrm{m}}+\sum_{ij}f^{\mathrm{m}}_{ij}(\omega_{0\mathrm{m}})\int \mathrm{d}\boldsymbol{r}\,[B_{ij}(\boldsymbol{r})\delta\phi^*_{1\mathrm{m}}+B^*_{ji}(\boldsymbol{r})\delta\phi^*_{2\mathrm{m}}]+\sum_{ij}f^{\mathrm{b}}_{ij}(\omega_{0\mathrm{m}})\int \mathrm{d}\boldsymbol{r}\,[D_{ij}(\boldsymbol{r})\delta\phi^*_{1\mathrm{m}}+D^*_{ji}(\boldsymbol{r})\delta\phi^*_{2\mathrm{m}}]. \tag{69}$$

By defining the matrix elements as

$$\mathscr{A}_{ij}=\int \mathrm{d}\boldsymbol{r}\,[A_{ij}(\boldsymbol{r})\delta\phi^*_{1\mathrm{b}}+A^*_{ji}(\boldsymbol{r})\delta\phi^*_{2\mathrm{b}}], \tag{70}$$

$$\mathscr{B}_{ij}=\int \mathrm{d}\boldsymbol{r}\,[B_{ij}(\boldsymbol{r})\delta\phi^*_{1\mathrm{m}}+B^*_{ji}(\boldsymbol{r})\delta\phi^*_{2\mathrm{m}}], \tag{71}$$

$$\mathscr{C}_{ij}=\int \mathrm{d}\boldsymbol{r}\,[C_{ij}(\boldsymbol{r})\delta\phi^*_{1\mathrm{b}}+C^*_{ji}(\boldsymbol{r})\delta\phi^*_{2\mathrm{b}}], \tag{72}$$

$$\mathscr{D}_{ij}=\int \mathrm{d}\boldsymbol{r}\,[D_{ij}(\boldsymbol{r})\delta\phi^*_{1\mathrm{m}}+D^*_{ji}(\boldsymbol{r})\delta\phi^*_{2\mathrm{m}}]. \tag{73}$$

Equations (52), (53), (68), and (69) can be written respectively as follows:

$$f^{\mathrm{b}}_{ij}(\omega)=\frac{(n^{\mathrm{b}}_i-n^{\mathrm{b}}_j)}{[\omega-(\varepsilon^{\mathrm{b}}_j-\varepsilon^{\mathrm{b}}_i)]}[\mathscr{A}^*_{ij}+\mathscr{D}^*_{ij}], \tag{74}$$

$$f^{\mathrm{m}}_{ij}(\omega)=\frac{(n^{\mathrm{m}}_i-n^{\mathrm{m}}_j)}{[\omega-(\varepsilon^{\mathrm{m}}_j-\varepsilon^{\mathrm{m}}_i)]}[\mathscr{B}^*_{ij}+\mathscr{C}^*_{ij}], \tag{75}$$

$$\omega_{\mathrm{b}}=\omega_{0\mathrm{b}}+\sum_{ij}f^{\mathrm{b}}_{ij}(\omega_{0\mathrm{b}})\mathscr{A}_{ij}+\sum_{ij}f^{\mathrm{m}}_{ij}(\omega_{0\mathrm{b}})\mathscr{C}_{ij}, \tag{76}$$

$$\omega_{\mathrm{m}}=\omega_{0\mathrm{m}}+\sum_{ij}f^{\mathrm{m}}_{ij}(\omega_{0\mathrm{m}})\mathscr{B}_{ij}+\sum_{ij}f^{\mathrm{b}}_{ij}(\omega_{0\mathrm{m}})\mathscr{D}_{ij}. \tag{77}$$

The damping rate is given by the imaginary part of ω. Landau damping comes from the process that a quasiparticle with the frequency ω_0 is absorbed by a thermal excitation E_i and the thermal excitation jumping to the energy $E_j = E_i + \omega_0$. This thermal process can be written as

$$\gamma_{\mathrm{Lb}}(\omega_{0\mathrm{b}}) = \pi \sum_{ij} (n_i^{\mathrm{b}} - n_j^{\mathrm{b}})\, \delta(\omega_{0\mathrm{b}} - (E_j^{\mathrm{b}} - E_i^{\mathrm{b}}))\left|\mathscr{A}_{ij}\right|^2, \tag{78}$$

$$\gamma_{\mathrm{Lm}}(\omega_{0\mathrm{m}}) = \pi \sum_{ij} (n_i^{\mathrm{m}} - n_j^{\mathrm{m}})\, \delta(\omega_{0\mathrm{m}} - (E_j^{\mathrm{m}} - E_i^{\mathrm{m}}))\left|\mathscr{B}_{ij}\right|^2, \tag{79}$$

$$\gamma_{\mathrm{Lbm}}(\omega_{0\mathrm{b}}) = \pi \sum_{ij} (n_i^{\mathrm{m}} - n_j^{\mathrm{m}})\, \delta(\omega_{0\mathrm{b}} - (E_j^{\mathrm{m}} - E_i^{\mathrm{m}}))\left|\mathscr{C}_{ij}\right|^2, \tag{80}$$

$$\gamma_{\mathrm{Lbm}}(\omega_{0\mathrm{m}}) = \pi \sum_{ij} (n_i^{\mathrm{b}} - n_j^{\mathrm{b}})\, \delta(\omega_{0\mathrm{m}} - (E_j^{\mathrm{b}} - E_i^{\mathrm{b}}))\left|\mathscr{D}_{ij}\right|^2. \tag{81}$$

Equations (78) and (79) are the Landau damping due to the boson–boson and molecule–molecule interactions respectively. In this mechanism of damping, a thermally excited boson (molecule) is scattered to the another one by absorbing a phonon which coincides with Eq. (26) of Ref. [25]. Equations (80) and (81) are known as Landau damping due to the boson–molecule coupling and in the absence of dipole–dipole interaction, which coincides with the finding of Ref. [10].

3. Landau damping in a homogeneous 2D dipolar Bose–Fermi mixture

In this section we focus on Landau damping due to the boson–molecule interactions in a homogeneous quasi-2D mixture of a dipolar Bose and superfluid Fermi gas. Quasi-2D gas is fabricated experimentally by a tight confinement along the axial (z) direction, the homogeneous quasi-2D gas is created when the trap potential in the x–y plane is eliminated. Because Bose gas is very dilute in the experiment, Landau damping due to the interaction between bosons can be disregarded in this case.

For a homogeneous system the condensate wave function at equilibrium is constant $\phi_{0\mathrm{b(m)}}(r) = \sqrt{n_{0\mathrm{b(m)}}}$, while the excitation and the fluctuation are plane wave functions

$$\delta\phi_{1\mathrm{b(m)}}(\boldsymbol{r}) = \frac{u_q^{\mathrm{b(m)}}}{\sqrt{V}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}, \quad \delta\phi_{2\mathrm{b(m)}}(\boldsymbol{r}) = \frac{v_q^{\mathrm{b(m)}}}{\sqrt{V}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}},$$
$$u_{\boldsymbol{p}}^{\mathrm{b(m)}}(\boldsymbol{r}) = \frac{u_p^{\mathrm{b(m)}}}{\sqrt{V}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}, \quad v_{\boldsymbol{p}}^{\mathrm{b(m)}}(\boldsymbol{r}) = \frac{v_p^{\mathrm{b(m)}}}{\sqrt{V}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}. \tag{82}$$

The coefficients $u_q^{\mathrm{b(m)}}$ and $v_q^{\mathrm{b(m)}}$ satisfy the usual Bogoliubov relations:

$$u_q^{\mathrm{b(m)}} = \sqrt{\frac{\varepsilon_q^{\mathrm{b(m)}} + n_0^{\mathrm{b(m)}} V(q)}{2E_q^{\mathrm{b(m)}}} + \frac{1}{2}}, \tag{83}$$

$$v_q^{\mathrm{b(m)}} = -\sqrt{\frac{\varepsilon_q^{\mathrm{b(m)}} + n_0^{\mathrm{b(m)}} V(q)}{2E_q^{\mathrm{b(m)}}} - \frac{1}{2}}, \tag{84}$$

where energy of the elementary excitation $E_q^{\mathrm{b(m)}}$ is

$$E_q^{\mathrm{b(m)}} = \sqrt{\varepsilon_q^{\mathrm{b(m)}2} + 2\varepsilon_q^{\mathrm{b(m)}m} n_0^{\mathrm{b(m)}} V_{\mathrm{b(m)}}(\boldsymbol{q})} \tag{85}$$

with $\varepsilon_q^{\mathrm{b(m)}} = \hbar^2 q^2 / 2m_{\mathrm{b(m)}}$ ($m_{\mathrm{m}} = 2m_{\mathrm{f}}$). The 2D Fourier transform of interaction potentials (5)–(7) becomes

$$V_{\mathrm{b(m)}}^{\mathrm{2D}}(k) = g_{\mathrm{b(m)}}^{\mathrm{2D}} + g_{\mathrm{db(m)}}^{\mathrm{2D}} F\left(\frac{k l_z}{\sqrt{2}}\right), \tag{86}$$

where $g_{\mathrm{b(m)}}^{\mathrm{2D}}$ is the contact interaction strength in two dimensions and l_z is a length scale on the order of the harmonic oscillator wavelength in the axial direction. It is not simply relevant to the s-wave scattering length, but will be treated as a parameter. The $g_{\mathrm{db(m)}}^{\mathrm{2D}}$ is the dipole–dipole interaction strength in quasi-2D and

$$F(x) = 1 - \frac{3}{2}\sqrt{\pi} x \operatorname{erfc}(x) \mathrm{e}^{x^2}. \tag{87}$$

By inserting Eqs. (82) and (86) into the right-hand side of Eq. (80) and integrating over space, the Landau damping rate due to the interaction between bosons and molecules can be written as

$$\gamma_{\mathrm{Lbm}}(\boldsymbol{q}) = -\pi n_{0\mathrm{b}} \sum_{\boldsymbol{p}\boldsymbol{p}'} (n_{p'}^{\mathrm{m}} - n_p^{\mathrm{m}}) \times \delta(\omega_q^{\mathrm{b}} - E_{p'}^{\mathrm{m}} + E_p^{\mathrm{m}}) \mathscr{C}_{\boldsymbol{p}\boldsymbol{p}'}^{\boldsymbol{q}} \mathscr{C}_{\boldsymbol{p}'\boldsymbol{p}}^{\boldsymbol{q}}, \tag{88}$$

where the matrix element $\mathscr{C}_{\boldsymbol{p}\boldsymbol{p}'}^{\boldsymbol{q}}$ reads

$$\mathscr{C}_{\boldsymbol{p}\boldsymbol{p}'}^{\boldsymbol{q}} = \delta_{\boldsymbol{q},\boldsymbol{p}'-\boldsymbol{p}} \{(u_q^{\mathrm{b}} + v_q^{\mathrm{b}}) V_{\mathrm{bm}}(q)(u_p^{\mathrm{m}} u_{p'}^{\mathrm{m}} + v_p^{\mathrm{m}} v_{p'}^{\mathrm{m}})\}. \tag{89}$$

For a dilute Bose gas, where the healing length of the bosons is much larger than their wavelength, the spectrum of the elementary excitations can be considered as a free-particle energy with quadratic dispersion, i.e., $E_q^b = \varepsilon_q^{\mathrm{b}}$ and u_q^{b} and v_q^{b} can be simplified into $u_q^{\mathrm{b}} = 1$ and $v_q^{\mathrm{b}} = 0$. For small values of $\boldsymbol{q}$, Landau damping (88) can be written in a dimensionless form as (we assume $m_{\mathrm{b}} = m_{\mathrm{f}} = m_{\mathrm{m}}/2$),

$$\frac{\hbar\gamma(y)}{\varepsilon_{\mathrm{F}}} = \frac{k_{\mathrm{F}}^2}{\pi n_0^{\mathrm{b}}} \frac{y V_{\mathrm{bm}}^2(y)}{\tau} \int_{x_{\mathrm{c}}}^{\infty} \frac{x\,\mathrm{d}x}{v_{\mathrm{gm}}(x)} \frac{\left(\mathrm{e}^{E^{\mathrm{m}}(x)/2\tau} - \mathrm{e}^{-E^{\mathrm{m}}(x)/2\tau}\right)^{-2}}{\sqrt{1 - (y/v_{\mathrm{gm}}(x))^2}} \left[\frac{x^2 + V(x)}{E^{\mathrm{m}}(x)} + \frac{y^2}{2v_{\mathrm{gm}}(x)} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2 + V(x)}{E^{\mathrm{m}}(x)}\right)\right]^2, \tag{90}$$

where $\tau = k_{\rm b}T/\varepsilon_{\rm F}$ is a reduced temperature, and $y = q/k_{\rm F}$. Conservation of energy in Eq. (88) $\omega_q^{\rm b} - E_{p'}^{\rm m} + E_p^{\rm m}$ gives a threshold for the Landau damping that the thermal excitations with $x \geq x_{\rm c}$ can participate in the damping,

$$x_{\rm c} = \frac{y}{2}\left(1 - \frac{y}{\tilde{c}_{\rm m}}\right), \tag{91}$$

where

$$\tilde{c}_{\rm m} = \sqrt{n_0^{\rm m}(g_{\rm m}^{\rm 2D} + g_{\rm dm}^{\rm 2D})/\varepsilon_{\rm F}}$$

is the dimensionless velocity of the sound wave in the molecular condensate. Equation (90) can be written as

$$\frac{\hbar\gamma(y)}{\varepsilon_{\rm F}} = \frac{k_{\rm F}^2}{\pi n_0^{\rm b}} I(\tau), \tag{92}$$

where $I(\tau)$ is the dimensionless function.

In Fig. 1, the temperature dependences of $I(\tau)$ is plotted for different values of $g_{\rm dm}^{\rm 2D}/g_{\rm m}^{\rm 2D}$ ratio. At a given temperature, the damping rate decreases by increasing the dipolar interaction. In fact, by increasing the strength of dipolar interaction, $x_{\rm c}$ increases. Then, the available thermal excitations involved in the Landau damping process decrease. For low temperatures $k_{\rm b}T < E_{\rm c}$ and large dipole–dipole interaction, the collective oscillations are undamped. The damping rate varies linearly with T at high temperatures.

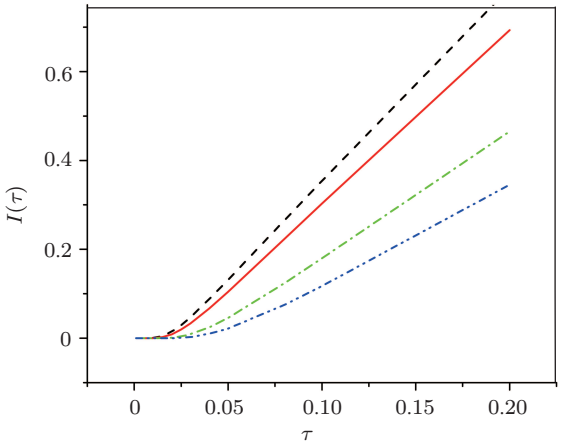


Fig. 1. (color online) Temperature dependences of the $I(\tau)$ versus $\tau = k_{\rm b}T/\varepsilon_{\rm F}$ for fixed $q/k_{\rm F} = 0.1$. The contact interaction is fixed in all the curves and the dimensionless ratio $g_{\rm dm}^{\rm 2D}/g_{\rm m}^{\rm 2D}$ is varied, 0.1 (black-dashed), 0.5 (red-solid), 2.5 (green-dot–dashed), and 5.0 (blue-dot–dot–dashed).

In Fig. 2, we plot the spectra of the elementary excitation of molecules corresponding to different ratios of $g_{\rm dm}^{\rm 2D}/g_{\rm m}^{\rm 2D}$. As we can see in Fig. 2, for sufficiently strong dipole–dipole interactions, the spectrum of energy shows a roton–maxon behavior at intermediate momentum.

The temperature dependences of the $I(\tau)$ for different values of $y = q/k_{\rm F}$ are plotted in Fig. 3. As we can see in Fig. 3, by increasing the energy of boson quasiparticles, Landau damping is suppressed over a broader temperature range. This is because the energy lower bound of Landau damping increases by increasing the energy of the boson quasiparticles according to Eq. (90). In a Bose gas with purely contact interaction, the energy-momentum conservation can always be satisfied for Bogoliubov dispersion. Therefore there is no energy lower bound nor exponential suppression at low temperatures. By contrast, in a dipolar Bose gas, there is a dramatic exponential suppression of Landau damping at low temperatures due to the energy lower bound.[25]

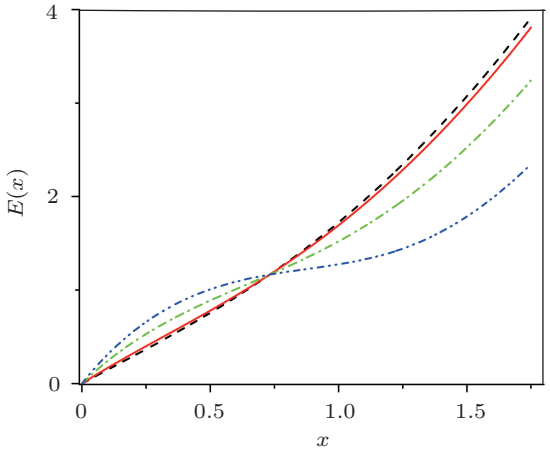


Fig. 2. (color online) Variations of energy with x for different values of ratio $g_{\rm dm}^{\rm 2D}/g_{\rm m}^{\rm 2D}$: 0.1 (black-dash), 0.5 (red-solid), 2.5 (green-dot–dash), and 5.0 (blue-dot–dot–dash).

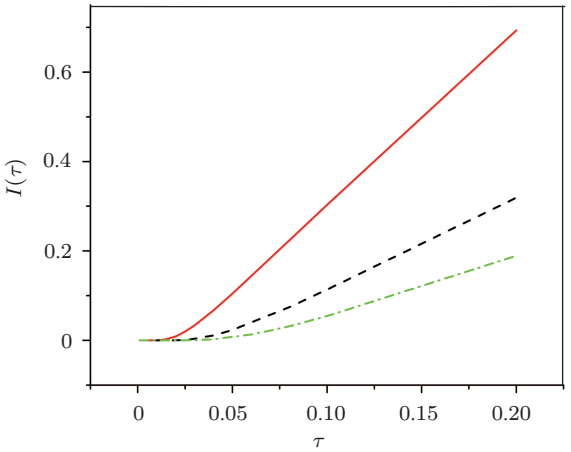


Fig. 3. (color online) Variations of integral $I(\tau)$ with $\tau = k_{\rm b}T/\varepsilon_{\rm F}$ for fixed $g_{\rm dm}^{\rm 2D}/g_{\rm m}^{\rm 2D} = 0.5$ and different values of $q/k_{\rm F}$: 0.1 (red-solid), 0.2 (black-dash), and 0.3 (green-dot–dash).

We should confirm that our calculations of temperature dependence in Landau damping are only reliable at low temperatures, because the condensate fraction in the Bose superfluid will decrease with the increase of temperature. For a general analysis of temperature dependence, the changing of the condensate fraction should be included in calculations. Anyway, at low temperatures the condensate fraction is not sensitive to the temperature and our result is reliable in this limit.

4. Conclusions

In this paper, damping of low-energy excitations in a Bose–Fermi superfluid mixture with dipole–dipole interac-

tions is studied at finite temperature in a collisionless regime. By using the time-dependent mean-field approach based on the Popov approximation, a set of coupled equations for the condensate and noncondensate components of bosons and molecules is obtained. An explicit expression for damping rate is derived, which in the absence of the boson–molecule interaction coincides with the finding of Ref. [25] and in the absence of the dipole–dipole interaction coincides with the finding of Ref. [10]. It is shown that the temperature dependence of Landau damping due to the boson–molecule interaction in a homogeneous 2D dipolar Bose–Fermi superfluid is linear at high temperatures and at low temperatures, exponentially suppressed. The temperature dependences of the Landau damping for different values of q/k_{F} and $g_{\mathrm{dm}}^{2\mathrm{D}}/g_{\mathrm{m}}^{2\mathrm{D}}$ are also calculated. At a given temperature, the damping rate decreases by increasing these two ratios. For a sufficiently large dipole–dipole interaction, the spectrum of energy has a roton–maxon characteristic. The conservation of energy $\omega_q^b - E_{p'}^{\mathrm{m}} + E_p^{\mathrm{m}}$ in the damping mechanism gives an energy lower bound. By increasing the strength of dipolar interaction ($g_{\mathrm{dm}}^{2\mathrm{D}}/g_{\mathrm{m}}^{2\mathrm{D}}$) and the energy of boson quasiparticles (q/k_{F}), energy lower bound increases. Then, the available thermal excitations involved in the Landau damping process decrease and Landau damping is exponentially suppressed.

In future it will be possible to realize a mixture of Bose and Fermi superfluids with dipolar atoms and our result should be observed in the experiment.

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