Stability analysis of a simple rheonomic nonholonomic constrained system*

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It is a difficult problem to study the stability of the rheonomic and nonholonomic mechanical systems. Especially it is difficult to construct the Lyapunov function directly from the differential equation. But the gradient system is exactly suitable to study the stability of a dynamical system with the aid of the Lyapunov function. The stability of the solution for a simple rheonomic nonholonomic constrained system is studied in this paper. Firstly, the differential equations of motion of the system are established. Secondly, a problem in which the generalized forces are exerted on the system such that the solution is stable is proposed. Finally, the stable solutions of the rheonomic nonholonomic system can be constructed by using the gradient systems.

Keywords: nonholonomic constrained system, stabillity, gradient system, Lyapunov function

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1. Introduction

It is an important and difficult problem to study the stability for a nonholonomic constrained system.[1–17] It is also difficult to construct the Lyapunov function directly from the differential equations of the dynamical system. The authors in Ref. [18] pointed out that an ordinary differential equation can be written in a linear-gradient system when its one or more first integrations or Lyapunov function is given. In the past few years, some research advances have been made in the stability analysis for the nonholonomic constrained system, such as those given in Refs. [1]–[8]. But unfortunately most of the researches are limited to scleronomic nonholonomic constrained systems, because the stability analysis of the rheonomic nonholonomic constrained system is especially difficult and there has existed no general theory. In this paper the problem of stability analysis for a simple rheonomic nonholonomic constrained system is studied. The configuration of the simple dynamical system is determined by two generalized coordinates, which is subjected to a rheonomic unintegrable differential constraint in the sense of Frobenius theorem. The problem is what kind of generalized force should be used to be exerted on the system in order to make the system stable or aymptotically stable. In order to solve the problem, several kinds of gradient systems are utilized to construct the generalized force for the simple rheonomic nonholonomic constrained system. Recently, the important achievements for the relation between gradient system and constrained mechanical system have been made, such as those given in Refs. [19]–[22].

2. Problem of stability for a simple rheonomic nonholonomic constrained system

The configuration of a dynamical system is determined by two generalized coordinates, whose motion is subjected to a rheonomic nonholonomic differential constraint equation

$$f = \dot{q}_2 - t\dot{q}_1 = 0. \tag{1}$$

The kinetic energy of the system is

$$T = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right), \tag{2}$$

which is subjected to generalized forces Q_1 and Q_2. The question is what kinds of generalized forces Q_1 and Q_2 should be used to be exerted on the system in order to make the solution $q_1 = \dot{q}_1 = 0$ stable or asymptotically stable. In order to solve the problem, the differential equations of motion for the dynamical system should be constructed first. The differential equations of motion with a Lagrange multiplier area follows:

$$\ddot{q}_1 = Q_1 - \lambda t, \tag{3a}$$

$$\ddot{q}_2 = Q_2 + \lambda, \tag{3b}$$

where λ is a Lagrange multiplier. Taking the derivation of Eq. (1) with respect to time, we obtain the following form:

$$\ddot{q}_2 = t\ddot{q}_1 + \dot{q}_1. \tag{4}$$

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Substituting Eq. (3) into Eq. (4), we have

$$\lambda = \frac{tQ_1 + \dot{q}_1}{1+t^2} - Q_2. \tag{5}$$

Suppose $Q_2 = 0$, then

$$\lambda = \frac{tQ_1 + \dot{q}_1}{1+t^2}. \tag{6}$$

Substituting Eq. (6) into Eq. (3a), the rheonomic nonholonomic system can be transformed into the holonomic nonconservative system:

$$\ddot{q}_1 = \frac{Q_1}{1+t^2} - \frac{t\dot{q}_1}{1+t^2} = F\left(t, q_1, \dot{q}_1\right). \tag{7}$$

If the suitable function F can be obtained and the generalized force Q_1 can be determined according to Eq. (7), we can obtain zero solution $q_1 = \dot{q}_1 = 0$ that is stable or asymptotically stable. Obviously, many of the functions F which can be used to let zero solution $q_1 = \dot{q}_1 = 0$ be stable or asymptotically stable, can be selected, for example, $F = -6\dot{q}_1 - 6q_1 + q_1^3$ makes zero solution $q_1 = \dot{q}_1 = 0$ asymptotically stable.

3. Constructing the generalized force by utilizing gradient system

The gradient system is suitable for discussing the stability of a dynamical system by utilizing the Lyapunov function. In this part, gradient system, skew-gradient system, gradient system with a symmetric negative matrix, generalized skew-gradient system and generalized gradient system with a symmetric negative matrix are used to construct the function F in Eq. (7) and furthermore to construct the generalized force.

3.1. Gradient system

The differential equations of a gradient system have the form

$$\dot{a}^\mu = -\frac{\partial V}{\partial a^\mu}, \quad (\mu = 1,2), \tag{8}$$

where V is a Lyapunov function which can let zero solution $a^1 = \dot{a}^1 = 0$ be stable. If letting

$$V = \left(a^1\right)^2 + \left(a^2\right)^2 - a^1 a^2 + \frac{1}{3}\left(a^1\right)^3, \tag{9}$$

we have

$$\dot{a}^1 = -2a^1 + a^2 - \left(a^1\right)^2,$$
$$\dot{a}^2 = -2a^2 + a^1,$$

which can be transformed into a second order ODE:

$$\ddot{a}^1 = -4\dot{a}^1 - 3a^1 - 2a^1\dot{a}^1 - 2\left(a^1\right)^2.$$

Suppose $a^1 = q^1$, then we will have

$$\ddot{q}^1 = -4\dot{q}^1 - 3q^1 - 2q^1\dot{q}^1 - 2\left(q^1\right)^2.$$

Let

$$F = -4\dot{q}^1 - 3q^1 - 2q^1\dot{q}^1 - 2\left(q^1\right)^2 \tag{10}$$

and substitute Eq. (10) into Eq. (7), then we will be able to obtain

$$Q^1 = t\dot{q}^1 + \left(1+t^2\right)\left[-4\dot{q}^1 - 3q^1 - 2q^1\dot{q}^1 - 2\left(q^1\right)^2\right]. \tag{11}$$

Obviously, zero solution $q^1 = \dot{q}^1 = 0$ of the rheonomic nonholonomic system is asymptotically stable under the action of the generalized force Q^1 of Eq. (11).

3.2. Skew-gradient system

The differential equations of the skew-gradient system have the following form:

$$\dot{a}^\mu = \omega_{\mu\nu}\frac{\partial V}{\partial a^\nu}, \quad \omega_{\mu\nu} = -\omega_{\nu\mu} \quad (\mu,\nu = 1,2). \tag{12}$$

According to Eq. (12), we can obtain $\dot{V} = 0$. Thus, if Lyapunov function V is positive definite in the neighborhood of the solution of the dynamical system, the zero solution is stable. If letting

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and}$$
$$V = \left(a^1\right)^2 + \left(a^2\right)^2 - a^1 a^2 + \frac{1}{3}\left(a^1\right)^3, \tag{13}$$

equation (12) can be transformed into a second order ODE

$$\ddot{a}^1 = -3a^1 - 2\left(a^1\right)^2.$$

Suppose $a^1 = q^1$, then

$$\ddot{q}^1 = -3q^1 - 2\left(q^1\right)^2. \tag{14}$$

Let

$$F = -3q^1 - 2\left(q^1\right)^2 \tag{15}$$

and substitute Eq. (15) into Eq. (7), then we will have

$$Q^1 = t\dot{q}^1 + \left(1+t^2\right)\left[-3q^1 - 2\left(q^1\right)^2\right]. \tag{16}$$

Obviously, zero solution $q^1 = \dot{q}^1 = 0$ of the rheonomic nonholonomic system is stable under the action of the generalized force Q^1 of Eq. (16).

3.3. Gradient system with a symmetric negative definite matrix

The differential equations of a gradient system with a symmetric negative matrix have the following form:

$$\dot{a}^{\mu} = \Lambda_{\mu\nu}\frac{\partial V}{\partial a^{\nu}}, \quad (\mu,\nu = 1,2), \tag{17}$$

where $(\Lambda_{\mu\nu})$ is a symmetric negative definite matrix. According to Eq. (17), we can obtain

$$\dot{V} = \frac{\partial V}{\partial a^{\mu}}\Lambda_{\mu\nu}\frac{\partial V}{\partial a^{\nu}}.$$

If V is positive definite, then $\dot{V}$ is negative definite. So the solution of the system is asymptotically stable.

Let

$$(\Lambda_{\mu\nu}) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and}$$
$$V = \left(a^1\right)^2 + \left(a^2\right)^2 - a^1a^2 + \frac{1}{3}\left(a^1\right)^3, \tag{18}$$

then we can obtain the second order ODE:

$$\ddot{a}^1 = -6\dot{a}^1 - 6a^1 - 2a^1\dot{a}^1 - 4\left(a^1\right)^2.$$

Suppose $a^1 = q^1$, then we will have

$$\ddot{q}^1 = -6\dot{q}^1 - 6q^1 - 2q^1\dot{q}^1 - 4\left(q^1\right)^2. \tag{19}$$

Let

$$F = -6\dot{q}^1 - 6q^1 - 2q^1\dot{q}^1 - 4\left(q^1\right)^2 \tag{20}$$

and substitute Eq. (20) into Eq. (7), then we will have

$$Q^1 = t\dot{q}^1 + \left(1+t^2\right)\left[-6\dot{q}^1 - 6q^1 - 2q^1\dot{q}^1 - 4\left(q^1\right)^2\right]. \tag{21}$$

Obviously, zero solution $q^1 = \dot{q}^1 = 0$ of the rheonomic nonholonomic system is stable under the action of the generalized force Q^1 of Eq. (21).

3.4. Generalized skew-gradient system

The differential equations of the generalized skew-gradient system have the following form:

$$\dot{a}^{\mu} = \Omega_{\mu\nu}(t,a)\frac{\partial V(t,a)}{\partial a^{\nu}}, \quad (\mu,\nu = 1,2), \tag{22}$$

where $\Omega_{\mu\nu}(t,a) = -\Omega_{\nu\mu}(t,a)$. According to Eq. (22), we can obtain $\dot{V} = \partial V/\partial t$. If V is positive definite and $\partial V/\partial t < 0$, the solution of the dynamical system is stable.

Let

$$(\Omega_{\mu\nu}) = \begin{pmatrix} 0 & 1+t\left(a^1\right)^2 \\ -1-t\left(a^1\right)^2 & 0 \end{pmatrix},$$
$$V = \frac{1}{2}\left(a^1\right)^2\left(1+\frac{1}{1+t}\right) + \frac{1}{2}\left(a^2\right)^2,$$

then we will be able to obtain a second order ODE:

$$\ddot{a}^1 = \left[2ta^1\dot{a}^1 + \left(a^1\right)^2\right]\frac{\dot{a}^1}{1+t\left(a^1\right)^2} - \left[1+t\left(a^1\right)^2\right]a^1. \tag{23}$$

Suppose $a^1 = q^1$, then we will have

$$\ddot{q}^1 = \left[2tq^1\dot{q}^1 + \left(q^1\right)^2\right]\frac{\dot{q}^1}{1+t\left(q^1\right)^2} - \left[1+t\left(q^1\right)^2\right]q^1. \tag{24}$$

Substituting Eq. (24) into Eq. (7), we have

$$Q^1 = t\dot{q}^1 + \left(1+t^2\right)\left\{\left[2tq^1\dot{q}^1 + \left(q^1\right)^2\right]\frac{\dot{q}^1}{1+t\left(q^1\right)^2} - \left[1+t\left(q^1\right)^2\right]^2 q^1\right\}. \tag{25}$$

Obviously, zero solution $q^1 = \dot{q}^1 = 0$ of the rheonomic nonholonomic system is stable under the action of the generalized force Q^1 of Eq. (25). Because V is positive definite and decreasing, zero solution $q^1 = \dot{q}^1 = 0$ is uniformly stable.

3.5. Generalized gradient system with a symmetric negative definite matrix

The differential equations of the generalized gradient system with a symmetric negative matrix have the following form:

$$\dot{a}^{\mu} = \Lambda_{\mu\nu}(t,a)\frac{\partial V(t,a)}{\partial a^{\nu}}, \quad (\mu,\nu = 1,2) \tag{26}$$

where $(\Lambda_{\mu\nu})$ is a symmetric negative definite matrix. According to Eq. (26), we have

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial a^{\mu}}\Lambda_{\mu\nu}\frac{\partial V}{\partial a^{\nu}}.$$

If V is positive definite in the neighborhood of the solution for the system and

$$\frac{\partial V}{\partial t} < 0,$$

the solution of the system is stable. If $\dot{V}$ is negative definite, the solution of the system is asymptotically stable.

Let

$$(\Lambda_{\mu\nu}) = \begin{pmatrix} -1 & 0 \\ 0 & -(2+\sin t) \end{pmatrix} \quad \text{and}$$
$$V\left(a^1\right)^2\left(1+\frac{1}{1+t}\right) + \left(a^2\right)^2 - a^1a^2,$$

we can obtain the following differential equations of the system :

$$\dot{a}^1 = -2a^1\left(1+\frac{1}{1+t}\right) + a^2,$$
$$\dot{a}^2 = -\left(2a^2 - a^1\right)(2+\sin t),$$

which can be transformed into a second order ODE:

$$\ddot{a}^1 = -2\dot{a}^1\left(3+\frac{1}{1+t}+\sin t\right)$$

$$-a^1\left[4\left(1+\frac{1}{1+t}\right)(2+\sin t)-\frac{2}{(1+t)^2}-2-\sin t\right].$$

Suppose $a^1=q^1$, then we will have

$$\begin{aligned}\ddot{q}^1 &= -2\dot{q}^1\left(3+\frac{1}{1+t}+\sin t\right)\\ &\quad -q^1\left[4\left(1+\frac{1}{1+t}\right)(2+\sin t)\right.\\ &\quad \left.-\frac{2}{(1+t)^2}-2-\sin t\right]. \qquad (27)\end{aligned}$$

Substituting Eq. (27) into Eq. (7), we have

$$\begin{aligned}Q^1 &= t\dot{q}^1+(1+t^2)\left\{-2\dot{q}^1\left(3+\frac{1}{1+t}+\sin t\right)\right.\\ &\quad -q^1\left[4\left(1+\frac{1}{1+t}\right)(2+\sin t)\right.\\ &\quad \left.\left.-\frac{2}{(1+t)^2}-2-\sin t\right]\right\}. \qquad (28)\end{aligned}$$

Obviously, zero solution $q^1=\dot{q}^1=0$ of the rheonomic nonholonomic system is also uniformly stable under the action of the generalized force Q^1 of Eq. (28).

4. Conclusions

By using the Lyapunov function, the stability analysis of the nonholonomic constrained system is very difficult. However, the gradient system is exactly suitable to study the stability of a dynamical system with the aid of the Lyapunov function. In this paper, gradient systems are utilized to study the stability of a simple rheonomic nonholonomic system, where 3.1–3.3 and 3.4–3.5 are scleronomic and rheonomic, respectively. Some conclusions are given for the constructing generalized force.

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