SPECIAL TOPIC — Acoustics

Higher-order harmonics of general limited diffraction Bessel beams*

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In this paper, we extensively study the higher-order harmonic generation of the general limited diffraction m-th-order Bessel beam. The analysis is based on successive approximations of the Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation. Asymptotic expansions are presented for higher-order harmonic Bessel beams in near and far fields. The validity of asymptotic approximation is also analyzed. The higher-order harmonic of the Bessel beam with the lowest zero-order is taken as a special example.

Keywords: harmonic generation, Bessel beam, limited diffraction

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1. Introduction

The Bessel beam has been widely investigated in the fields of acoustics and optics.[1–17] Ideally, an unusual feature of this beam is that it does not change its shape during propagation, since its amplitude and profile are independent of the propagation distance. The nonlinearity of Bessel beams has also received considerable attention, because of their potential applications in harmonic imaging of ultrasound as well as in second and higher harmonic generation in nonlinear optics. In a series of papers the acoustical nonlinearity in Bessel beams has been considered.[18–25] In the present paper, we further extend it and supplement a result for the high harmonics of the zero-order Bessel beam. The extended analysis, based on asymptotic approximation, is given for the cases of the m-th-order Bessel beam in near and far fields. The validity range of asymptotic approximation is also analyzed.

2. Theory and results

Our analysis starts with the perturbation solutions of the Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation.[18–25] Assume that a sound source $f_n(\xi')\mathrm{e}^{\mathrm{i}n\phi'}$ with an arbitrary radial distribution oscillates harmonically at angular frequency ω, and is located in the (r,ϕ) plane normal to the z axis, which is the direction of the beam propagation. According to previous results,[24–29] the fundamental (primary) sound field is given by

$$\bar{q}_1(\xi,\eta) = \frac{2}{\mathrm{i}\eta}\int_0^\infty \exp\left(\mathrm{i}\frac{\xi^2+\xi'^2}{\eta}\right)\mathrm{J}_m\left(\frac{2\xi\xi'}{\eta}\right)f_m(\xi')\xi'\mathrm{d}\xi', \quad (1)$$

and the n-th-order harmonic beam is given as

$$\bar{q}_n(\xi,\eta) = \sum_{\ell=1}^{n-1}\bar{q}_{n\ell j}(\xi,\eta), \quad (2a)$$

where

$$\bar{q}_{n\ell j}(\xi,\eta) = \frac{n^2}{8}\int_{\eta'=0}^{\eta}\int_{\xi'=0}^{\infty}\frac{\xi'}{\eta-\eta'}\exp\left(\frac{\mathrm{i}n(\xi^2+\xi'^2)}{\eta-\eta'}\right) \times \mathrm{J}_{n\times m}\left(\frac{2n\xi\xi'}{\eta-\eta'}\right)\bar{q}_\ell(\xi',\eta')\bar{q}_j(\xi',\eta')\mathrm{d}\xi'\mathrm{d}\eta'. \quad (2b)$$

The subscripts ℓ, j, and n denote the orders of harmonics, and it is always true that $n=\ell+j$. The product $n\times m$ is the order number of the Bessel function. The terms $\mathrm{e}^{\mathrm{i}m\phi}$ and $\mathrm{e}^{\mathrm{i}nm\phi}$ are omitted. Note that the lower order ($|\ell-j|$) harmonics caused by nonlinear interaction of the harmonics of the order numbers ℓ and j are assumed to be much smaller than the n-th-order harmonic from these harmonics. This is of course valid only under the condition of relatively weak nonlinearity. In Eqs. (1) and (2), $\xi=r/a$ and $\eta=2z/ka^2$ are the radial and axial dimensionless coordinates, respectively; $f_m(\xi')$ represents the radial distribution function at the source plane $\eta=0$ or $z=0$. Equations (1) and (2) may be considered as the complex-valued (pressure) amplitudes in dimensionless form.

We now consider the high-order (n) harmonic generation of a general m-th-order Bessel beam. Suppose that on the source, the m-th-order Bessel beam with radial form is given as

$$f_m(\xi') = \mathrm{J}_m(\alpha\xi'). \quad (3)$$

Here a scaling parameter α is the dimensionless radial wave number of this beam. Assuming that the aperture of the source

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extends infinitely, hence the well-known Bessel beam field at fundamental frequency, from Eq. (1), is written as

$$\bar{q}_1(\xi,\eta) = \mathrm{J}_m(\alpha\xi)\exp\left(-\frac{\mathrm{i}\alpha^2}{4}\eta\right) \tag{4}$$

for a limited-diffraction Bessel beam. The second-harmonic generation of the m-th-order Bessel beam is given as

$$\bar{q}_2(\xi,\eta) = -\frac{1-\mathrm{i}}{4\pi^{1/2}\alpha}\mathrm{J}_{2m}(2\alpha\xi)\eta^{1/2}\exp\left(-\frac{\mathrm{i}\alpha^2}{2}\eta\right) \tag{5a}$$

under the approximation that the propagation distance satisfies an asymptotic condition of $\alpha^2\eta/2 > 2\pi$. When $0 \le \alpha^2\eta/2 < \pi/2$, the second harmonic beam distribution is

$$\bar{q}_2(\xi,\eta) = \frac{\mathrm{i}}{8}\mathrm{J}_m^2(\alpha\xi)\eta\exp\left(-\frac{\mathrm{i}\alpha^2\eta}{2}\right). \tag{5b}$$

If the distance η is varied, there is an apparent difference between the second harmonic beam profile terms in the above two equations. Beam profile is approximated by $\mathrm{J}_m^2(\alpha\xi)$ in the near field, where $0 \le \alpha^2\eta/2 < \pi/2$, and by $\mathrm{J}_{2m}(2\alpha\xi)$ in the far field, where $\alpha^2\eta/2 > 2\pi$. Between the near-field and far-field regions, depending on propagation distance η, the profile evolves from the near-field term $\mathrm{J}_m^2(\alpha\xi)$ to the far-field term $\mathrm{J}_m(2\alpha\xi)$. The profile difference of the second harmonic Bessel beam in different regions can be physically explained as follows.[25] Evolution of the second harmonic beam profile from approximately $\mathrm{J}_0^2(\alpha\xi)$ in the near field to $\mathrm{J}_0(2\alpha\xi)$ in the far field is due to the combination effects of nonlinearity and diffraction. The profile of the primary beam is $\mathrm{J}_0(\alpha\xi)$ throughout the entire field, and because the square of this field generates the second harmonic, it behaves as a volume distribution of virtual sources with shading proportional to $\mathrm{J}_0^2(\alpha\xi)$. Radiation from a source plane with a $\mathrm{J}_0^2(\alpha\xi)$ amplitude profile diffracts as it propagates. Diffraction causes a wave radiated from a given virtual source plane to eventually diverge, decrease in amplitude, and shift slightly in phase with respect to the primary beam. It is the superposition of the diffracting radiation from virtual sources far from the observation point, and the second-harmonic components generated close to the observation point and therefore having profiles proportional to $\mathrm{J}_0^2(\alpha\xi)$ and producing the $\mathrm{J}_0(2\alpha\xi)$ beam profile in the far field. In contrast, a spherically spreading sound in a lossless fluid always generates a second harmonic with a far directivity proportional to the square of the primary-beam directivity, because the far-field contribution always dominates the near-field contribution at sufficiently large distance. What makes Bessel beams different is that the collimation of the field establishes a balance between these near-field and far-field contributions at the observation point.[25] Thus, the explanation is also suitable to the circumstance of the higher-order harmonics of general Bessel beams.

For each harmonic of the order n, two points are defined: γ_{1n} and γ_{2n}. These two points mark the boundaries between the near field and the far field. According to the value of α^2/η, we divide the region of beam propagation into three parts for a general n-th harmonic of the Bessel beam. The region satisfying the following condition

$$0 \le \alpha^2\eta/2 < \gamma_{1n} \tag{6a}$$

is called the near field (or near diffractionless region). In this region, the radial beam profile is almost independent of the propagation distance. The transition region is defined by

$$\gamma_{1n} \le \alpha^2\eta/2 < \gamma_{2n}. \tag{6b}$$

The far field (or far diffractionless region) is given by

$$\alpha^2\eta/2 \ge \gamma_{2n}, \tag{6c}$$

where the radial profile is also nearly independent of the distance. The values of γ_{1n} and γ_{2n} each are generally a function of the order of harmonics and other factors (such as the absorption coefficients of media). In the present analysis, the absorption is ignored.

In what follows, we will derive a more general result, namely that the n-th-order harmonic component in the m-th-order Bessel beam has a $\mathrm{J}_{n\times m}(n\alpha\xi)$ function distribution in the far field and has a $\mathrm{J}_m^n(\alpha\xi)$ function distribution in the near field. We first consider the asymptotic case of a sufficiently large $\alpha^2\eta \gg \gamma_{2n}$. This condition can be relaxed to $\alpha^2\eta/2 > \gamma_{2n}$ as will be seen later.

From Eqs. (4) and (5) for $n = 1$ and 2, we assume in general that ℓ-th-order and j-th-order harmonic components may be represented by

$$\bar{q}_\ell(\xi,\eta) = A_\ell \mathrm{J}_{\ell\times m}(l\alpha\xi)\eta^{(\ell-1)/2}\exp\left(-\frac{\mathrm{i}\ell\alpha^2}{4}\eta\right) \tag{7a}$$

and

$$\bar{q}_j(\xi,\eta) = A_j \mathrm{J}_{j\times m}(\ell\alpha\xi)\eta^{(j-1)/2}\exp\left(-\frac{\mathrm{i}j\alpha^2}{4}\eta\right). \tag{7b}$$

From Eq. (2b), we have

$$\begin{aligned}&\hat{q}_{n\ell j}(\xi,\eta)\\ &= \int_{\eta'=0}^{\eta}\int_{\xi'=0}^{\infty}\frac{\xi'}{\eta-\eta'}\exp\left(\frac{\mathrm{i}n(\xi^2+\xi'^2)}{\eta-\eta'}\right)\\ &\quad\times \mathrm{J}_{n\times m}\left(\frac{2n\xi\xi'}{\eta-\eta'}\right)\mathrm{J}_{\ell\times m}(\ell\alpha\xi')\mathrm{J}_{j\times m}(j\alpha\xi')\\ &\quad\times \exp\left(-\frac{\mathrm{i}n}{4}\alpha^2\eta'\right)\eta'^{n/2-1}\mathrm{d}\xi'\mathrm{d}\eta'.\end{aligned} \tag{8}$$

Applying Eq. (A4) from Appendix A to Eq. (8), a triple integral is obtained as

$$\begin{aligned}&\hat{q}_{n\ell j}(\xi,\eta)\\ &= \frac{1}{2\pi}\int_{\eta'=0}^{\eta}\int_{\xi'=0}^{\infty}\int_{t'=-\pi}^{\pi}\frac{\xi'}{\eta-\eta'}\end{aligned}$$

$$\times \exp\left(\frac{\mathrm{i}n(\xi^2+\xi'^2)}{\eta-\eta'}\right)\mathrm{J}_{n\times m}\left(\frac{2n\xi\xi'}{\eta-\eta'}\right)\mathrm{J}_{n\times m}(\lambda\alpha\xi')$$
$$\times \exp\left(-\frac{\mathrm{i}n}{4}\alpha^2\eta'\right)\eta'^{n/2-1}F(t')\,\mathrm{d}\xi'\,\mathrm{d}\eta'\,\mathrm{d}t', \quad (9)$$

where $\lambda=(\ell^2+j^2-2\ell j\cos t')^{1/2}$. Note that $\hat{q}_{m\ell j}$ differs from $\bar{q}_{m\ell j}$ by certain constants. By a procedure similar to that of Ref. [24], integration over ξ' yields

$$\hat{q}_{n\ell j}(\xi,\eta) = \frac{\mathrm{i}}{4\pi n}\int_{\eta'=0}^{\eta}\int_{t'=-\pi}^{\pi}\mathrm{J}_{n\times m}(\lambda\alpha\xi)\exp\left(-\frac{\mathrm{i}\alpha^2\lambda^2}{4n}\eta\right)$$
$$\times \exp\left[-\frac{\mathrm{i}\alpha^2}{4}\eta'\left(\frac{\lambda^2}{n}-n\right)\right]\eta'^{n/2-1}F(t')\,\mathrm{d}\eta'\,\mathrm{d}t'. \quad (10)$$

Furthermore, integrating with respect to η', we find

$$\hat{q}_{n\ell j}(\xi,\eta) = \frac{\mathrm{i}}{\pi n}\exp\left(-\frac{\mathrm{i}n\alpha^2}{4}\eta\right)\int_{t=-\pi/2}^{\pi/2}\mathrm{J}_{n\times m}(\lambda\alpha\xi)F(t')b^{-n/2}$$
$$\times \exp(b\eta)\Gamma\left(\frac{n}{2}\right)P\left(\frac{n}{2},b\eta\right)\mathrm{d}t \quad (11)$$

with $b=\mathrm{i}\alpha^2(\ell j/n)\cos^2 t$ and $t=t'/2$.

As shown in Appendix A, when the propagating beam is in the asymptotical far-field region of $\alpha^2\eta/2>\gamma_{2n}$, the n-th-order harmonic of the J_m Bessel beam is well approximated by

$$\bar{q}_n(\xi,\eta)=A_n\mathrm{J}_{n\times m}(n\alpha\xi)\eta^{(n-1)/2}\exp\left(-\frac{\mathrm{i}n}{4}\alpha^2\eta\right) \quad (12a)$$

and

$$A_n=n^2\left(\frac{\mathrm{e}^{\mathrm{i}3\pi/4}}{8\alpha}\right)^{n-1}(4n)^{(n-1)/4}\frac{n^{n-3}}{n!}. \quad (12b)$$

We next treat the other limiting case where $\alpha^2\eta/2<\gamma_{1n}$. In this near field, the second harmonic of the J_m Bessel beam is described by Eq. (5b). From Eqs. (4) and (5b), we assume that the ℓ-th-order harmonic of the m-th-order Bessel beam is

$$\bar{q}_\ell(\xi,\eta)=A_\ell\mathrm{J}_m^\ell(\alpha\xi)\eta^{\ell-1}\exp\left(-\frac{\mathrm{i}\ell\alpha^2}{4}\eta\right) \quad (13)$$

and that the j-th-order harmonic of the m-th-order Bessel beam is

$$\bar{q}_j(\xi,\eta)=A_\ell\mathrm{J}_m^j(\alpha\xi)\eta^{j-1}\exp\left(-\frac{\mathrm{i}j\alpha^2}{4}\eta\right). \quad (14)$$

Inserting Eqs. (13) and (14) into Eq. (2b) gives

$$\hat{q}_{n\ell j}(\xi,\eta)=\frac{\mathrm{i}}{2n(2n-1)}\mathrm{J}_m^n(\alpha\xi)\eta^{n-1}\exp\left(-\frac{\mathrm{i}n\alpha^2}{4}\eta\right). \quad (15)$$

From Eq. (2a), we obtain

$$\bar{q}_n(\xi,\eta)=A_n\mathrm{J}_m^n(\alpha\xi)\exp\left(-\frac{\mathrm{i}n}{4}\alpha^2\eta\right)\eta^{n-1} \quad (16a)$$

and

$$A_n=\frac{1}{n!}\left(\frac{\mathrm{i}n}{8}\right)^{n-1}. \quad (16b)$$

This derivation accounts for beam distribution if one excludes beam diffraction. The detailed derivation of A_n is presented in Appendix A.

3. Discussion

Now we come to examine the validity of our asymptotic approximation. We point out that the above asymptotic conditions of $\alpha^2\eta/2\gg\gamma_{2n}$ and $\alpha^2\eta/2\ll\gamma_{1n}$ may be relaxed to $\alpha^2\eta/2>\gamma_{2n}$ and $\alpha^2\eta/2<\gamma_{1n}$, respectively. From the result of Cunningham and Hamilton[25] (taking $\alpha^2\eta/2$ to correspond to σ in Ref. [25]), it is easy to show that $\gamma_{1n}\approx 1$. Then the amplitude of the on-axis beam in the near field $0\le\alpha^2\eta/2<\gamma_{1n}\approx 1$ is approximated by

$$|\bar{q}_n(0,\eta)|=\frac{1}{n!}\left(\frac{n}{8}\right)^{n-1}\eta^{n-1}, \quad (17)$$

and in the far field $\alpha^2\eta/2>\gamma_{2n}$ is approximated by

$$|\bar{q}_n(0,\eta)|=\left(\frac{1}{8\alpha}\right)^{n-1}(4n)^{(n-1)/4}\frac{n^{n-1}}{n!}\eta^{(n-1)/2}. \quad (18)$$

These two approximations are not valid in the transition region. To find the location of a transition point, let the above two amplitudes be equal and let γ_{2n} be determined roughly. It is found that the transition point is $\alpha^2\eta_c/2=\gamma_c=\sqrt{n}$. As a result, $\gamma_{2n}\approx C\sqrt{n}$, where C is almost not affected by any of the factors such as the order number n of the harmonic, α, aperture size, etc. To obtain C, we consider the cases of the fundamental and second harmonic amplitudes. For the fundamental amplitude (n=1, where $\gamma_{2n}=0$ in the entire region), no transition point exists. For the second harmonic amplitude ($n=2$), $\gamma_{2n}\approx 2\pi$. It is reasonable to deduce that $\gamma_{2n}\approx 2\pi\sqrt{n-1}$ and $\gamma_{2n}\approx 2\pi\sqrt{n}$ for a large n.

It is noted that the present analysis is based on the ideal case under which the aperture of the sound source, i.e., the Bessel beam function, is infinite. For a real Bessel beam, we should take into account the finite dimension of its aperture, which limits the field depth and introduces diffractive effects. Also we must clarify the validities of Eqs. (12) and (16), because they are derived by the so-called quasilinear (or successive) approximation method. Neither of these two equations is a uniformly accurate expression for the n-th-order harmonic component of the Bessel beam. Finally, our analysis ignores the absorption of sound in medium. As pointed out by Cunningham and Hamilton, for sufficiently large values of $\alpha^2\eta/2$, the absorption has a strong influence on the radial distribution of sound beams; this leads to deviation from the beam profile $\mathrm{J}_{m\times n}(n\alpha\xi)$ in the far field.

Diffraction, strong nonlinearity, and absorption make the problem much more complicated and are beyond the scope of this work. A full analysis may be based on the numerical solution method of the KZK equation developed by Hamilton *et al.*[25,27,28]

4. Conclusions

We obtain asymptotic expansions for high-order harmonic Bessel beams in near and far fields. For each harmonic of the order n, two points γ_{1n} and γ_{2n} are defined. These two points mark the boundaries between the near field and the far field. The validities of asymptotic approximations, equations (12) and (16), are also analyzed.

Appendix A

The derivations of the coefficient A_n for the cases where $\alpha^2\eta/2 < \gamma_{1n}$ and $\alpha^2\eta/2 > \gamma_{2n}$ are given in the following. From Eqs. (2a) and (15), A_n satisfies the recursive relation

$$A_n = \frac{\mathrm{i}n}{16(n-1)} \sum_{\ell=1}^{n-1} A_\ell A_m. \tag{A1}$$

Setting $A_k = k^2 B_k (\mathrm{i}/8)^{k-1}$ and substituting it into Eq. (A1) lead to

$$B_n = \frac{1}{2n(n-1)} \sum_{\ell=1}^{n-1} \ell^2 m^2 B_\ell B_m. \tag{A2}$$

It is easy to verify that

$$B_n = \frac{n^{n-3}}{n!} \tag{A3}$$

satisfies Eq. (A2). Accordingly, A_n in Eq. (16b) for the cases of $\alpha^2\eta/2 < \gamma_{1n}$ is obtained.

When $\alpha^2\eta/2$ is sufficiently large, the integrand

$$F(t')b^{-n/2}\exp(b\eta)\Gamma\left(\frac{n}{2}\right)P\left(\frac{n}{2}, b\eta\right)$$

in Eq. (11) resembles function (14) in Ref. [24]. The real and imaginary parts of this integrand are extremely similar to the delta function. To obtain Eq. (12) explicitly, one may use this property as it was used in previous work.[24] Alternatively, we may express the integral in Eq. (11) as a Bessel function expansion and then take its asymptotic approximation.

Some useful formulas will be used and listed below

$$2\pi \mathrm{J}_\mu(X)\mathrm{J}_\nu(x) = \int_{-\pi}^{\pi} F(\vartheta)\mathrm{J}_{\mu+\nu}(\varpi)\,\mathrm{d}\vartheta \tag{A4a}$$

with

$$F(\vartheta) = \mathrm{e}^{\mathrm{i}\nu\vartheta}\left(\frac{X - x\mathrm{e}^{-\mathrm{i}\vartheta}}{X - x\mathrm{e}^{\mathrm{i}\vartheta}}\right)^{(\mu+\nu)/2} \cos\nu\pi \tag{A4b}$$

and

$$\varpi = (X^2 + x^2 - 2Xx\cos\vartheta)^{1/2}. \tag{A4c}$$

The identity

$$\mathrm{e}^{\mathrm{i}\nu\psi}\mathrm{J}_\nu(m\varpi) = \sum_{k=-\infty}^{\infty} \mathrm{J}_k(mX)\mathrm{J}_{\nu+k}(mx)\,\mathrm{e}^{\mathrm{i}k\varphi} \tag{A5a}$$

with

$$\mathrm{e}^{2\mathrm{i}\psi} = \frac{X - x\mathrm{e}^{-\mathrm{i}\varphi}}{X - x\mathrm{e}^{\mathrm{i}\varphi}} \tag{A5b}$$

being a special form of Formula (8.530) on page 979 in Ref. [30].

If equation (A8) uses the following identity:

$$z^{-a}P(a,z) = \gamma*(a,z), \tag{A6}$$

where

$$\gamma*(a,z) = \mathrm{e}^{-z}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(a+n+1)}, \tag{A7}$$

then the integral in Eq. (11), denoted by $X(\xi,\eta)$, is written as

$$\begin{aligned} X(\xi,\eta) &= \Gamma\left(\frac{n}{2}\right)\eta^{n/2}\int_{-\pi/2}^{\pi/2} F(t')\mathrm{J}_{n\times m}(\lambda\alpha\xi) \\ &\quad\times \exp(b\eta)\gamma*\left(\frac{n}{2}, b\eta\right)\mathrm{d}t. \end{aligned} \tag{A8}$$

The foregoing equation is further simplified by means of Eqs. (A5) and (A7) as

$$\begin{aligned} X(\xi,\eta) &= \Gamma\left(\frac{n}{2}\right)\eta^{n/2}\int_{-\pi/2}^{\pi/2}\cos L\pi\,\mathrm{e}^{\mathrm{i}2Lt} \\ &\quad\times\left[\sum_{k=-\infty}^{\infty}\mathrm{J}_k(\ell\alpha\xi)\mathrm{J}_{M+k}(j\alpha\xi)\,\mathrm{e}^{\mathrm{i}2kt}\right] \\ &\quad\times\left[\sum_{p=0}^{\infty}\frac{(z\cos^2 t)^p}{\Gamma\left(\frac{n}{2}+1+p\right)}\right]\mathrm{d}t \\ &= 2\Gamma(n/2)\,\eta^{n/2}\int_0^{\pi/2}\cos L\pi \\ &\quad\times\left[\sum_{k'=-\infty}^{\infty}\mathrm{J}_{k'-L}(\ell\alpha\xi)\mathrm{J}_{M+k'-L}(j\alpha\xi)\,\mathrm{e}^{\mathrm{i}2k't}\right] \\ &\quad\times\left[\sum_{p=0}^{\infty}\frac{(z\cos^2 t)^p}{\Gamma(n/2+1+p)}\right]\mathrm{d}t. \end{aligned} \tag{A9}$$

where $k' = k + L$, $L = j \times m$, $M = n \times m$, and $z = \mathrm{i}(\ell j/n)\alpha^2\eta$. After performing integration with respect to t, equation (A9) becomes

$$\begin{aligned} X(\xi,\eta) &= 2\Gamma\left(\frac{n}{2}\right)\eta^{n/2}\sum_{k'=-\infty}^{\infty}\mathrm{J}_{k'-L}(\ell\alpha\xi) \\ &\quad\times\mathrm{J}_{M+k'-L}(j\alpha\xi)F_{k',n/2}(z) \end{aligned} \tag{A10}$$

and

$$\begin{aligned} &F_{k',n/2}(z) \\ &= \frac{\pi z^{k'}}{2^{2k'+1}\Gamma(k'+\frac{n}{2}+1)} \end{aligned}$$

$$\times {}_2\mathrm{F}_2\left(k'+\frac{1}{2},k'+1;k'+\frac{n}{2}+1,2k'+1;z\right). \quad \text{(A11)}$$

Taking the first leading term of the asymptotic expansion of Eq. (A11) and using the summation formula for a Bessel function, we have asymptotically

$$\hat{q}_{n\ell j}(\xi,\eta) = \frac{\mathrm{e}^{\mathrm{i}3\pi/4}\Gamma(n/2)}{2n\alpha\Gamma((n+1)/2)}\sqrt{\frac{n}{\ell j}}\exp\left(-\frac{\mathrm{i}n}{4}\alpha^2\eta\right) \times \mathrm{J}_{n\times m}(n\alpha\xi)\eta^{(n-1)/2}. \quad \text{(A12)}$$

From the above equation, the recursive relation is taken as

$$A_n = \left(\frac{\mathrm{e}^{\mathrm{i}3\pi/4}}{16\alpha}\right)n^{3/2}\frac{\Gamma(n/2)}{\Gamma((n+1)/2)}\sum_{\ell=1}^{n-1}A_\ell A_m\sqrt{\frac{1}{\ell m}}. \quad \text{(A13)}$$

This corrects A_n in Eq. (17) of Ref. [24]. Setting $A_k = k^2(\mathrm{e}^{\mathrm{i}3\pi/4}/8\alpha)^{k-1}C_k$ and inserting it into Eq. (A13), we then obtain

$$C_n = \frac{1}{2\sqrt{n}}\frac{\Gamma(n/2)}{\Gamma((n+1)/2)}\sum_{\ell=1}^{n-1}C_\ell C_m \ell^{3/2}m^{3/2}. \quad \text{(A14)}$$

The numerical values of C_n are listed in Table 1. From the table, it is seen that

$$\frac{C_n}{B_n} \approx (2\sqrt{n})^{(n-1)/2}, \quad \text{(A15)}$$

where B_n is given in Eq. (A3). Thus, we can obtain the recursive relation (12b).

Table 1. Numerical values of C_n and C_n/B_n.

n	C_n in Eq. (A14)	$\sqrt{n}C_n/[(2\sqrt{n})^{(n-1)/2}B_n]$	$\sqrt{n}$	C_n in Eq. (A15)
1	1	1	1	1
2	$\frac{1}{\sqrt{2\pi}} = 0.3989$	1.27	1.414	0.4204
3	$\frac{1}{\sqrt{3}} = 0.5774$	1.732	1.732	0.5774
4	$\frac{1}{3\sqrt{\pi}}\left(6+\frac{4}{\pi}\right) = 1.3678$	2.03	2	1.3333
5	$\frac{3}{\sqrt{5}}\left(\frac{11}{4}+\frac{4}{3\pi}\right) = 4.2589$	2.260	2.236	4.1667
6	$\frac{8}{15\sqrt{6\pi}}\left(91+\frac{104}{\pi}+\frac{128}{3\pi^2}+\frac{1}{2}\right) = 15.8377$	2.4434	2.4494	15.9362

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