SPECIAL TOPIC — Acoustics

Rayleigh reciprocity relations: Applications

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Classical reciprocity relations have wide applications in acoustics, from field representation to generalized optical theorem. In this paper we introduce our recent results on the applications and generalization of classical Rayleigh reciprocity relation: higher derivative reciprocity relations as a generalization of the classical one and a theoretical proof on the Green's function retrieval from volume noises.

Keywords: Rayleigh reciprocity relation, higher derivative reciprocity, Green's function retrieval, symplectic structure

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1. Introduction

Classical Rayleigh reciprocity relation is indispensable in a text on fundamentals of acoustics, which interrelates two admissible linear acoustic states of systems in the same domain. Hendrik Lorentz first derived the Lorentz reciprocity for classical electromagnetism, then Lord Rayleigh and Helmholtz made analogous works regarding sound and light respectively, and it is commonly known as Rayleigh–Carson reciprocity for acoustics. In the linear elastic wave field, a similar theorem Maxwell–Betti reciprocal work theorem is known, which is the basis of the boundary element method.[1–3] Reciprocity theorems have been applied widely in acoustic, electromagnetic and geophysical fields: wave propagation modeling,[4] inverse problem modeling,[4] acoustic holography,[5] extraction of Green's functions,[6–8] and time reversal mirror.[9]

An acoustic wave equation in four-dimensional (4D) space–time form can be written as

$$c^{-2}(\boldsymbol{x})\partial_t^2\phi - \nabla^2\phi = q(\boldsymbol{X}), \tag{1}$$

where $\phi \in C^2(R^4)$ denotes the velocity potential, $c(\boldsymbol{x}) \in C^\infty(R^3)$ is the sound speed distribution, which is assumed to be time-independent, $q(\boldsymbol{X}) \in C(R^4)$ is the source term, and $\boldsymbol{X} = (t, \boldsymbol{x})$. Denote two acoustic states by (ϕ_1, q_1) and (ϕ_2, q_2) respectively in the same medium, after applying simple arithmetic operations one has

$$c^{-2}\partial_t[\varphi_2\partial_t\varphi_1 - \varphi_1\partial_t\varphi_2] - \nabla\cdot[\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2] = q_1\varphi_2 - q_2\varphi_1, \tag{2}$$

where ∇ stands for the gradient operator in three-dimensional (3D) spatial space. Equation (2) is known as Lagrange identity in partial differential equation, which can be recast to the following form

$$\partial_t\rho + \nabla\cdot v = \sigma \quad \text{or} \quad \partial^\mu J_\mu = \sigma, \tag{3}$$

where $J = (\rho, \boldsymbol{v})$ is a 4D current, and Einstein's summation convention is adopted for repeated lowercase subscripts throughout the paper. For a finite 4D region Σ, equation (3) has the integral form by employing Gauss's divergence theorem

$$\int_{\partial\Sigma} J_\mu \,\mathrm{d}S^\mu = \int_\Sigma \mathrm{d}V\sigma. \tag{4}$$

It is seen that equations (3) and (4) define a flux conservation law; the left-hand side denotes the flux of the current and the right-hand side is the source generating the flux. Moreover, equations (2) and (3) in fact denote the 4D local Rayleigh reciprocity relation, and equation (4) is its integral form.

In this paper, we would like to reveal the reciprocity relation from some different perspectives, especially bearing in mind with possible applications to ocean acoustics. The reciprocity relation is simple in form, while we shall show that the simple but excellent relation has a rich mathematical structure and physical applications. Figure 1 gives a diagram where different concepts or mathematical structures are related to the reciprocity relation. Some of them are well known, e.g., the field representation theorem can be derived from the reciprocity relation and Green's formula[3] (see also Eq. (7)), and some of them are known seldom, e.g., geometric structure of the reciprocity and its relation with Peierls bracket.

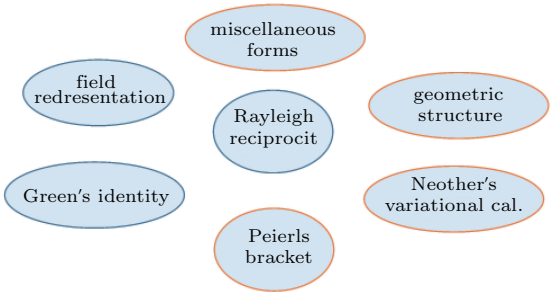


Fig. 1. (color online) Reciprocity and interconnections.

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For general references on the applications of the reciprocity relation on acoustics and geophysics, we refer interested readers to Ref. [3]. This is a special issue on acoustics, so we restrict ourselves to the acoustic wave equation.

We review and summarize our recent results on reciprocity relations in brief. The structure of the paper is organized as follows. In Section 2, we explain higher derivative reciprocity relations as a generalization of the classical one. The symplectic and Hamiltonian structure of the reciprocity relation is illustrated in Section 3, and furthermore, its relation with the Peierls bracket is built. We also give a physical interpretation of the associating field-theoretical symplectic structure. In Section 4, the main contribution, i.e., a balance condition for Green's function retrieval of volume random noise sources, is given. Section 5 comes to the summary and discussion.

2. Higher derivative reciprocity relation

In this section, we consider a generalization of the reciprocity relation. For simplicity, we restrict ourselves to a time-harmonics wave equation, i.e., the Helmholtz equation which is the wave equation in the frequency domain.

Two acoustic states are denoted by $\phi_A, \phi_B \in C^2(R^3)$, we are concerned with the following interaction quantities between the two states

$$J_k(\phi_A,\phi_B) \equiv [\partial_k\phi_A\phi_B - \partial_k\phi_B\phi_A](\boldsymbol{x}), \quad k=1,2,3, \tag{5}$$

and its higher order extension

$$\begin{aligned} J_{k,k_1,\ldots,k_n}(\phi_A,\phi_B) \equiv &\big[\partial_k\phi_A(\partial_{k_1}\cdots\partial_{k_n}\phi_B) \\ &-(\partial_k\partial_{k_1}\cdots\partial_{k_n}\phi_B)\phi_A\big](\boldsymbol{x}), \\ &k,k_1,\ldots,k_n=1,2,3, \end{aligned} \tag{6}$$

where ∂_k stands for the partial derivative in the x^k $(k=1,2,3)$ direction.

Both equations (5) and (6) define 3D tensor current components. In the time-independent case equation (5) reduces to

$$\nabla\cdot J(\phi_A,\phi_B) = q_A\phi_B - q_B\phi_A. \tag{7}$$

Clearly, this takes the standard conserved theorem form. It implies that the whole flux of the interaction vector through a closed surface inside the source free being zero, is conserved. Note that equation (7) is satisfied in the medium whether it is homogeneous or not.

For a homogeneous medium with $c(\mathrm{x})$ is constant, it is easily seen that any arbitrary higher derivative of wave field also satisfies the same wave equation, hence a new state can be defined. It turns out that the higher interaction quantities defined in Eq. (6) are all conserved when taking the divergence with respect to the left most index. For example,

$$\begin{aligned} \partial^k J_{k,k_1,\ldots,k_n}(\varphi_A,\varphi_B) = &\big[(\partial^k\partial_k\varphi_A)(\partial_{k_1,\ldots,k_n}\varphi_B) \\ &-(\partial^k\partial_k(\partial_{k_1,\ldots,k_n}\varphi_B))\varphi_A\big](\mathrm{x}) = 0, \end{aligned} \tag{8}$$

where we have used the conventions:

$$\partial_k\partial_k = \nabla^2, \quad \partial_{k_1}\cdots\partial_{k_n} \equiv \partial_{k_1\cdots k_n}.$$

For the homogeneous case these higher quantities are trivial in the above sense. However, these higher conserved relations will be broken down when the underlying medium becomes inhomogeneous. To see this explicitly, we investigate the first higher one $(n=1)$ in a source free domain

$$\partial^k J_{k,k_1}(\phi_A,\phi_B) = \omega^2\phi_A\phi_B\partial_{k_1}[1/c(\boldsymbol{x})^2]. \tag{9}$$

It is seen that the inhomogeneity arises in the right-hand side as a "source" term.

Let wave states be defined in a domain $\Sigma \in R^3$ which is bounded by $\partial\Sigma \in R^3$ (Fig. 2). The outward pointing normal to $\partial\Sigma$ is represented by $\boldsymbol{n}$. The complement of $\Sigma\cup\partial\Sigma$ in R^3 is denoted by Σ', where the medium is assumed to be homogeneous. We consider two wave states denoted by the superscripts A and B, respectively, with the same constitutive parameter distribution. An inhomogeneous region is embedded inside of the domain (shown by the shadowed area).

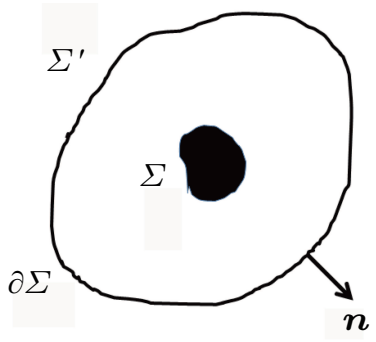


Fig. 2. A de-Hoop's egg.

Now let us consider two acoustic states, which are generated by two different sources q_A and q_B, respectively. Taking divergence and integrating over the volume V at both sides, then applying Gauss's divergence theorem, equation (6) can be modified to

$$\begin{aligned} \iint_{\partial V} \mathrm{d}\boldsymbol{S}\cdot[(\nabla\phi_A)(\partial_{k_1,\ldots,k_n}\phi_B) - (\nabla(\partial_{k_1,\ldots,k_n}\phi_B))\phi_A](\boldsymbol{x}) &= \omega^2\iiint_V \mathrm{d}V\left[-\frac{\phi_A}{c^2}(\partial_{k_1,\ldots,k_n}\phi_B) + \left(\partial_{k_1,\ldots,k_n}\left(\frac{\phi_B}{c^2}\right)\right)\phi_A\right](\boldsymbol{x}), \\ &+\iiint_V \mathrm{d}V\left[q_A(\boldsymbol{x})\partial_{k_1,\ldots,k_n}\phi_B(\boldsymbol{x}) - \phi_A(\boldsymbol{x})\partial_{k_1,\ldots,k_n}q_B(\boldsymbol{x})\right] \end{aligned} \tag{10}$$

for $n = 0, 1, \ldots$. This relation defines the basic form for higher reciprocity theorem. The $n = 0$ case corresponds to the conventional one. As usual, the left-hand side is interpreted as a current flux, and the right-hand side belongs to higher order "source" terms. The important difference is the additional source term coming from medium inhomogeneity, which we refer to as the secondary source. Thus a higher reciprocity relation characterizes the higher derivative wave field interacting with the higher derivatives of the source and the medium inhomogeneity. Various applications are determined through choosing different configurations of sources, inhomogeneity, and observers.

3. Geometric structure of the reciprocity relation

In this section we interpret the geometric structure of the frequency-space type reciprocity relations. It is not surprising that the acoustic wave equation is variational in the sense which can be derived through the Euler–Lagrange equation of a variational action. The acoustic wave equation has a canonical Hamiltonian structure from the field-theoretical point of view. As it is known that a Hamiltonian structure is usually associated with the temporal variable, it seems that there is no place to say a Hamiltonian or symplectic structure for a time-independent system. However, we shall show in this section that a general symplectic structure, even a Hamiltonian structure, can be indeed defined for the Helmholtz equation. On the other hand, the symplectic structure is known as a pure mathematical concept. After being combined with the Peierls bracket, a symplectic structure can be considered as a physics origin of a kind of flux conservation law, therefore the flux through a surface can be defined by a symplectic structure. To the best of our knowledge, it seems that there is scarcely discussion on this topic.

We start with the following formal action for the Helmholtz equation,

$$\begin{aligned} I[\phi,\phi^*] &\equiv \int_\Sigma \mathrm{d}^3\boldsymbol{x} L(\phi,\nabla\phi) \\ &= \int_\Sigma \mathrm{d}^3\boldsymbol{x}\left[-\nabla\phi\cdot\nabla\phi^* + \frac{\omega^2}{c(\boldsymbol{x})^2}\phi\phi^*\right], \end{aligned} \tag{11}$$

where ϕ and ϕ^* denote the independent sound field variables, symbol $*$ denotes the complex conjugate and Σ is a finite spatial domain. The equation of motion is determined by the variational stationary, and the Euler–Lagrange equation of $L(\phi,\nabla\phi)$ can be given by

$$\frac{\partial L}{\partial\phi^*} - \nabla\cdot\frac{\partial L}{\partial(\nabla\phi^*)} \equiv EL(\phi) = \Delta\phi + k^2(\boldsymbol{x})\phi = 0, \tag{12}$$

where EL denotes the Euler–Lagrange operator, and the complex conjugate of Eq. (12) has a similar form. After simplifying the expression, we can obtain

$$EL(\phi) = \Delta\phi + k^2(x)\phi = 0,$$

where k is wavenumber. We now explain why a symplectic structure appears in the time-independent problem. To this end, two independent arbitrary variations around ϕ are defined as

$$\phi \to \phi + \delta_\alpha\phi, \quad \phi^* \to \phi^* + \delta_\alpha\phi^*$$

with $\alpha = 1, 2$, respectively, where ϕ, ϕ^* are arbitrary. Taking the first cross-variational derivative $\delta_1\delta_2$ to the action in Eq. (11) one has

$$\begin{aligned} \delta_1\delta_2 I = \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \mathrm{d}^3\boldsymbol{x}\big\{ &\delta_2\phi EL(\delta_1\phi^*) + \delta_2\phi^* EL(\delta_1\phi) \\ &- \nabla\cdot\left[\delta_2\phi\nabla\delta_1\phi^* + \delta_2\phi^*\nabla\delta_1\phi\right]\big\}. \end{aligned}$$

By following the similar steps, one can have $\delta_1\delta_2 I$. Note that in the above derivation we have used the fact that the spatial derivatives and variational derivatives are commutative as operators. Next, along with the integrability condition $[\delta_2,\ \delta_1] = 0$, we have

$$\begin{aligned} 0 = \int_\Sigma \mathrm{d}^3\boldsymbol{x}\big\{ &\left[\delta_1\phi EL(\delta_2\phi^*) + \delta_1\phi^* EL(\delta_2\phi) - (1\leftrightarrow 2)\right] \\ &- \nabla\cdot\left[\delta_1\phi\nabla\delta_2\phi^* + \delta_1\phi^*\nabla\delta_2\phi - (1\leftrightarrow 2)\right]\big\}. \end{aligned} \tag{13}$$

It is noted that the variations in Eq. (13) are arbitrary, which need not be a solution of the Euler–Lagrange equation. Both the integral and boundary terms (full derivatives) are bilinear and anti-symmetric with respect to the variations, hence they define a second-order antisymmetric tensor in the space of solution. In Eq. (13), it turns out that the boundary term is conserved if the variations are solutions of the Euler–Lagrange equation. This is nothing but a restatement of the reciprocity relation Eq. (7) written in the symmetric form for its complex conjugate. The boundary term above defines the (pre-)symplectic structure of the infinite dimensional system, and the field-theoretical space is considered as an infinite dimensional vector space, in which infinitesimal variations are the differentials along the space. This kind of reasoning is the fundamental idea of the multisymplectic or polysymplectic and covariant phase space approach.[10–12] Although in general the boundary $\partial\Sigma$ is assumed as the space-like in the above articles, it will be shown that the boundary in the present case can be an arbitrary 2D surface. Another difference from the time dependent field theories discussed in Refs. [11] and [12] is that for the Helmholtz equation the boundary value and its normal derivative are not independent, therefore a Dirichlet-to-Neumann map or Neumann-to-Dirichlet map is needed for the mapping purpose. In order to achieve this mapping object we have to define the physical phase space and states of the field carefully. These are beyond the scope of this paper.

We have thus seen that the interaction quantity integrated over a closed surface in geometry is the symplectic two-form of the Lagrange system Eq. (12). Next we investigate the relation between the symplectic structure and a time-independent analogy of the Peierls bracket. We now need the reduced form of Eq. (10) for the case of $n = 0$ with one Green's function replaced by its complex conjugate, thus equation (10) is modified as the following form

$$\iint_{\partial\Sigma} \mathrm{d}\boldsymbol{S}\cdot\left[\nabla G(\boldsymbol{x},\boldsymbol{x}_A)G(\boldsymbol{x},\boldsymbol{x}_B)^* - \nabla G(\boldsymbol{x},\boldsymbol{x}_B)^* G(\boldsymbol{x},\boldsymbol{x}_A)\right] = G(\boldsymbol{x}_B,\boldsymbol{x}_A) - G(\boldsymbol{x}_B,\boldsymbol{x}_A)^*,\ \ \boldsymbol{x}_A,\boldsymbol{x}_B\in\Sigma. \tag{14}$$

This relation has a straightforward interpretation in terms of the time reverse symmetry. Let us consider the following thought experiment. First a point source at $\boldsymbol{x}_A$ in the domain Σ generated an acoustic field at some past time and then was removed. An observer-α receives the field distribution $G(x,x_A)$ on the boundary $x\in\partial\Sigma$, then the received signals are time-reversed, which is shown as $G(x,x_A)^*$ (corresponding to the complex conjugate in frequency domain), and retransmitted as a boundary source is located at the same position of observer-α. The problem is what signals can be seen by observerβ at the observed point x_B. By virtue of the time reversal symmetry of the acoustic wave equation, observer-β will observe an incoming Green's function $G(x_A,x_B)^*$ at first, which is followed by an outgoing Green's function $G(x_A,x_B)$. It is noted that there is no source at x_A when observer-β works, so the incoming wave has to go away again to assure no singularity at x_A. Equation (14) just describes the thought experiment. The flux through the boundary corresponding to the LHS, focuses at x_B as described by the RHS. This is also the basic principle of the time-reversal mirror technique.[13]

After the above preparation, we are now ready to show that the Green's formula Eq. (14) connects the Peierls bracket[14,15] and the field-theoretical symplectic structure defined above. To this end, we define the symbol $\tilde{G}(k;\boldsymbol{x}',\boldsymbol{x}")$ as

$$\tilde{G}(k;\boldsymbol{x}',\boldsymbol{x}'') \equiv G^*(k;\boldsymbol{x}',\boldsymbol{x}'') - G(k;\boldsymbol{x}',\boldsymbol{x}''), \tag{15}$$

and define the bilinear form for two arbitrary functions F,G as

$$(F,G) \equiv \int_\Sigma \mathrm{d}^3\boldsymbol{x}'\mathrm{d}^3\boldsymbol{x}''\frac{\delta F}{\delta\phi^*(k,\boldsymbol{x}')}\tilde{G}(k;\boldsymbol{x}',\boldsymbol{x}'')\frac{\delta G}{\delta\phi(k,\boldsymbol{x}'')}. \tag{16}$$

In order to define the functional derivation in Eq. (16), the local function F is set to be the function of functions ϕ, ϕ^* and their derivative, and given by

$$F[\phi] = \int_\Sigma \mathrm{d}^3\boldsymbol{x} f(\phi,\nabla\phi,\phi^*,\nabla\phi^*).$$

Then its functional derivative is defined by

$$\frac{\delta F}{\delta\phi(\boldsymbol{x})} = \frac{\partial f}{\partial\phi}.$$

The following equality can be easily verified

$$(G,F) = -(F,G)^*. \tag{17}$$

We define a time-independent Peierls bracket as a bilinear form $\{\ ,\ \}$:

$$\{F,G\} \equiv (F,G) + (F,G)^*, \tag{18}$$

which satisfies the following three axioms.

(i) Skew-symmetry:

$$\{G,F\} = (G,F) + (G,F)^* = -(F,G)^* - (F,G) = -\{F,G\}; \tag{19}$$

(ii) Leibniz rule:

$$\begin{aligned}\{FG,H\} &= F\{G,H\} + \{F,H\}G,\\ \{F,GH\} &= \{F,G\}H + G\{F,H\};\end{aligned} \tag{20}$$

(iii) Jacobian identity:

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0. \tag{21}$$

The first two are clear from its defining relations, the third Jacobian identity (for a Peierls bracket it is also called the Peierls identity) can also be easily demonstrated, the details are not given here. By virtue of Green's formula Eq. (14), the Peierls bracket can be rewritten as the boundary terms and the details are presented below.

Let

$$\begin{aligned}F_{\mathrm{in}}(\boldsymbol{x}) &\equiv \int_\Sigma \mathrm{d}^3\boldsymbol{x}' G^*(k;\boldsymbol{x},\boldsymbol{x}')\frac{\delta F}{\delta\phi(\boldsymbol{x}')^*},\\ G_{\mathrm{out}}(\boldsymbol{x}) &\equiv \int_\Sigma \mathrm{d}^3\boldsymbol{x}'' G(k;\boldsymbol{x},\boldsymbol{x}'')\frac{\delta G}{\delta\phi(\boldsymbol{x}'')},\end{aligned} \tag{22}$$

then by virtue of the Green formula, we have

$$(F,G) \equiv -\int_{\partial\Sigma}\mathrm{d}\boldsymbol{S}\cdot\left[\nabla F_{\mathrm{in}}G_{\mathrm{out}} - \nabla G_{\mathrm{out}}F_{\mathrm{in}}\right]. \tag{23}$$

Physically F_{in} and G_{out} define the incoming and outgoing waves generated by the source density $f(\phi)$, $g(\phi)$ respectively. We have thus seen that the Peierls bracket and the field-theoretical symplectic structures are related by Green's formula. The Peierls bracket is defined over the 3D volume Σ, and the field-theoretical structure is defined on the boundary $\partial\Sigma$. The equivalence of two Poisson structures has the natural physical interpretation in Eq. (14).

We have seen in this section that geometrically the Rayleigh reciprocity relation has a close relation with field-theoretical symplectic structure. The Green formula has built a connection between a covariant Peierls bracket and a covariant phase-space symplectic structure. It remains open how to apply these mathematical structures to acoustics.

4. Green's function retrieval revisited

The extraction of the Green function from ambient noises for acoustic and elastic waves has recently received a great deal of attention, see e.g., Ref. [15]. The extraction of Green's function has been applied to ultrasound, to crustal seismology to exploration seismology, to structural engineering, and so on.[16–33] Early derivations of the principle for Green's function retrieval were based mainly on the notion of the modal equipartition of diffuse fields[18] for closed systems. However this concept fails to be applied directly to open/infinite systems. For open systems, several derivations based on representation theorems,[17,19] on the superposition of incoming plane waves,[21–23] on time-reversal invariance,[24,25] and on the principle of stationary phase,[26] have been explored. The physical hypotheses behind these theoretical investigations, for open systems, may be classified roughly into two large categories: field distributions, e.g., local diffuse fields in disorder media or in the local equilibrium with ambient diffuse fields, and the distributions on uncorrelated noise sources. In the former the common basis is the diffuse field, although detailed mathematical derivations are different. For the latter, the Rayleigh reciprocity is powerful, which provides a unified formulation for various linear systems.[6] However, as to the reciprocity theorems used in these studies, almost all are formulated in the space-frequency domain. One remarkable limit in the space-frequency approach is that the distribution of uncorrelated noise sources has to be limited to be a spatially surface distribution. As its result, for a lossless system, volume distributed noise sources cannot be dealt with by reciprocity theorems. To prove the emergence of the Green function in the case of volume sources for open systems, Weaver & Lobiks[21] introduced a small attenuation ε to generate a volume integral term of a compact spatial domain, and finally taking the limit $\varepsilon \to 0$ formally. The approach is called ε-prescription in classical field theory. We have worked out an alternative derivation of the principle of the Green function retrieval in the case of volume noise sources for open systems. It was shown that instead of using the space-frequency reciprocity theorems, by properly employing a space-time reciprocity relation we can handle the problem within the framework of reciprocity relation. There are no assumptions on underlying media, the only one that we suppose is an ergodicity: the sample ensemble average on uncorrelated noises can be replaced by a time-averaged.

First we rewrite explicitly Eq. (4) by considering Eq. (2) in the following form,

$$\int_D \mathrm{d}V \left[q_B\phi_A - q_A\phi_B\right] = \oint_{\partial D} \left\{ n^t \left[\phi_B \frac{\partial \phi_A}{c^2 \partial t} - \phi_A \frac{\partial \phi_B}{c^2 \partial t}\right] - \boldsymbol{n} \cdot \left[\phi_B \nabla \phi_A - \phi_A \nabla \phi_B\right] \right\} \mathrm{d}S, \tag{24}$$

where $(n^t, \boldsymbol{n})$ stands for the 4D outward normal vector of the hypersurface ∂D. The space-time reciprocity, which is different from the space-frequency domain ones, is defined in the space-time domain. In this paper, we will consider a special 4D domain bounded by two instants t and T marked by red lines with $T > t$ and spatially extended unbounded (see Fig. 3). In this case, the two boundaries consist of the three dimensional space at the instants t and T. For this space-time domain, equation (24) reduces to

$$\int_D \left[q_B\phi_A - q_A\phi_B\right] \mathrm{d}V = -\int \mathrm{d}^3\boldsymbol{x} \left[\phi_B \frac{1}{c^2}\partial_t\phi_A - \phi_A \frac{1}{c^2}\partial_t\phi_B\right]_t + \int \mathrm{d}^3\boldsymbol{x} \left[\phi_B \frac{1}{c^2}\partial_t\phi_A - \phi_A \frac{1}{c^2}\partial_t\phi_B\right]_T. \tag{25}$$

It is known that for a hyperbolic partial differential equation, there exist two Green's functions: causal and anti-causal Green's functions. They satisfy the same equation,

$$\frac{\partial^2 G^{\pm}(t,t';\boldsymbol{x},\boldsymbol{x}')}{c^2\partial t^2} - \nabla^2 G^{\pm}(t,t';\boldsymbol{x},\boldsymbol{x}') = \delta(t-t')\delta^3(\boldsymbol{x}-\boldsymbol{x}'), \tag{26}$$

but with different supports

$$G^{\pm}(t,t';\boldsymbol{x},\boldsymbol{x}') = \begin{cases} 0, & (t',\boldsymbol{x}') \in \Gamma^{-}(t,\boldsymbol{x}), \\ 0, & (t',\boldsymbol{x}') \in \Gamma^{+}(t,\boldsymbol{x}), \end{cases}$$

where $\Gamma^{\pm}(t,\boldsymbol{x})$ denote the future (+) and past (–) wave light cones of the point $(t,\boldsymbol{x})$, respectively. For time-invariant media, the two Green's functions are connected by the following relations,

$$\begin{aligned} G^{+}(t,t';\boldsymbol{x},\boldsymbol{x}') &= G^{+}(t'-t;\boldsymbol{x},\boldsymbol{x}') = G^{+}(t'-t;\boldsymbol{x}',\boldsymbol{x}) \\ &= G^{-}(t-t';\boldsymbol{x},\boldsymbol{x}'). \end{aligned} \tag{27}$$

This relation is not anything other than the reciprocity theorem which is related to the Green functions.

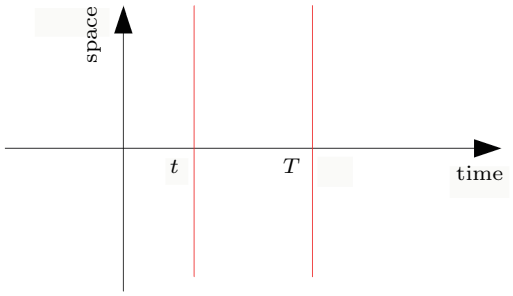


Fig. 3. (color online) Integrated domain.

The two states are defined as follows:

$$q_A = \delta(t-t')\delta^3(\boldsymbol{x}-\boldsymbol{x}'),\quad \phi_A = G^+(t,t';\boldsymbol{x},\boldsymbol{x}'),$$

$$q_B = \delta(t-t'')\delta^3(\boldsymbol{x}-\boldsymbol{x}''),\quad \phi_B = G^-(t'',t;\boldsymbol{x}'',\boldsymbol{x}),$$

and substituting these into Eq. (25) yields

$$\begin{aligned} G^-(t'-t'';\boldsymbol{x}'',\boldsymbol{x}') - G^+(t'-t'';\boldsymbol{x}'',\boldsymbol{x}') = \int \mathrm{d}^3\boldsymbol{x}\left[G^+(t''-t,\boldsymbol{x}'',\boldsymbol{x})\frac{\partial^2}{c^2\partial t^2}G^-(t-t';\boldsymbol{x},\boldsymbol{x}') - G^-(t-t';\boldsymbol{x},\boldsymbol{x}')\right.\\ \left.\times\frac{\partial^2}{c^2\partial t^2}G^+(t''-t,\boldsymbol{x}'',\boldsymbol{x})\right] - \int \mathrm{d}^3\boldsymbol{x}\left[G^+(T-t'',\boldsymbol{x}'',\boldsymbol{x})\frac{\partial^2}{c^2\partial t^2}G^-(t'-T;\boldsymbol{x},\boldsymbol{x}') - G^-(t'-T;\boldsymbol{x},\boldsymbol{x}')\right.\\ \left.\times\frac{\partial^2}{c^2\partial t^2}G^+(T-t'',\boldsymbol{x}'',\boldsymbol{x})\right], \end{aligned} \tag{28}$$

where $T > t'$, $t'' > t$, that is, instants t' and t'' fall in the time interval $[t,T]$. In the above derivation, we have used the time-invariance and reciprocity theorem. We now prove that the second line on the right-hand side of Eq. (28) vanishes. First, we remind the reader that the causal Green function satisfies the well-known propagation finiteness: $G^+(t,t';\boldsymbol{x},\boldsymbol{x}') = 0$, while $\|\boldsymbol{x}-\boldsymbol{x}'\|_{R^3} > c_{\max}(t'-t)$. It is mean when the distance between two positions is larger than the distance that wave propagates among the time interval $t-t'$, the causal Green's function equals zero. By considering the above condition it implies immediately that the second line on the right-hand side of Eq. (28) will be vanished whatever T is as long as $T > t', t''$. Consequently, only the first term on the right-hand side is left. It is important to note that the variable t in Eq. (28) is arbitrary. We can give a time-reversal interpretation of Eq. (28) as follows: a point source at the spatial location $\boldsymbol{x}''$ and the instant t generates a sound field, which will be received at the instant t'' over the total space. The first line at the right-hand side of Eq. (28) reverses the received datum in time, which is considered as the initial value generating the Green functions defined at the left-hand side.

It is noted that the left-hand side of Eq. (28) is in fact independent of the parameter t, while the right-hand side is a function of the parameter. We now apply this property to rewrite Eq. (28) in a form comparable with a field–field correlation generated by some volume noise sources.

We set $\tau \equiv t'-t$, $\Delta t = t'-t''$, with the τ value domain being $[0,T']$, a time interval. Integrating with respect to τ over the interval $[0,T']$ at both sides, one obtains

$$\begin{aligned} T'[G^-(\Delta t,\boldsymbol{x}'',\boldsymbol{x}') - G^+(\Delta t;\boldsymbol{x}'',\boldsymbol{x}')] = -\int \mathrm{d}^3\boldsymbol{x}\left[\frac{1}{c^2}\,G^-(\Delta t-\tau;\boldsymbol{x}'',\boldsymbol{x})G^+(\tau;\boldsymbol{x},\boldsymbol{x}')\Big|_{\tau=0}^{\tau=T'}\right]\\ -2\frac{\partial}{\partial\Delta t}\int \mathrm{d}^3\boldsymbol{x}\int_0^{T'}\mathrm{d}\tau\frac{1}{c^2}G^+(\tau,\boldsymbol{x},\boldsymbol{x}')G^-(\Delta t-\tau;\boldsymbol{x}'',\boldsymbol{x}). \end{aligned} \tag{29}$$

Equation (29) can be rewritten as

$$\begin{aligned} [G^-(\Delta t,\boldsymbol{x}'',\boldsymbol{x}') - G^+(\Delta t;\boldsymbol{x}'',\boldsymbol{x}')] = \frac{-\int \mathrm{d}^3\boldsymbol{x}\frac{1}{c^2}\left[G^-(\Delta t-\tau;\boldsymbol{x}'',\boldsymbol{x})G^+(\tau;\boldsymbol{x},\boldsymbol{x}')\big|_{\tau=0}^{\tau=T'}\right]}{T'}\\ -2\frac{\frac{\partial}{\partial\Delta t}\int \mathrm{d}^3\boldsymbol{x}\int_0^{T'}\mathrm{d}\tau\frac{1}{c^2}G^+(\tau,\boldsymbol{x},\boldsymbol{x}')G^-(\Delta t-\tau;\boldsymbol{x}'',\boldsymbol{x})}{T'}. \end{aligned} \tag{30}$$

The first term on the right-hand side vanishes as $T' \to \infty$. We explain this in two different ways. First the physical meaning of this term is that the integrand can be explained as the delayed products of fields generated at the instants $\tau = 0$ and T' by the point sources located at $\boldsymbol{x}'$ and $\boldsymbol{x}''$, respectively, and the integral is taken over the whole space. This quantity must be finite for all τ-values. Second, technically the first integral also appears in the second term as a spatial integral over the whole space. To assure Eq. (30) is satisfied in both sides, the spatial integral has to be finite and consequently the first term vanishes as $T' \to \infty$, then equation (30) reduces to

$$[G^-(\Delta t,\boldsymbol{x}'',\boldsymbol{x}') - G^+(\Delta t;\boldsymbol{x}'',\boldsymbol{x}')] = -2\frac{\frac{\partial}{\partial\Delta t}\int \mathrm{d}^3\boldsymbol{x}\int_0^{T'}\mathrm{d}\tau\frac{1}{c^2}G^+(\tau,\boldsymbol{x},\boldsymbol{x}')G^-(\Delta t-\tau;\boldsymbol{x}'',\boldsymbol{x})}{T'} + o\left(\frac{1}{T'}\right). \tag{31}$$

Expressed in terms of the causal Green function, this can also be rewritten as,

$$[G^-(\Delta t,\boldsymbol{x}'',\boldsymbol{x}') - G^+(\Delta t;\boldsymbol{x}'',\boldsymbol{x}')] = -2\frac{\frac{\partial}{\partial\Delta t}\int \mathrm{d}^3\boldsymbol{x}\int_0^{T'}\mathrm{d}\tau\frac{1}{c^2}G^+(\tau,\boldsymbol{x},\boldsymbol{x}')G^+(\tau-\Delta t;\boldsymbol{x}'',\boldsymbol{x})}{T'} + o\left(\frac{1}{T'}\right). \tag{32}$$

Now let us show that the above relations emergence as the field–field correlation of some volume noise sources. We assume that the density of the noise sources satisfies the following conditions,

$$\rho(t,\boldsymbol{x})\,,\ \langle\rho(t,\boldsymbol{x}),\rho(t',\boldsymbol{x}')\rangle\ =\sigma\frac{1}{c^2}\delta(t-t')\delta^3(\boldsymbol{x}-\boldsymbol{x}'),\quad(33)$$

where $\langle\,,\rangle$ denotes the sample average. These generate the random sound field

$$\phi(t,\boldsymbol{x})=\int_{R^4}\rho(t',\boldsymbol{x}')G^+(t',t;\boldsymbol{x}',\boldsymbol{x})\,\mathrm{d}t'\,\mathrm{d}^3\boldsymbol{x}',$$

and its time-reversal field is given by

$$\phi^-(t,\boldsymbol{x})=\int_{R^4}\rho(t',\boldsymbol{x}')G^-(t',t;\boldsymbol{x}',\boldsymbol{x})\,\mathrm{d}t'\,\mathrm{d}^3\boldsymbol{x}'.$$

The field–field correlation is then given by

$$\begin{aligned}\langle\phi(t',\boldsymbol{x}')\phi'(t'',\boldsymbol{x}'')\rangle&=\int_{R^4}\int_{R^4}G^+(t_1,t',\boldsymbol{x}_1,\boldsymbol{x}')G^-(t_2,t'',\boldsymbol{x}_2,\boldsymbol{x}'')\,\langle\rho(t_1,\boldsymbol{x}_1)\rho(t_2,\boldsymbol{x}_2)\rangle\,\mathrm{d}t_1\,\mathrm{d}^3\boldsymbol{x}_1\,\mathrm{d}t_2\,\mathrm{d}^3\boldsymbol{x}_2\\&=-\sigma\int_{R^4}\frac{1}{c^2}G^+(\tau,\boldsymbol{x}_1,\boldsymbol{x}')G^-(\Delta t-\tau,\boldsymbol{x}_1,\boldsymbol{x}'')\,\mathrm{d}\tau\,\mathrm{d}^3\boldsymbol{x}_1.\end{aligned}\quad(34)$$

In practice this average is replaced by averaging over nonoverlapping time windows.[15] Comparing this relation with Eq. (32) it is found that there are differences in the factor $1/T'$ and the integral upper limit. To compensate these differences we apply the ergodicity assumption, i.e., the sample ensemble average may be replaced by a long time average, which is defined by

$$\langle\rho(t',\boldsymbol{x}')\rho(\Delta t+t',\boldsymbol{x}'')\rangle\equiv\frac{\int_0^{T_1}\rho(t',\boldsymbol{x}')\rho(\Delta t+t',\boldsymbol{x}'')\,\mathrm{d}t'}{T_1}.\quad(35)$$

Substituting this into the above relation Eq. (34) by replacing the sample average and taking T' and T_1 to be infinite while keeping T'/T_1 finite yields

$$\sigma[G^-(\Delta t,\boldsymbol{x}'',\boldsymbol{x}')-G^+(\Delta t;\boldsymbol{x}'',\boldsymbol{x}')]\propto 2\frac{\partial\,\langle\phi(t',\boldsymbol{x}')\phi'(\Delta t+t',\boldsymbol{x}'')\rangle}{\partial\Delta t}+o\left(\frac{1}{T}\right).\quad(36)$$

Note that in order to extract an exact Green's function, the uncorrelated volume sources must satisfy the matching/balance condition Eq. (33) with the additional factor $1/c^2$. This factor is necessary to compensate for different radiations due to the space inhomogeneity, and to create a state of global equipartition.

In practical situations, the volume sources may not be balanced in such a way that the expression in Eq. (36) is not satisfied. Such an incompleteness in the source distribution leads to an extracted Green's function that may display spurious arrivals[34,35] or lacks some waves which are present in the real Green's function. Indeed, the Green's functions extracted in crustal seismology often are deficient in the body wave amplitude.[27]

5. Summary

In this paper we have introduced and summarized the classical Rayleigh reciprocity relation and two applications: higher derivative reciprocity relations and a theoretical proof on the Green function retrieval from some volume noises. A new mathematical structure of field-theoretical symplectic structure and time-independent Peierls bracket has been introduced, moreover its relation with Green's formula has been investigated. Finally in connection with the application of Green's function retrieval, it is still open how to extend linear problems to nonlinear systems.

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