Synthesization of high-capacity auto-associative memories using complex-valued neural networks*

Yu-Jiao Huang(黄玉娇)†, Xiao-Yan Wang(汪晓妍), Hai-Xia Long(龙海霞), and Xu-Hua Yang(杨旭华)

College of Computer Science and Technology, Zhejiang University of Technology, Hangzhou 310023, *China*

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In this paper, a novel design procedure is proposed for synthesizing high-capacity auto-associative memories based on complex-valued neural networks with real-imaginary-type activation functions and constant delays. Stability criteria dependent on external inputs of neural networks are derived. The designed networks can retrieve the stored patterns by external inputs rather than initial conditions. The derivation can memorize the desired patterns with lower-dimensional neural networks than real-valued neural networks, and eliminate spurious equilibria of complex-valued neural networks. One numerical example is provided to show the effectiveness and superiority of the presented results.

Keywords: associative memory, complex-valued neural network, real-imaginary-type activation function, external input

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1. Introduction

Since Hopfield's inspirational work on the stability of the Hopfield neural network using an energy function,[1] numerous studies on the dynamics of various neural networks have been reported.[2–13] Due to strong learning ability, high classification and prediction accuracy, the dynamics of neural networks have been investigated extensively to achieve associative memory and pattern recognition. Design procedures for synthesizing associative memories have been proposed based on real-valued neural networks (RVNNs)[14–20] and complex-valued neural networks (CVNNs).[21–26]

In the design procedures based on RVNNs, coexistence of many equilibria is a necessary feature in the applications of neural networks to associative memory storage or pattern recognition. Extensive investigations have been done to study the existence and stability of multiple equilibria of neural networks.[27–33] Due to the dependence on initial states of neural networks, these works resulted in the appearance of spurious equilibria. It is difficult to avoid spurious equilibria, and accurate pattern recalling cannot be guaranteed by using local stability of neural networks. In order to solve this problem, design procedures were presented for synthesizing associative memories based on RVNNs by assuring that each trajectory globally converged to a unique equilibrium point, which depended on the external inputs of neural networks, for example, see Refs. [15], [16], [19], and [20]. These designed networks defined a nonlinear mapping from the space of external inputs to that of steady state outputs. In the design procedures based on CVNNs, neural networks have been discussed as multistate associative memory models by employing a class of amplitude-phase-type activation functions, i.e., multilevel sigmoid functions. Although such dynamics are favorable for gray-level image reconstruction, it often suffers from convergence to an undesirable spurious pattern. Most works have been done to overcome this shortcoming.[21,23,24] As is well known, it is more efficient to avoid spurious patterns when the convergence of neural networks depends on external inputs instead of initial states in associative memories. However, to the best of the authors' knowledge, less attention has been paid to designing associative memory procedures based on CVNNs dependent on external inputs.

It is well known that the activation functions play an important role in the dynamical analysis of neural networks. In RVNNs, the activation functions are usually chosen as smooth and bounded functions. However, this is not suitable in CVNNs. Since every bounded entire function must be constant in the complex domain according to Liouville's theorem. Therefore, the choice of activation functions is the main challenge for CVNNs. Two kinds of complex-valued activation functions have been presented in previous studies, i.e., real-imaginary-type activation functions[34] and amplitude-phase-type activation functions.[23,25] Many works have been done to study the design procedure of associative memories based on CVNNs with amplitude-phase-type activation functions[21–25] and the convergence of CVNNs with real-imaginary-type activation functions.[6,34] To the best of the authors' knowledge, less associative memory results based on CVNNs with real-imaginary-type activation functions have been published. In fact, this is a very meaningful topic. First, the real-imaginary-type activation functions are suitable for dealing with complex information that must have symmetry concerning, or a cer-

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†Corresponding author. E-mail: hyj0507@zjut.edu.cn

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tain special meaning on the real and imaginary axes. Second, the real parts and the imaginary parts of activation functions can be some functions with multiple constant values, which is helpful in realizing the coexistence of multiple equilibrium points of the neural networks. Moreover, one of the advantages of using the real-imaginary-type activation functions is that the whole CVRNNs can be analyzed by dealing with real and imaginary parts separately and independently, which reduces the complexity of computation and analysis.

In electronic implementation of analog neural networks, time delays always exist due to the transmission of signal and the finite switching speed of amplifiers. The existence of time delays may lead to instability and oscillation in a neural network. Therefore, it is necessary to introduce delays into neural network models to synthesize the associative memories.

Motivated by the above discussions, the main goal of this paper is to investigate the design procedure of auto-associative memories based on complex-valued neural networks with real-imaginary-type activation functions and constant delays. The first main work is that we construct a class of the real-imaginary-type activation functions, which are piecewise linear functions corresponding to their real parts and imaginary parts, respectively. Secondly, four lemmas are presented to track the dynamics of solution flows of CVNNs. Based on these lemmas, global exponential stability criteria of CVNNs are established, which depend on external inputs of neural networks. Moreover, one high-capacity design procedure is proposed for synthesizing auto-associative memories. There are two difficulties on analysis and design of high-capacity auto-associative memories based on CVNNs with the real-imaginary-type activation functions.

(i) One difficulty is how to establish conditions to make neuron states accessible to the regions corresponding to external inputs. It is a key problem to obtain auto-associative memories.

(ii) Another difficulty is how to deal with the coupling relation between real and imaginary parts of CVNNs in order to study the stability of equilibrium points.

Compared with the existing relevant results, the main advantages of this paper include the following two points.

i) The design procedure can retrieve the stored patterns by feeding probes via external inputs rather than initial conditions, which results in eliminating spurious equilibria.

ii) In applications, the dimension of the designed CVNNs is smaller than that of RVNNs. Under the same dimension, the storage capacity of the designed CVNNs is larger than that of RVNNs.

The remainder of this paper consists of the following sections. Section 2 describes the problem formulations. The main results are presented in Section 3. Section 4 gives an example to show the effectiveness of the obtained results. Finally, in Section 5, the conclusion is drawn.

2. Preliminaries and problem formulations

Denote $\{-1,1\}^{2n}$ as the set of $2n$-dimensional bipolar vectors, i.e., $\{-1,1\}^{2n}=\{x\in\mathbb{R}^{2n}, x=(x_1,x_2,\ldots,x_{2n})^{\mathrm{T}}, x_i=1 \text{ or } -1, i=1,2,\ldots,2n\}$. Denote i as the imaginary unit, that is $\mathrm{i}=\sqrt{-1}$. We are now in a position to address the following design problem.

Design problem Given m $(m\le 2^{2n})$ vectors $\alpha^{(1)}$, $\alpha^{(2)},\ldots,\alpha^{(m)}$, where $\alpha^{(k)}\in\{-1,1\}^{2n}$, $k=1,2,\ldots,m$, design an auto-associative memory based on CVNNs such that if $(\alpha_1^{(k)}+\mathrm{i}\alpha_2^{(k)},\alpha_3^{(k)}+\mathrm{i}\alpha_4^{(k)},\ldots,\alpha_{2n-1}^{(k)}+\mathrm{i}\alpha_{2n}^{(k)})^{\mathrm{T}}$ is fed to the associative memory from its input as a probe, then the output vector of the neural network converges to corresponding pattern $(\alpha_1^{(k)}+\mathrm{i}\alpha_2^{(k)},\alpha_3^{(k)}+\mathrm{i}\alpha_4^{(k)},\ldots,\alpha_{2n-1}^{(k)}+\mathrm{i}\alpha_{2n}^{(k)})^{\mathrm{T}}$, $k\in\{1,2,\ldots,m\}$.

Consider a complex-valued neural network with multiple delays described by

$$\dot{z}_i(t)=-d_iz_i(t)+\sum_{j=1}^{n}a_{ij}f_j(z_j(t))+\sum_{j=1}^{n}b_{ij}f_j(z_j(t-\tau_{ij}))+I_i, \tag{1}$$

where $z(t)=(z_1(t),z_2(t),\ldots,z_n(t))^{\mathrm{T}}\in\mathbb{C}^n$ is the neuron state vector; $D=\mathrm{diag}(d_1,d_2,\ldots,d_n)$ is a diagonal matrix, $d_i\in\mathbb{R}$ with $d_i>0$ represents the self-inhibition with which the i-th neuron will reset its potential to the resting state in isolations when disconnected from the network; $A=(a_{ij})_{n\times n}\in\mathbb{C}^{n\times n}$ and $B=(b_{ij})_{n\times n}\in\mathbb{C}^{n\times n}$ are the connection weight matrices those are not assumed to be symmetric; τ_{ij} is the constant delay with $0<\tau_{ij}\le\tau$ ($\tau=\max_{1\le i,j\le n}\{\tau_{ij}\}$); $f(z(t))=(f_1(z_1(t)),f_2(z_2(t)),\ldots,f_n(z_n(t)))^{\mathrm{T}}:\mathbb{C}^n\to\mathbb{C}^n$ denotes the neuron activation function; $I=(I_1,I_2,\ldots,I_n)^{\mathrm{T}}\in\mathbb{C}^n$ stands for the external constant input.

Let $|z|=\sqrt{z^*z}$, where $*$ indicates the conjugate transpose. The initial state associated with Eq. (1) is assumed to be $\phi(s)=(\phi_1(s),\phi_2(s),\ldots,\phi_n(s))^{\mathrm{T}}:[-\tau,0]\to\mathbb{C}^n$. Denote the norm of ϕ as

$$\|\phi\|=\max_{1\le i\le n}\{\sup_{s\in[-\tau,0]}|\phi_i(s)|\}.$$

In order to synthesize the auto-associative memory in the design problem of this paper, the following hypothesis about the activation function is needed.

Hypothesis 1 Suppose the activation function can be expressed by

$$f_j(z)=f_j^R(\mathrm{Re}(z))+\mathrm{i}f_j^I(\mathrm{Im}(z)),$$

where

$$f_j^R(\mathrm{Re}(z))=\begin{cases}1, & \mathrm{Re}(z)\in(1,+\infty),\\ \mathrm{Re}(z), & \mathrm{Re}(z)\in[-1,1],\\ -1, & \mathrm{Re}(z)\in(-\infty,-1),\end{cases}$$

$$f_j^I(\mathrm{Im}(z))=\begin{cases}1, & \mathrm{Im}(z)\in(1,+\infty),\\ \mathrm{Im}(z), & \mathrm{Im}(z)\in[-1,1],\\ -1, & \mathrm{Im}(z)\in(-\infty,-1),\end{cases}$$

with $j=1,2,\ldots,n$.

Since the real part and imaginary part of the activation function under Hypothesis 1 have individual properties, we can separate system (1) into its real and imaginary parts. By separating the state, the connection weight, the activation function, and the external input, we have that, $x_i(t)$ and $y_i(t)$ are the real part and the imaginary part of $z_i(t)$, respectively; a_{ij}^{R} and a_{ij}^{I} are the real part and the imaginary part of a_{ij}, respectively; b_{ij}^{R} and b_{ij}^{I} are the real part and the imaginary part of b_{ij}, respectively; I_i^{R} and I_i^{I} are the real part and the imaginary part of I_i, respectively. Therefore, the system (1) can be rewritten as follows:

$$\begin{aligned}\dot{x}_i(t) = & -d_i x_i(t) + \sum_{j=1}^{n}[a_{ij}^{\mathrm{R}} f_j^{\mathrm{R}}(x_j(t)) - a_{ij}^{\mathrm{I}} f_j^{I}(y_j(t)) \\ & + b_{ij}^{\mathrm{R}} f_j^{\mathrm{R}}(x_j(t-\tau_{ij})) - b_{ij}^{\mathrm{I}} f_j^{\mathrm{I}}(y_j(t-\tau_{ij}))] + I_i^{\mathrm{R}}, \quad (2a)\\ \dot{y}_i(t) = & -d_i y_i(t) + \sum_{j=1}^{n}[a_{ij}^{\mathrm{R}} f_j^{\mathrm{I}}(y_j(t)) + a_{ij}^{\mathrm{I}} f_j^{\mathrm{R}}(x_j(t)) \\ & + b_{ij}^{\mathrm{R}} f_j^{\mathrm{I}}(y_j(t-\tau_{ij})) + b_{ij}^{\mathrm{I}} f_j^{\mathrm{R}}(x_j(t-\tau_{ij}))] + I_i^{\mathrm{I}}. \quad (2b)\end{aligned}$$

The definition of the exponential stability is given as follows.

Definition 1 The equilibrium point $\bar{z}$ of the system (1) is said to be globally exponentially stable, if there exist constants $\alpha > 0$, $\beta > 0$ such that for any $t \geq 0$,

$$\|z(t;\phi) - \bar{z}\| \leq \beta \|\phi - \bar{z}\| \, \mathrm{e}^{-\alpha t},$$

where $z(t;\phi)$ is the solution of system (1) with any initial condition $\phi(s)$, $s \in [-\tau, 0]$.

In order to obtain the main results, we need the following lemma.

Lemma 1 (see Ref. [35]): Let $\mathbb{D}$ be a bounded and closed set in $\mathbb{R}^n$, and H be a mapping on complete matric space $(\mathbb{D}, \|\cdot\|)$, where for any $x, y \in \mathbb{D}$, $\|x-y\| = \max_{1\leq i\leq n}\{|x_i - y_i|\}$ is measurement in $\mathbb{D}$. If $H(\mathbb{D}) \subset \mathbb{D}$ and there exists a positive constant $\alpha < 1$ such that for any $x, y \in \mathbb{D}$, $\|H(x) - H(y)\| \leq \alpha\|x-y\|$, then there exists a unique $\bar{x} \in \mathbb{D}$ such that $H(\bar{x}) = \bar{x}$.

3. Main results

The purpose of this paper is to design an auto-associative memory based on CVNNs. Firstly, we will introduce four lemmas according to Ref. [29]. Through these four lemmas, the dynamics of each state component of system (1) will be tracked, which depend on external inputs. Denote

$$\Delta_i = \sum_{j=1, j\neq i}^{n} |a_{ij}^{\mathrm{R}}| + \sum_{j=1}^{n}[|a_{ij}^{\mathrm{I}}| + |b_{ij}^{\mathrm{R}}| + |b_{ij}^{\mathrm{I}}|].$$

Lemma 2 The state component $z_i(t)$ of system (1) will flow into the region $\{z_i = x_i + \mathrm{i}y_i | x_i \in (-\infty, -1),\ y_i \in (-\infty, -1)\}$, and stay in it afterwards, if for any $i \in \{1, 2, \ldots, n\}$,

$$I_i^{\mathrm{R}} < -|-d_i + a_{ii}^{\mathrm{R}}| - \Delta_i,\ \ I_i^{\mathrm{I}} < -|-d_i + a_{ii}^{\mathrm{R}}| - \Delta_i. \quad (3)$$

Lemma 3 The state component $z_i(t)$ of system (1) will flow into the region $\{z_i = x_i + \mathrm{i}y_i | x_i \in (-\infty, -1),\ y_i \in (1, +\infty)\}$, and stay in it afterwards, if for any $i \in \{1, 2, \ldots, n\}$,

$$I_i^{\mathrm{R}} < -|-d_i + a_{ii}^{\mathrm{R}}| - \Delta_i,\ \ I_i^{\mathrm{I}} > |-d_i + a_{ii}^{\mathrm{R}}| + \Delta_i. \quad (4)$$

Lemma 4 The state component $z_i(t)$ of system (1) will flow into the region $\{z_i = x_i + \mathrm{i}y_i | x_i \in (1, +\infty),\ y_i \in (-\infty, -1)\}$, and stay in it afterwards, if for any $i \in \{1, 2, \ldots, n\}$,

$$I_i^{\mathrm{R}} > |-d_i + a_{ii}^{\mathrm{R}}| + \Delta_i,\ \ I_i^{\mathrm{I}} < -|-d_i + a_{ii}^{\mathrm{R}}| - \Delta_i. \quad (5)$$

Lemma 5 The state component $z_i(t)$ of system (1) will flow into the region $\{z_i = x_i + \mathrm{i}y_i | x_i \in (1, +\infty),\ y_i \in (1, +\infty)\}$, and stay in it afterwards, if for any $i \in \{1, 2, \ldots, n\}$,

$$I_i^{\mathrm{R}} > |-d_i + a_{ii}^{\mathrm{R}}| + \Delta_i,\ \ I_i^{\mathrm{I}} > |-d_i + a_{ii}^{\mathrm{R}}| + \Delta_i. \quad (6)$$

The proofs of Lemmas 2–5 are similar to the proof of Lemma 2 in Ref. [29], and thus are omitted here for brevity.

Denote four index subsets as follows:

$$\begin{aligned} N_1 &= \{i | i \in \{1, 2, \ldots, n\},\ I_i^{\mathrm{R}} < -|-d_i + a_{ii}^{\mathrm{R}}| - \Delta_i,\ I_i^{\mathrm{I}} < -|-d_i + a_{ii}^{\mathrm{R}}| - \Delta_i\}, \\ N_2 &= \{i | i \in \{1, 2, \ldots, n\},\ I_i^{\mathrm{R}} < -|-d_i + a_{ii}^{\mathrm{R}}| - \Delta_i,\ I_i^{\mathrm{I}} > |-d_i + a_{ii}^{\mathrm{R}}| + \Delta_i\}, \\ N_3 &= \{i | i \in \{1, 2, \ldots, n\},\ I_i^{\mathrm{R}} > |-d_i + a_{ii}^{\mathrm{R}}| + \Delta_i,\ I_i^{\mathrm{I}} < -|-d_i + a_{ii}^{\mathrm{R}}| - \Delta_i\}, \\ N_4 &= \{i | i \in \{1, 2, \ldots, n\},\ I_i^{\mathrm{R}} > |-d_i + a_{ii}^{\mathrm{R}}| + \Delta_i,\ I_i^{\mathrm{I}} > |-d_i + a_{ii}^{\mathrm{R}}| + \Delta_i\}. \end{aligned}$$

It is obvious that $N_1 \cap N_2 \cap N_3 \cap N_4 = \emptyset$. Denote

$$\begin{aligned}\Omega = & \prod_{i\in N_1}\{z_i = x_i + \mathrm{i}y_i | x_i \in (-\infty, -1),\ y_i \in (-\infty, -1)\} \\ & \times \prod_{i\in N_2}\{z_i = x_i + \mathrm{i}y_i | x_i \in (-\infty, -1),\ y_i \in (1, +\infty)\} \\ & \times \prod_{i\in N_3}\{z_i = x_i + \mathrm{i}y_i | x_i \in (1, +\infty),\ y_i \in (-\infty, -1)\} \\ & \times \prod_{i\in N_4}\{z_i = x_i + \mathrm{i}y_i | x_i \in (1, +\infty),\ y_i \in (1, +\infty)\}.\end{aligned}$$

Now, we will show the existence of the globally exponen-

tially stable equilibrium point of system (1).

Theorem 1 Suppose $N_1 \cup N_2 \cup N_3 \cup N_4 = \{1,2,\ldots,n\}$. For any $i \in \{1,2,\ldots,n\}$, if

$$d_i > \sum_{j=1}^{n}(|a_{ij}^{\mathrm{R}}| + |a_{ij}^{\mathrm{I}}| + |b_{ij}^{\mathrm{R}}| + |b_{ij}^{\mathrm{I}}|), \tag{7}$$

then there exists a globally exponentially stable equilibrium point for system (1), and the stable output vector belongs to $\{1+\mathrm{i}, 1-\mathrm{i}, -1+\mathrm{i}, -1-\mathrm{i}\}^n$.

Proof We will study the existence, uniqueness, and global exponential stability of the equilibrium point for system (1) in two steps.

Step 1 We show that there exists a unique equilibrium point for system (1). Applying Lemmas 2–5, for any initial state $\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^{\mathrm{T}}$, $\exists\, T > 0$, when $t > T$, we have the following four results.

1) The real part of $z_i(t)$ will flow into $(-\infty, -1)$, the imaginary part of $z_i(t)$ will flow into $(-\infty, -1)$ for $i \in N_1$.

2) The real part of $z_i(t)$ will flow into $(-\infty, -1)$, the imaginary part of $z_i(t)$ will flow into $(1, +\infty)$ for $i \in N_2$.

3) The real part of $z_i(t)$ will flow into $(1, +\infty)$, the imaginary part of $z_i(t)$ will flow into $(-\infty, -1)$ for $i \in N_3$.

4) The real part of $z_i(t)$ will flow into $(1, +\infty)$, the imaginary part of $z_i(t)$ will flow into $(1, +\infty)$ for $i \in N_4$.

The neuron state will stay in them afterwards. That is, when the state $z(t)$ of system (1) flows into Ω, it will not escape from Ω. Therefore, we only need to prove the existence of a unique equilibrium point with initial state $\phi(s) \in \Omega$. Let $H(x) = (H_1(x), H_2(x), \ldots, H_n(x))^{\mathrm{T}}$ and $G(y) = (G_1(y), G_2(y), \cdots, G_n(y))^{\mathrm{T}}$, where

$$H_i(x) = \frac{1}{d_i}\Big[\sum_{j=1}^{n} a_{ij}^{\mathrm{R}} f_j^{\mathrm{R}}(x_j) - \sum_{j=1}^{n} a_{ij}^{\mathrm{I}} f_j^{\mathrm{I}}(y_j) + \sum_{j=1}^{n} b_{ij}^{\mathrm{R}} f_j^{\mathrm{R}}(x_j) - \sum_{j=1}^{n} b_{ij}^{\mathrm{I}} f_j^{\mathrm{I}}(y_j) + I_i^{\mathrm{R}}\Big], \tag{8}$$

$$G_i(y) = \frac{1}{d_i}\Big[\sum_{j=1}^{n} a_{ij}^{\mathrm{R}} f_j^{\mathrm{I}}(y_j) + \sum_{j=1}^{n} a_{ij}^{\mathrm{I}} f_j^{\mathrm{R}}(x_j) + \sum_{j=1}^{n} b_{ij}^{\mathrm{R}} f_j^{\mathrm{I}}(y_j) + \sum_{j=1}^{n} b_{ij}^{\mathrm{I}} f_j^{\mathrm{R}}(x_j) + I_i^{\mathrm{I}}\Big]. \tag{9}$$

For any $i \in N_1$, there exists a small enough constant $\varepsilon_1 > 0$ such that

$$d_i - a_{ii}^{\mathrm{R}} + \Delta_i + I_i^{\mathrm{R}} < -d_i\varepsilon_1,\ d_i - a_{ii}^{\mathrm{R}} + \Delta_i + I_i^{\mathrm{I}} < -d_i\varepsilon_1.$$

For any $i \in N_2$, there exists a small enough constant $\varepsilon_2 > 0$ such that

$$d_i - a_{ii}^{\mathrm{R}} + \Delta_i + I_i^{\mathrm{R}} < -d_i\varepsilon_2,\ -d_i + a_{ii}^{\mathrm{R}} - \Delta_i + I_i^{\mathrm{I}} > d_i\varepsilon_2.$$

For any $i \in N_3$, there exists a small enough constant $\varepsilon_3 > 0$ such that

$$-d_i + a_{ii}^{\mathrm{R}} - \Delta_i + I_i^{\mathrm{R}} > d_i\varepsilon_3,\ d_i - a_{ii}^{\mathrm{R}} + \Delta_i + I_i^{\mathrm{I}} < -d_i\varepsilon_3.$$

For any $i \in N_4$, there exists a small enough constant $\varepsilon_4 > 0$ such that

$$-d_i + a_{ii}^{\mathrm{R}} - \Delta_i + I_i^{\mathrm{R}} > d_i\varepsilon_4,\ -d_i + a_{ii}^{\mathrm{R}} - \Delta_i + I_i^{\mathrm{I}} > d_i\varepsilon_4.$$

Consider any one small enough positive constant ε with $0 < \varepsilon \ll \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$. Denote

$$\begin{aligned}\Omega_\varepsilon = &\prod_{i\in N_1}\{z_i = x_i + \mathrm{i}y_i | x_i \in [-1/\varepsilon, -1-\varepsilon],\ y_i \in [-1/\varepsilon, -1-\varepsilon]\}\prod_{i\in N_2}\{z_i = x_i + \mathrm{i}y_i | x_i \in [-1/\varepsilon, -1-\varepsilon],\ y_i \in [1+\varepsilon, 1/\varepsilon]\}\\ &\times\prod_{i\in N_3}\{z_i = x_i + \mathrm{i}y_i | x_i \in [1+\varepsilon, 1/\varepsilon],\ y_i \in [-1/\varepsilon, -1-\varepsilon]\}\prod_{i\in N_4}\{z_i = x_i + \mathrm{i}y_i | x_i \in [1+\varepsilon, 1/\varepsilon],\ y_i \in [1+\varepsilon, 1/\varepsilon]\}.\end{aligned}$$

It is obvious that Ω_ε is a bounded and closed set of Ω. When $x_i \in [-1/\varepsilon, -1-\varepsilon]$, it can be derived that

$$H_i(x) = \frac{1}{d_i}[-a_{ii}^{\mathrm{R}} + \sum_{j=1, j\neq i}^{n} a_{ij}^{\mathrm{R}} f_j^{\mathrm{R}}(x_j) - \sum_{j=1}^{n} a_{ij}^{\mathrm{I}} f_j^{\mathrm{I}}(y_j) + \sum_{j=1}^{n} b_{ij}^{\mathrm{R}} f_j^{\mathrm{R}}(x_j) - \sum_{j=1}^{n} b_{ij}^{\mathrm{I}} f_j^{\mathrm{I}}(y_j) + I_i^{\mathrm{R}}] \le \frac{1}{d_i}[-a_{ii}^{\mathrm{R}} + \Delta_i + I_i^{\mathrm{R}}] < -1-\varepsilon.$$

When $x_i \in [1+\varepsilon, 1/\varepsilon]$, it can be derived that

$$H_i(x) = \frac{1}{d_i}[a_{ii}^{\mathrm{R}} + \sum_{j=1, j\neq i}^{n} a_{ij}^{\mathrm{R}} f_j^{\mathrm{R}}(x_j) - \sum_{j=1}^{n} a_{ij}^{\mathrm{I}} f_j^{\mathrm{I}}(y_j) + \sum_{j=1}^{n} b_{ij}^{\mathrm{R}} f_j^{\mathrm{R}}(x_j) - \sum_{j=1}^{n} b_{ij}^{\mathrm{I}} f_j^{\mathrm{I}}(y_j) + I_i^{\mathrm{R}}] \ge \frac{1}{d_i}[a_{ii}^{\mathrm{R}} - \Delta_i + I_i^{\mathrm{R}}] > 1+\varepsilon.$$

Similarly, when $y_i \in [-1/\varepsilon, -1-\varepsilon]$, we have $G_i(y) < -1-\varepsilon$; when $y_i \in [1+\varepsilon, 1/\varepsilon]$, we have $G_i(y) > 1+\varepsilon$. Because ε is small enough, we get

$$H\left(\prod_{i\in N_1\cup N_2}[-1/\varepsilon, -1-\varepsilon] \times \prod_{i\in N_3\cup N_4}[1+\varepsilon, 1/\varepsilon]\right) \subset \prod_{i\in N_1\cup N_2}[-1/\varepsilon, -1-\varepsilon] \times \prod_{i\in N_3\cup N_4}[1+\varepsilon, 1/\varepsilon],$$

$$G\left(\prod_{i\in N_1\cup N_3}[-1/\varepsilon, -1-\varepsilon] \times \prod_{i\in N_2\cup N_4}[1+\varepsilon, 1/\varepsilon]\right) \subset \prod_{i\in N_1\cup N_3}[-1/\varepsilon, -1-\varepsilon] \times \prod_{i\in N_2\cup N_4}[1+\varepsilon, 1/\varepsilon].$$

For any

$$x^{(1)},\, x^{(2)} \in \prod_{i\in N_1\cup N_2}[-1/\varepsilon,-1-\varepsilon]\times\prod_{i\in N_3\cup N_4}[1+\varepsilon,1/\varepsilon],$$

$x^{(1)} = (x_1^{(1)},x_2^{(1)},\ldots,x_n^{(1)})^{\mathrm{T}}$, $x^{(2)} = (x_1^{(2)},x_2^{(2)},\ldots,x_n^{(2)})^{\mathrm{T}}$, it holds that $f_i^{\mathrm{R}}(x_i^{(1)}) - f_i^{\mathrm{R}}(x_i^{(2)}) = 0$ for $i = 1,2,\ldots,n$. Since we only discuss the existence of equilibrium points in Ω, when fixing $i \in \{1,2,\ldots,n\}$, we have $f_i^{\mathrm{I}}(y_i^{(1)}) - f_i^{\mathrm{I}}(y_i^{(2)}) = 0$ for any $y_i^{(1)}$, $y_i^{(2)}$. Choose $\alpha = 0.5$. We have $|H_i(x^{(1)}) - H_i(x^{(2)})| \le \alpha\|x^{(1)} - x^{(2)}\|$. Therefore, $\|H(x^{(1)}) - H(x^{(2)})\| \le \alpha\|x^{(1)} - x^{(2)}\|$. According to Lemma 1, there exists a unique fixed point $\bar{x}$ such that $H(\bar{x}) = \bar{x}$. By a similar analysis, there exists a unique fixed point $\bar{y}$ such that $G(\bar{y}) = \bar{y}$. Based on the above discussion, we can derive that $y = (y_1,y_2,\ldots,y_n)^{\mathrm{T}}$ in Eq. (8) and $x = (x_1,x_2,\cdots,x_n)^{\mathrm{T}}$ in Eq. (9) are allowed to be arbitrary. So, we can make $y = \bar{y}$ in Eq. (8) and $x = \bar{x}$ in Eq. (9). Then, $\bar{z} = \bar{x} + \mathrm{i}\bar{y}$ is a unique equilibrium point of system (1) in Ω_ε. By arbitrariness of ε, $\bar{z}$ is a unique equilibrium point of system (1) in $\mathbb{C}^n$.

Step 2 We investigate the global exponential stability of equilibrium point $\bar{z} = \bar{x} + \mathrm{i}\bar{y}$ of system (1). With translation $v(t) = z(t) - \bar{z}$, systems (2a) and (2b) become

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{Re}(v_i(t))] =\ & -d_i[\mathrm{Re}(v_i(t))] + \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}[f_j^{\mathrm{R}}(x_j(t)) - f_j^{\mathrm{R}}(\bar{x}_j)] - \sum_{j=1}^{n} a_{ij}^{\mathrm{I}}[f_j^{\mathrm{I}}(y_j(t)) - f_j^{\mathrm{I}}(\bar{y}_j)] \\ & + \sum_{j=1}^{n} b_{ij}^{\mathrm{R}}[f_j^{\mathrm{R}}(x_j(t-\tau_{ij})) - f_j^{\mathrm{R}}(\bar{x}_j)] - \sum_{j=1}^{n} b_{ij}^{\mathrm{I}}[f_j^{\mathrm{I}}(y_j(t-\tau_{ij})) - f_j^{\mathrm{I}}(\bar{y}_j)], \qquad (10a)\\ \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{Im}(v_i(t))] =\ & -d_i[\mathrm{Im}(v_i(t))] + \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}[f_j^{\mathrm{I}}(y_j(t)) - f_j^{\mathrm{I}}(\bar{y}_j)] + \sum_{j=1}^{n} a_{ij}^{\mathrm{I}}[f_j^{\mathrm{R}}(x_j(t)) - f_j^{\mathrm{R}}(\bar{x}_j)] \\ & + \sum_{j=1}^{n} b_{ij}^{\mathrm{R}}[f_j^{\mathrm{I}}(y_j(t-\tau_{ij})) - f_j^{\mathrm{I}}(\bar{y}_j)] + \sum_{j=1}^{n} b_{ij}^{\mathrm{I}}[f_j^{\mathrm{R}}(x_j(t-\tau_{ij})) - f_j^{\mathrm{R}}(\bar{x}_j)], \qquad (10b)\end{aligned}$$

where $v = (v_1,v_2,\ldots,v_n)^{\mathrm{T}}$. For each $i \in \{1,2,\ldots,n\}$, consider the single-variable function

$$\begin{aligned}F_i(\zeta) =\ & -\zeta + d_i - \sum_{j=1}^{n}|a_{ij}^{\mathrm{R}}| - \sum_{j=1}^{n}|a_{ij}^{\mathrm{I}}| - \sum_{j=1}^{n}|b_{ij}^{\mathrm{R}}|\,\mathrm{e}^{\zeta\tau_{ij}} \\ & - \sum_{j=1}^{n}|b_{ij}^{\mathrm{I}}|\,\mathrm{e}^{\zeta\tau_{ij}}.\end{aligned}$$

From Eq. (7), we can get $F_i(0) > 0$, and there exists a positive constant λ such that $F_i(\lambda) > 0$. Consider $w_i(t) = \mathrm{e}^{\lambda t}|\mathrm{Re}(v_i(t))|$, $u_i(t) = \mathrm{e}^{\lambda t}|\mathrm{Im}(v_i(t))|$, $i = 1,2,\ldots,n$. Let $\delta > 1$ and

$$K = \max_{1\le i\le n}\{\sup_{s\in[-\tau,0]}[|\mathrm{Re}(\phi_i(s)) - \bar{x}_i|^2 + |\mathrm{Im}(\phi_i(s)) - \bar{y}_i|^2]\}.$$

Therefore, $w_i^2(t) + u_i^2(t) < K\delta$ for $t \in [-\tau,0]$. We will show that $w_i^2(t) + u_i^2(t) < K\delta$ for all $t > 0$, $i = 1,2,\ldots,n$. Suppose this is not true. Then there exist $k \in \{1,2,\ldots,n\}$, $t_1 > 0$ such that $w_i^2(t) + u_i^2(t) < K\delta$ for $t \in [-\tau,t_1]$, $i \ne k$, $w_k^2(t) + u_k^2(t) < K\delta$ for $t \in [-\tau,t_1)$, $w_k^2(t_1) + u_k^2(t_1) = K\delta$, and $D^+(w_k^2(t_1) + u_k^2(t_1)) \ge 0$, where D^+ denotes the upper right Dini-derivative. From Eqs. (10a) and (10b), we compute that

$$\begin{aligned}D^+|\mathrm{Re}(v_k(t_1))| =\ & \mathrm{sgn}(\mathrm{Re}(v_k(t_1)))\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{Re}(v_k(t_1))] \\ \le\ & -d_k|\mathrm{Re}(v_k(t_1))| + \sum_{j=1}^{n}|a_{kj}^{\mathrm{R}}||\mathrm{Re}(v_j(t_1))| + \sum_{j=1}^{n}|a_{kj}^{\mathrm{I}}||\mathrm{Im}(v_j(t_1))| \\ & + \sum_{j=1}^{n}|b_{kj}^{\mathrm{R}}||\mathrm{Re}(v_j(t_1-\tau_{kj}))| + \sum_{j=1}^{n}|b_{kj}^{\mathrm{I}}||\mathrm{Im}(v_j(t_1-\tau_{kj}))|, \\ D^+|\mathrm{Im}(v_k(t_1))| =\ & \mathrm{sgn}(\mathrm{Im}(v_k(t_1)))\frac{d}{dt}[\mathrm{Im}(v_k(t_1))] \\ \le\ & -d_k|\mathrm{Im}(v_k(t_1))| + \sum_{j=1}^{n}|a_{kj}^{\mathrm{R}}||\mathrm{Im}(v_j(t_1))| + \sum_{j=1}^{n}|a_{kj}^{\mathrm{I}}||\mathrm{Re}(v_j(t_1))| \\ & + \sum_{j=1}^{n}|b_{kj}^{\mathrm{R}}||\mathrm{Im}(v_j(t_1-\tau_{kj}))| + \sum_{j=1}^{n}|b_{kj}^{\mathrm{I}}||\mathrm{Re}(v_j(t_1-\tau_{kj}))|.\end{aligned}$$

Hence,

$$\begin{aligned}D^+[w_k^2(t_1) + u_k^2(t_1)] =\ & 2w_k(t_1)D^+w_k(t_1) + 2u_k(t_1)D^+u_k(t_1) = 2w_k(t_1)[\lambda\,\mathrm{e}^{\lambda t_1}|\mathrm{Re}(v_k(t_1))| + \mathrm{e}^{\lambda t_1}D^+|\mathrm{Re}(v_k(t_1))|] \\ & + 2u_k(t_1)[\lambda\,\mathrm{e}^{\lambda t_1}|\mathrm{Im}(v_k(t_1))| + \mathrm{e}^{\lambda t_1}D^+|\mathrm{Im}(v_k(t_1))|]\end{aligned}$$

$$
\begin{aligned}
&\le 2[(\lambda-d_k)w_k^2(t_1)+\sum_{j=1}^{n}|a_{kj}^{\mathrm{R}}|w_k(t_1)w_j(t_1)+\sum_{j=1}^{n}|a_{kj}^{\mathrm{I}}|w_k(t_1)u_j(t_1)\\
&\quad+\sum_{j=1}^{n}|b_{kj}^{\mathrm{R}}|\mathrm{e}^{\lambda\tau_{kj}}w_k(t_1)w_j(t_1-\tau_{kj})+\sum_{j=1}^{n}|b_{kj}^{\mathrm{I}}|\mathrm{e}^{\lambda\tau_{kj}}w_k(t_1)u_j(t_1-\tau_{kj})\\
&\quad+(\lambda-d_k)u_k^2(t_1)+\sum_{j=1}^{n}|a_{kj}^{\mathrm{R}}|u_k(t_1)u_j(t_1)+\sum_{j=1}^{n}|a_{kj}^{\mathrm{I}}|u_k(t_1)w_j(t_1)\\
&\quad+\sum_{j=1}^{n}|b_{kj}^{\mathrm{R}}|\mathrm{e}^{\lambda\tau_{kj}}u_k(t_1)u_j(t_1-\tau_{kj})+\sum_{j=1}^{n}|b_{kj}^{\mathrm{I}}|\mathrm{e}^{\lambda\tau_{kj}}u_k(t_1)w_j(t_1-\tau_{kj})]\\
&\le 2[\lambda-d_k+\sum_{j=1}^{n}|a_{kj}^{\mathrm{R}}|+\sum_{j=1}^{n}|a_{kj}^{\mathrm{I}}|+\sum_{j=1}^{n}|b_{kj}^{\mathrm{R}}|\mathrm{e}^{\lambda\tau_{kj}}+\sum_{j=1}^{n}|b_{kj}^{\mathrm{I}}|\mathrm{e}^{\lambda\tau_{kj}}]K\delta\\
&< 0.
\end{aligned}
$$

This yields a contradiction to $D^+(w_k^2(t_1)+u_k^2(t_1))\ge 0$. So, $w_i^2(t)+u_i^2(t)<K\delta$ for all $t>0$, $i=1,2,\ldots,n$. By taking $\delta\to 1^+$, we obtain $w_i^2(t)+u_i^2(t)\le K$, which leads to

$$
|x_i(t)-\bar{x}_i|^2+|y_i(t)-\bar{y}_i|^2\le \mathrm{e}^{-2\lambda t}\max_{1\le i\le n}\{\sup_{s\in[-\tau,0]}[|\mathrm{Re}(\phi_i(s))-\bar{x}_i|^2+|\mathrm{Im}(\phi_i(s))-\bar{y}_i|^2]\}
$$

for any $t>0$. Therefore, $z(t)$ converges to $\bar{z}$ exponentially. Since the equilibrium point $\bar{z}$ lies in Ω, it is easy to derive the stable output vector of system (1) belonging to $\{1+\mathrm{i},1-\mathrm{i},-1+\mathrm{i},-1-\mathrm{i}\}^n$. □

Remark 1 Among the conditions of Theorem 1, $N_1\cup N_2\cup N_3\cup N_4=\{1,2,\ldots,n\}$ determines the range of the external inputs, which ensures neuron states can flow into Ω according to Lemmas 2–5. In fact, Ω is the attractive region of the globally exponentially stable equilibrium point of system (1).

Remark 2 Equilibrium states of neurons of system (1) depend on external inputs of the neural network. Therefore, equilibrium states of system (1) are different when external inputs are different.

Remark 3 According to Theorem 1, we can realize associative memories based on CVNNs. We need to use the information about a lot of pairwise vectors of desired patterns and inputs to design a recurrent neural network. When the retrieval probe is determined as the input of the neural network, the output vector will converge to the desired pattern. Based on this idea, a design process of associative memories will be put forward after the following corollaries.

For the case of system (1) without delay, i.e., $b_{ij}=0$, $i,j=1,2,\ldots,n$, denote

$$
\begin{aligned}
&\Lambda_i=|-d_i+a_{ii}^{\mathrm{R}}|+\sum_{j=1,j\ne i}^{n}|a_{ij}^{\mathrm{R}}|+\sum_{j=1}^{n}|a_{ij}^{\mathrm{I}}|,\\
&N_5=\{i|i\in\{1,2,\ldots,n\},\ I_i^{\mathrm{R}}<-\Lambda_i,\ I_i^{\mathrm{I}}<-\Lambda_i\},\\
&N_6=\{i|i\in\{1,2,\ldots,n\},\ I_i^{\mathrm{R}}<-\Lambda_i,\ I_i^{\mathrm{I}}>\Lambda_i\},\\
&N_7=\{i|i\in\{1,2,\ldots,n\},\ I_i^{\mathrm{R}}>\Lambda_i,\ I_i^{\mathrm{I}}<-\Lambda_i\},\\
&N_8=\{i|i\in\{1,2,\ldots,n\},\ I_i^{\mathrm{R}}>\Lambda_i,\ I_i^{\mathrm{I}}>\Lambda_i\}.
\end{aligned}
$$

We can derive the following corollary.

Corollary 1 Suppose $N_5\cup N_6\cup N_7\cup N_8=\{1,2,\ldots,n\}$. For any $i\in\{1,2,\ldots,n\}$, if

$$
d_i>\sum_{j=1}^{n}(|a_{ij}^{\mathrm{R}}|+|a_{ij}^{\mathrm{I}}|),\tag{11}
$$

then there exists a globally exponentially stable equilibrium point for system (1) without delay, and the stable output vector belongs to $\{1+\mathrm{i},1-\mathrm{i},-1+\mathrm{i},-1-\mathrm{i}\}^n$.

In particular, when $y_i=0$, $a_{ij}^{\mathrm{I}}=0$, $b_{ij}^{\mathrm{I}}=0$, $f_j^{\mathrm{I}}(\cdot)=0$, $I_i^{\mathrm{I}}=0$, system (1) can be rewritten as the following RVNNs,

$$
\dot{x}_i(t)=-d_ix_i(t)+\sum_{i=1}^{n}[a_{ij}^{\mathrm{R}}f_j^{\mathrm{R}}(x_j(t))+b_{ij}^{\mathrm{R}}f_j^{\mathrm{R}}(x_j(t-\tau_{ij}))]+I_i^{\mathrm{R}}.\tag{12}
$$

Denote

$$
\begin{aligned}
&\Gamma_i=|-d_i+a_{ii}^{\mathrm{R}}|+\sum_{j=1,j\ne i}^{n}|a_{ij}^{\mathrm{R}}|+\sum_{j=1}^{n}|b_{ij}^{\mathrm{R}}|,\\
&N_9=\{i|i\in\{1,2,\ldots,n\},\ I_i^{\mathrm{R}}<-\Gamma_i\},\\
&N_{10}=\{i|i\in\{1,2,\ldots,n\},\ I_i^{\mathrm{R}}>\Gamma_i\}.
\end{aligned}
$$

We can derive the following corollary.

Corollary 2 Suppose $N_9\cup N_{10}=\{1,2,\ldots,n\}$. For any $i\in\{1,2,\ldots,n\}$, if

$$
d_i>\sum_{j=1}^{n}(|a_{ij}^{\mathrm{R}}|+|b_{ij}^{\mathrm{R}}|),\tag{13}
$$

then there exists a globally exponentially stable equilibrium point for system (12), and the stable output vector belongs to $\{1,-1\}^n$.

Based on the above discussion, one can obtain the following design procedure of auto-associative memory.

Design procedure

(I) Denote m vectors of the desired patterns as $\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(m)}$, where $\alpha^{(k)}\in\{-1,1\}^{2n}$, $k=1,2,\ldots,m$.

(II) Determine D, A, and B such that equation (7) and $d_i < a_{ii}^{\mathrm{R}} + 0.5$ $(i = 1, 2, \ldots, n)$ hold.

(III) Synthesize recurrent neural network (1) with the parameters D, A, B.

In the application of associative memory, the parameters of CVNNs are designed according to the information about pairwise vectors of desired patterns and inputs. In this paper, the input vectors are designed to be the same as the vectors of desired patterns. According to the above design procedure, the parameters D, A, B can be designed. Hence, CVNNs can be derived. Once the neural networks are designed, the parameters will not change. The stable output vectors are only relevant to the input vectors. If one input is fed as a probe, the corresponding desired pattern will be memorized. It can be noticed that the choice of the parameters of neural networks depends on some inequalities. Therefore, the parameters can be changed in a certain range, which do not affect the result of associative memories.

Remark 4 In the design procedure, the conditions $d_i < a_{ii}^{\mathrm{R}} + 0.5$ $(i = 1, 2, \ldots, n)$ are necessary. From Eq. (7) and $d_i < a_{ii} + 0.5$ $(i = 1, 2, \ldots, n)$, we can derive that equation (3) holds for $I_i^R = \alpha_{2i-1}^{(k)} = -1$ and $I_i^I = \alpha_{2i}^{(k)} = -1$; equation (4) holds for $I_i^R = \alpha_{2i-1}^{(k)} = -1$ and $I_i^I = \alpha_{2i}^{(k)} = 1$; equation (5) holds for $I_i^R = \alpha_{2i-1}^{(k)} = 1$ and $I_i^I = \alpha_{2i}^{(k)} = -1$; equation (6) holds for $I_i^R = \alpha_{2i-1}^{(k)} = 1$ and $I_i^I = \alpha_{2i}^{(k)} = 1$. That is $N_1 \cup N_2 \cup N_3 \cup N_4 = \{1, 2, \ldots, n\}$.

Remark 5 Compared with the existing design procedures of auto-associative memories based on CVNNs, there exist the following advantages for the design procedure in this paper.

1) In Refs. [21]–[26], activation functions were chosen as amplitude-phase-type functions. However, when dealing with complex information that must have symmetry concerning, or a certain special meaning on the real and imaginary axes, the design procedure with real-imaginary-type activation function can work well.[2]

2) In this paper, the design procedure of auto-associative memories depends on external inputs of neural networks instead of initial states of neurons, which ensures the global stability of CVNNs. The method does not yield spurious memory patterns. It is superior to the methods dependent on initial states of neurons (e.g., Refs. [14], [29], and [31]).

Remark 6 Compared with the existing design procedures of auto-associative memories based on RVNNs, there exist the following advantages for the design procedure in this paper.

a) A CVNN originates the fact that the multiplication of the synaptic weights can reduce the degree of freedom in learning and self-organization, in comparison with a double-dimensional RVNN.[36]

b) Due to two-dimensionality of the complex plane, RVNNs can only be seen as one special case of CVNNs.

c) As regards the memory patterns presented by bipolar vectors, n-dimensional CVNNs can store 2^{2n} patterns. n-dimensional RVNNs can store 2^n patterns. It is obvious that the storage capacity of CVNNs is larger than that of RVNNs.

d) When memorizing the same pattern, the number of dimensions of our designed CVNNs is smaller than that of RVNNs. This can help to reduce the curse of dimensionality of RVNNs.

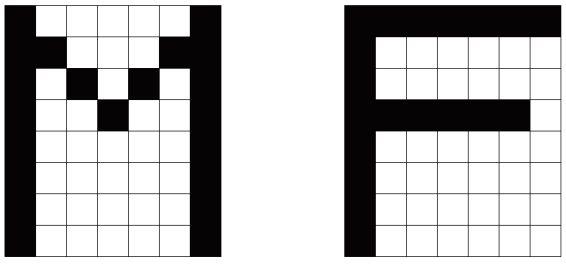
e) In Ref. [17], the design procedure of associative memories depended on external inputs and the initial states were fixed at zero. Whereas, our criteria guarantee the existence of global exponential stability for any initial states. Therefore, our method is less conservative than Ref. [17].

f) Compared with Refs. [15] and [20], our method does not strictly limit the parameters of neural networks. For example, the self-inhibition is restricted to be equal to 1 in Refs. [15] and [20]. There do not exist these limitations in this paper.

4. Illustrative examples

In this section, we will give one example to illustrate the effectiveness of our results.

Example 1 Consider two to-be-stored patterns "M" and "F", which are represented by two (8×7)-pixel images (black pixel = -1; white pixel = 1), as shown in Fig. 1.



$\alpha^{(1)} = I^{(1)}$ $\alpha^{(2)} = I^{(2)}$

Fig. 1. Desired output patterns.

Since the external inputs are equal to the desired patterns in the auto-associative memory, we can give the input vectors as follows:

$$
\begin{aligned}
I^{(1)} = [&-1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, -1-\mathrm{i}, -1+\mathrm{i}, 1+\mathrm{i}, -1-\mathrm{i}, -1+\mathrm{i}, -1+\mathrm{i}, -1+\mathrm{i},\\
&-1-\mathrm{i}, 1+\mathrm{i}, -1+\mathrm{i}, 1-\mathrm{i}, -1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, -1-\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, 1-\mathrm{i},\\
&-1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, -1-\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, 1-\mathrm{i}]^{\mathrm{T}},\\
I^{(2)} = [&-1-\mathrm{i}, -1-\mathrm{i}, -1-\mathrm{i}, -1-\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, -1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i},\\
&1-\mathrm{i}, -1-\mathrm{i}, -1-\mathrm{i}, -1+\mathrm{i}, -1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, 1-\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i},\\
&-1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, 1-\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i}]^{\mathrm{T}}.
\end{aligned}
$$

According to the design procedure mentioned in Section 3, we consider neural network (1) with 28 neurons, and the parameters are determined as follows for $i, j = 1, 2, \ldots, 28$,

$$d_i = 0.6,\ a_{ij} = \begin{cases} 0.2 + 0.003\mathrm{i}, & i = j, \\ 0.003 + 0.003\mathrm{i}, & i \neq j, \end{cases}\ b_{ij} = 0.003 + 0.003\mathrm{i}.$$

Let $\tau_{ij} = 0.1$, $f_j(\xi) = (|\xi + 1| - |\xi - 1|)/2$.

When the input probe is set to be $I^{(1)}$, the desired pattern $\alpha^{(1)}$ will be memorized. When the input probe is set to be $I^{(2)}$, the desired pattern $\alpha^{(2)}$ will be memorized. Simulations show the effectiveness of the results with four random initial values in Figs. 2 and 3.



Fig. 2. Typical evolutions of output variables with input probe $I^{(1)}$ under four random initial values.



Fig. 3. Typical evolutions of output variables with input probe $I^{(2)}$ under four random initial values.

5. Conclusion

In this paper, a new design procedure for synthesizing auto-associative memories has been proposed based on CVNNs with real-imaginary-type activation functions and constant delays. Global exponential stability criteria have been derived. The stability results depend on external inputs of neural networks. Compared with previous results, the design method for synthesizing auto-associative memories is suitable for dealing with complex information that must have symmetry concerning, or a certain special meaning on the real and imaginary axes in the applications to associative memory, eliminates spurious memory patterns, relaxes the relationship of parameters of recurrent neural networks, and has high storage capacity of auto-associative memories.

Several extensions would be welcome as future work:

I) Design of multi-valued associative memories based on CVNNs with real-imaginary-type activation functions with multiple constant values.

II) Design of hetero-associative memories dependent on external inputs of CVNNs.

III) Analysis of stability and design of associative memories for CVNNs with amplitude-phase-type activation functions.

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