Thermodynamics and geometrothermodynamics of regular black hole with nonlinear electrodynamics*

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In this paper we investigate the phase transition and geometrothermodynamics of regular electrically charged black hole in nonlinear electrodynamics theory coupled to general relativity. We analyze the types of phase transition of the thermodynamic system by calculating its temperature, heat capacity, and free energy, etc. We find that there are second-order phase transitions from the heat capacity for a large value of S. In addition, employing the geometrothermodynamics, we obtain a Legendre invariance metric and find the relationship between the thermodynamical phase transition and the singularity of the curvature scalar in the regular black hole with the nonlinear electrodynamics.

Keywords: thermodynamic, phase transition, geometrothermodynamics

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1. Introduction

Many scholars have focused on the black hole thermodynamical properties[1–12] since the discovery of Hawking radiation, i.e., its corresponding temperature. The quantities of black hole, including horizon area, mass, surface gravity, and so on, are similar to thermodynamic variables, such as energy, temperature, entropy, and so on, which can describe the thermodynamical system. The laws of thermodynamics have been translated into the laws of black hole thermodynamics.[13]

In very recent years the thermodynamics and geometrothermodynamics have been studied in many places in connection with AdS/CFT[14,15] correspondence, which provides a very useful framework to investigate such a geometry via the equivalence between gravitational theories and the conformal field theories (CFT).[16] The first applications of the geometric approach in black hole thermodynamics were given by Hermann[17] and Mrugala.[18,19] Then, Weinhold[20] proposed a second approach with the purpose of considering the associated Riemannian structure and the physical properties of various thermodynamic systems. Those studies were further developed by Ruppeiner[21] who introduced a metric which is conformally equivalent to Weinhold's metric. Both methods have been intensively applied to analyze characters of the black hole thermodynamics, but they have been separately used for different thermodynamic systems. The most recent theory called geometrothermodynamics[22–24] can solve the problem that the two above approaches can be unified into a single approach by using purely mathematical considerations. There is a simple method that derives the conditions from these metrics to be invariant with respect to arbitrary Legendre transformations, making the contact structure compatible with the Riemannian structure. This paper will employ this approach in an attempt to obtain a curvature scalar, which potentially explains the singularity or thermodynamic properties of the regular black hole.

In recent years, various new regular black holes have been introduced and have attracted some authors to research them. It is well known that the Bardeen black hole[25] was the first regular black hole solution, and other researchers had constructed a series of regular black hole metrics by employing the mass distribution function.[26–29] So regular black holes have become a research hot-spot, especially in the aspect of thermodynamics. Some charged regular black holes, whose metrics and curvature invariants are both regular, satisfy the weak energy condition (WEC).[28–30] By means of coupling gravity and nonlinear electrodynamics theories (NED), several regular black hole metrics are constructed.[29–35] At the same time, lots of authors tend to study the phase transition, stability, and geometrothermodynamics of thermodynamics for those regular black holes.[5,36–39]

In this paper, we focus on the thermodynamic behaviors and geometrothermodynamics of a regular charged black hole found by NED coupled to general relativity.[40] Many efforts have been devoted to studying the variation of thermodynamic quantities of this regular black hole. Ehrenfest had attempted to classify the phase transition, which reveals whether the thermodynamic quantities are continuous or not, by analyzing their Gibbs free energy. If it is continuous, it means that there exists the first order phase transition. Otherwise, the second-order phase transition appears. Furthermore we will consider

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thermodynamical parameters, i.e., temperature, heat capacity, and Gibbs free energy as the function of entropy to classify the phase transition. Using a Legendre-invariant metric proposed by Quevedo[41] in the context of geometrothermodynamics, we formulate an invariant geometric representation of a regular black hole with nonlinear electrodynamics, which can explain the singularity or thermodynamic properties of the regular black hole.

This paper is organized as follows: In Section 2 we briefly introduce nonlinear electrodynamics theories and regular black holes. In Section 3, we calculate the black hole mass, temperature, heat capacity, and Gibbs free energy as functions of the radial coordinate. In Section 4, in order to study the phase transition of the regular black hole, we employ a new geometric approach, i.e., geometrothermodynamics, to study the black hole thermodynamic behaviors. A brief conclusion is given in the last section.

2. Regular black hole with nonlinear electrodynamics

With the nonlinear electrodynamics minimally coupled to gravity, its corresponding action is $(G=c=1)$[40]

$$S=\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}(R-\mathscr{L}(F)). \tag{1}$$

The metric defined by the symmetry of a source term $T_t^t=T_r^r$ takes the form as follows:

$$\mathrm{d}s^2=g(r)\mathrm{d}t^2-\frac{\mathrm{d}r^2}{g(r)}-r^2\mathrm{d}\Omega^2. \tag{2}$$

The metric function and the mass function are given by

$$g(r)=1-\frac{\mathscr{M}(r)}{r}, \tag{3}$$

$$\mathscr{M}(r)=\frac{1}{2}\int_0^r \rho(x)x^2\mathrm{d}x. \tag{4}$$

The above electrically charged solutions are typically found in the alternative form of NED obtained by the Legendre transformation: one introduces the tensor $P_{\mu\nu}=\mathscr{L}_F F_{\mu\nu}$ with its invariant $P=P_{\mu\nu}P^{\mu\nu}$ and considers Hamiltonian-like function $\mathscr{H}(P)=2F\mathscr{L}_F-\mathscr{L}$ as a function of P; the theory is then reformulated in terms of P and specified by $\mathscr{H}(P)$. The P frame is related with the F frame by

$$\mathscr{L}=2P\mathscr{H}_P-\mathscr{H}, \tag{5}$$

$$\mathscr{L}_F\mathscr{H}_P-1, \tag{6}$$

$$F=P\mathscr{H}_P^2. \tag{7}$$

Here $\mathscr{H}_P=\mathrm{d}\mathscr{H}/\mathrm{d}P$. The electric invariant is

$$P=-\frac{2q^2}{r^4}. \tag{8}$$

So the density in this P frame is calculated as

$$\rho(r)=-\frac{1}{2}\mathscr{H}. \tag{9}$$

In search of a new exact electric NED, we select the function $\mathscr{H}(P)$ in the form

$$\mathscr{H}(P)=-\frac{P}{(1+\alpha\sqrt{-P})^2}, \tag{10}$$

where α is a characteristic parameter of the NED theory, and then

$$\mathscr{H}_P=-\frac{P}{(1+\alpha\sqrt{-P})^3}. \tag{11}$$

Then by substituting Eqs. (5)–(7) into the above equation, we obtain

$$\mathscr{H}=-\frac{2q^2}{(r_0^2+r^2)^2}, \tag{12}$$

$$\mathscr{H}_P=-\frac{r^6}{(r_0^2+r^2)^3}, \tag{13}$$

where the parameter $r_0^2=\alpha\sqrt{2q^2}$, and the tangential pressure

$$p_\perp=-T_\theta^\theta=-T_\phi^\phi=-\rho-(r/2)\rho',$$

the density and the pressure function are expressed as follows:

$$\rho(r)=\frac{q^2}{(r_0^2+r^2)^2}, \tag{14}$$

$$\rho_\perp=\frac{q^2(r_0^2-r^2)}{(r_0^2+r^2)^2}. \tag{15}$$

Integrating Eqs. (3) and (4) with density profile Eqs. (14) and (15), we can obtain the metric

$$f(r)=1-\frac{4m}{\pi r}\left(\arctan\frac{r}{r_0}-\frac{r_0 r}{r_0^2+r^2}\right), \tag{16}$$

when $r\ll r_0$ and $r\to\infty$, it will reduce to the Reissner–Nordström black hole.

3. Thermodynamics of regular black hole with nonlinear electrodynamics

In this section, we will investigate the thermodynamic stability of the regular black hole by computing thermodynamic quantities and analyzing its behaviors which contain temperature, heat capacity, and Helmholtz free energy. For the new exact nonlinear electrodynamics solution, by solving Eq. (16) in terms of r_+ $(f(r_+)=0)$, we can easily obtain the mass function

$$M=\frac{\pi r_+(r_+^2+r_0^2)}{4\left(r_+^2\arctan\dfrac{r_+}{r_0}+r_0^2\arctan\dfrac{r_+}{r_0}-r_0r_+\right)}. \tag{17}$$

Bekenstein[42,43] proposed that an entropy should be associated with a black hole, that is proportional to the area of

the event horizon and Hawking temperature[44] $T=\hbar/8\pi GM$, and fixes the co-efficient to be one-quarter. In natural units ($\hbar=G=1$)

$$S=\frac{1}{4}\frac{A}{\hbar G}, \tag{18}$$

where A is the event horizon area of the black hole, and $A=4\pi r_+^2$. Hence the relation between entropy S and event horizon radius r_+ is: $S=\pi r_+^2$,[45] the mass of this black hole will be obtained in terms of the entropy as,

$$M=\frac{\sqrt{\pi S}(S_0+S)}{4\left(-\sqrt{SS_0}+S_0\arctan\sqrt{\frac{S}{S_0}}+S\arctan\sqrt{\frac{S}{S_0}}\right)}. \tag{19}$$

The other thermodynamic parameters can be calculated by using the above expression, such as the temperature of this regular black hole is given in terms of $T=\partial M/\partial S$ as follows:

$$T=\frac{\sqrt{\pi}\left[(S+S_0)^2\arctan\left(\sqrt{\frac{S}{S_0}}\right)-\left(\sqrt{S_0S}S+\left(2\sqrt{\frac{S}{S_0}}S+\sqrt{S_0S}\right)S_0\right)\right]}{8\sqrt{S}\left[\sqrt{S_0S}-(S+S_0)\arctan\left(\sqrt{\frac{S}{S_0}}\right)\right]^2}. \tag{20}$$

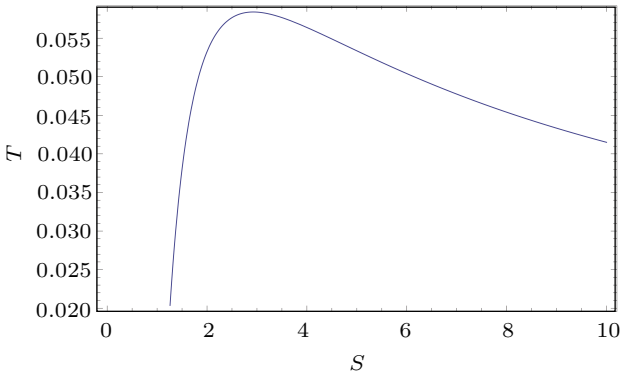


Fig. 1. (color online) The variation of the temperature as a function of the entropy with state parameter $S_0=0.33$.

We have simulated the variation of the temperature vs the entropy in Fig. 1. First of all, we all know that the sign of the temperature determines whether the thermodynamical system is physical or non-physical. In other words, the negativity of the temperature denotes a non-physical system of a black hole. We can see that there is a critical point at $S_c=1.10$ where $T=0$, in which for $S_+<S_c$ the temperature of the system is negative. Therefore, in this region solutions are non-physical. Since for $S_+>S_c$ the system has positive temperature, the horizon radius or entropy of physical black holes is located in this region. The black hole temperature varies continuously in this curve without any discontinuity and undergoes a highest point $T=0.0058$ corresponding to $S=2.48$, which means that a phase transition does not occur from a thermodynamically unstable state to a thermodynamically stable state.

It is well known that the thermodynamically stable system should have a positive heat capacity. It means that the positivity of the heat capacity guarantees the local thermal stability of the black holes. To this end, we will calculate the heat capacity and study its behavior in the next moment. Based on the standard thermodynamic relation $\mathrm{d}U=T\,\mathrm{d}S-P\mathrm{d}V$, we obtain the heat capacity $C_P=T\partial S/\partial T$ at constant pressure as follows:

$$C_P=\frac{A}{B}, \tag{21}$$

where

$$\begin{aligned}A=&-2\sqrt{S_0S^3}\left((S+S_0)\arctan\left(\sqrt{\frac{S}{S_0}}\right)-\sqrt{S_0S}\right)\\&\times\left\{(S+S_0)^2\arctan\left(\sqrt{\frac{S}{S_0}}\right)-\left[S\sqrt{S_0S}+S_0\left(2S\sqrt{\frac{S}{S_0}}+\sqrt{S_0S}\right)\right]\right\},\end{aligned} \tag{22}$$

$$\begin{aligned}B=&\,S_0\left(-4S^3+2S_0S^2-3S_0^2S+S_0^2S\right)\arctan\left(\sqrt{\frac{S}{S_0}}\right)\\&+S_0S\left(3S_0^2\sqrt{\frac{S}{S_0}}-2S_0\sqrt{S_0S}-3SS_0\sqrt{\frac{S}{S_0}}-2S\sqrt{S_0S}\right)+\sqrt{S_0S}(S+S_0)^3\arctan\left(\sqrt{\frac{S}{S_0}}\right)^2.\end{aligned} \tag{23}$$

We have plotted the heat capacity C_P versus the entropy S in Fig. 2. In context of black holes, we argued that the root of the heat capacity ($T=0$) represents a border line between physical ($T>0$) and non-physical ($T<0$) black holes. From the figure we can see that the heat capacity C_P is negative and has a local lowest value at the point $S=0.60$ in left regions, this indicates that the system is in the thermodynamically unstable state. Interestingly, the position of $C_P=0$ coincides

with the zero temperature at S_c, which is called the physical limitation point. Our result proves once again that the region for $S_+ < S_c$ corresponds to the unstable branch of the black hole with a negative heat capacity, and there exists the locally stable state after S_c. In the middle regions of S, the C_P curve has a divergence point at $S = 2.48$ implying that the divergence of C_P corresponds to the maximum Hawking temperature. This indicates that only the intermediate black hole with positive heat capacity is thermodynamically stable and, it is believed that the divergency of the heat capacity represents the phase transition of the black hole. This type of phase transition is known as the second-order phase transition.[46] Whereas the smaller and larger black holes with negative heat capacity are unstable. In addition, the heat capacity in the right regions of S reaches a local peak from negative infinity, and then continues to decrease slowly.

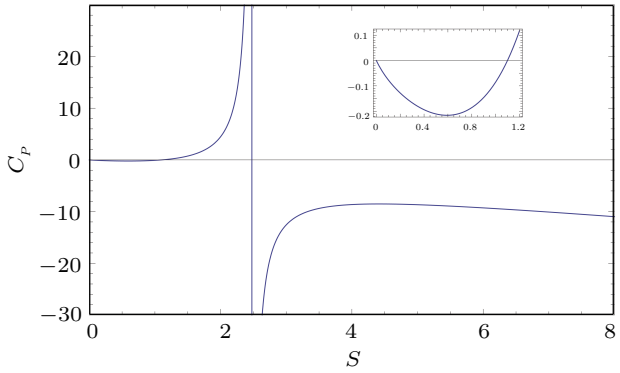


Fig. 2. (color online) The variation of the heat capacity at constant pressure as a function of the entropy with state parameter $S_0 = 0.33$.

Furthermore, we analyze the Helmholtz free energy $F = M - TS$, which is written as the following form,

$$F = \frac{\sqrt{\pi S}\left[\left(-2S_0^2\sqrt{\frac{S}{S_0}} + S_0\sqrt{S_0 S} + S\sqrt{S_0 S}\right) + (S+S_0)^2 \arctan\left(\sqrt{\frac{S}{S_0}}\right)\right]}{8\left[\sqrt{S_0 S} - (S+S_0)\arctan\left(\sqrt{\frac{S}{S_0}}\right)\right]^2}. \tag{24}$$

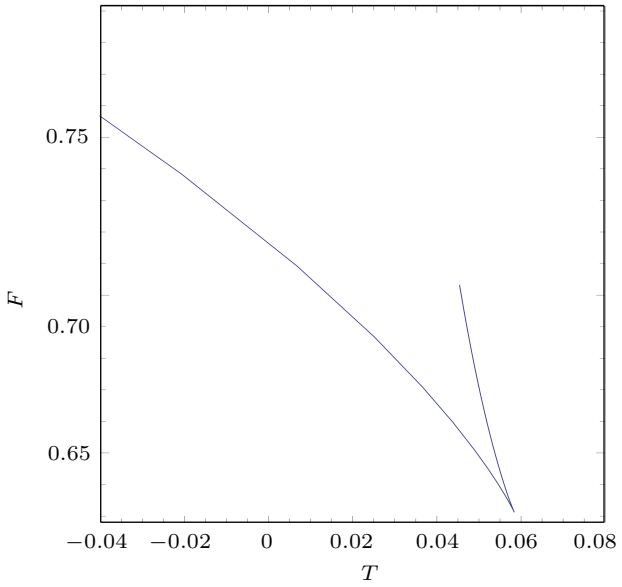


Fig. 3. (color online) The variation of the free energy as a function of temperature with state parameter $S_0 = 0.33$.

Based on the thermodynamical theory, we know that there are two methods to determine whether the black hole thermodynamic system will undergo a second-order phase transition or not. The first method is to investigate the behaviors of the heat capacity C_P versus entropy S, i.e., a discontinuity of the heat capacity C_P possesses the existence of the second-order phase transition in the thermodynamic system. The second method is by studying the behaviors of the free energy F as a function of temperature T to see its change. The sign change of the free energy indicates the appearance of the Hawking–Page phase transition. In Fig. 3, we can see that the free energy with respect to entropy shows a cusp, which undergoes a lowest point at $T = 0.0058$ or $S = 2.48$. The free energy F via entropy S decreases before this point and increases rapidly after that point, and always keeps the positive value in the process of this change. We can conclude that there exists a signal for the second-order phase transition in terms of the sign change of the free energy.

4. Geometrothermodynamics and phase transition of regular black hole

In this section, we will employ a geometrothermodynamics (GTD)[22] approach to study the stability and phase transition of the regular black hole thermodynamic system. We describe a thermodynamics system by using the concepts of geometrothermodynamics and investigate the thermodynamic properties of the regular black hole with nonlinear electrodynamics. First we define a thermodynamics phase space as a Riemannian contact manifold, and then the physical properties of a thermodynamics system in an equilibrium state can be described in terms of the geometric properties of the corresponding equilibrium state. While other methods such as Weinhold,[20] Ruppeiner,[21] and Quevedo[41] metrics, can only solve local problems of thermodynamic systems.

We now briefly introduce the origin of the thermodynamic metric. The differential geometry in thermodynamics

is defined as the (2n+1)-dimensional thermodynamic phase space $\mathscr{T}$. The coordinative set is given by

$$Z^A = \{\Phi, E^a, I^a\};\ A = 0,\ldots,2n;\ a = 1,\ldots,n, \tag{25}$$

where Θ, E^a, I^a signify the thermodynamic potential, the extensive variable, and the intensive variable, respectively. The fundamental Gibbs 1-form is given by

$$\Theta_{\rm G} = {\rm d}\Phi - \delta_{ab} I^a {\rm d}E^b, \quad \delta_{ab} = {\rm diag}(1,1\ldots,1). \tag{26}$$

The pair $(\mathscr{T}, \Theta_{\rm G})$ is called a contact manifold if $\mathscr{T}$ is differentiable and $\Theta_{\rm G}$ satisfies the condition $\Theta_{\rm G} \wedge ({\rm d}\Theta_{\rm G}) \neq 0$. Then we take into account an n-dimensional space, which can be achieved via the smooth mapping $\varphi : \varepsilon \to \mathscr{T}$, so that the space of thermodynamic equilibrium state $(\varphi^*(\Theta_{\rm G}) = \varphi^*({\rm d}\Phi - \delta_{ab} I^a {\rm d}E^b) = 0)$ can infer the first law of thermodynamics ${\rm d}\Phi - \delta_{ab} I^a {\rm d}E^b = 0$.

Furthermore, we use a partial Legendre transformation to obtain a new coordinate and Gibbs 1-form by means of a similar method. The thermodynamic metric, which induces a Riemannian structure on the thermodynamic phase space, can be described as follows:

$$G = {\rm d}\Phi^2 - 2\delta_{ab} I^a {\rm d}E^b {\rm d}\Phi + \delta_{ab}\delta_{cd} I^a I^c {\rm d}E^b {\rm d}E^d. \tag{27}$$

We compute the corresponding metric g by using the pullback of the mapping $\varphi : \varepsilon \to \mathscr{T}$:

$$g^Q = \varphi^*(G) = \left(E^c \frac{\partial \Phi}{\partial E^c}\right)\left(\eta_{ab}\delta^{bc}\frac{\partial \Phi^2}{\partial E^c \partial E^d}{\rm d}E^a {\rm d}E^d\right). \tag{28}$$

Therefore, we obtain the metric form as follows:

$$g^{GTD} = (SM_S + qM_q)\begin{pmatrix} -M_{SS} & 0 \\ 0 & M_{qq} \end{pmatrix}, \tag{29}$$

where M_ι and $M_{\iota\iota}$ respectively indicate the first and the second derivatives of the mass function. The curvature scalar is a long complicated expression with less physical interest, so it will not be given here.

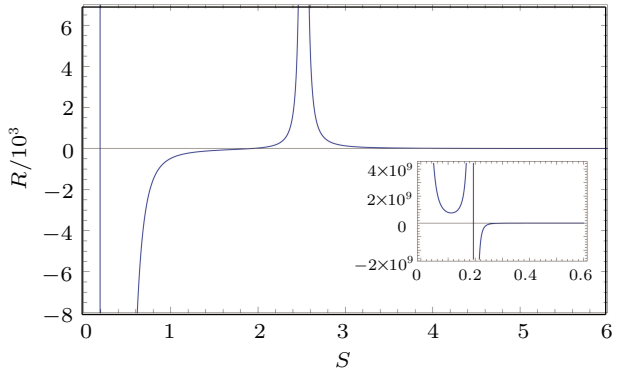


Fig. 4. (color online) The variation of curvature scalar as a function of entropy with the state parameter $q = 0.5$, $\alpha = 0.1$.

Next, it can easily be found that there is respectively an infinite discontinuity at $S = 0$, $S = 0.21$, and $S = 2.50$ in the curvature scalar function of entropy S in Fig. 4. It should be pointed out that $S < 1.10$ corresponds to the situation of negative Hawking temperature, this nonphysical situation is not our main concern and will not be considered in detail in the following discussion. Note that there exist two additional singularities, i.e., $S = 0$ and $S = 0.21$, they both correspond to the regions of negative Hawking temperature. In addition, the divergence of the scalar curvature at $S = 0$ is the intrinsic singularity of the regular black hole metric with nonlinear electrodynamics. Then, the point, $S = 2.48$, undergoes singularity in the figure of curvature scalar[38] and heat capacity.[46] In other words, this singularity shows that the behavior of the scalar curvature is the same as that of the specific heat. It indicates that the regular black hole has a second-order phase transition. This approach, using geometrothermodynamics for our purpose of constructing the Legendre invariant metric, is able to reveal thermodynamic properties and phase transition behaviors of this regular black hole again.

5. Conclusion

In this paper, we have investigated the thermodynamics and geometrothermodynamics of the regular black hole with nonlinear electrodynamics. First of all, we have obtained the radial function which is able to calculate the thermodynamic quantities, such as the function of mass, temperature, heat capacity, and Gibbs free energy for this black hole thermodynamics system. Then, we have plotted the curve of these thermodynamics and analyzed each plot carefully in order to confirm the types of phase transitions. The temperature curve shows that there is a smooth curve without discontinuity, which implies that there exists a first-order phase transition. But the heat capacity curve has a local lowest value at the point $S = 0.60$, and indicates a singularity at $S = 2.48$, which means that the black hole changes from an unstable state to a stable state, and shows a second-order phase transition in this thermodynamic system finally. Also, the heat capacity C_P reaches a local peak from negative infinity after the discontinue point, and then continues to decrease slowly. The free energy F shows a cusp, which undergoes a lowest point at $T = 0.0058$ or $S = 2.48$. Then it demonstrates one value of F corresponding to the two points of entropy S, which is the signals of a second-order phase transition. Both of the above methods reveal that this regular black hole has a second-order phase transition. Furthermore, we employed another method (GTD) to investigate the phase transition in the context of this black hole thermodynamics. In this approach, the result indicates that the divergence of thermodynamical curvature indeed

is related to the divergence of the specific heat, which exhibits the information of phase transition behaviors of the regular black hole again. These studies are helpful to further understand the relation between phase transition and divergence of the thermodynamical curvature.

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