Quantum process discrimination with information from environment*

Yuan-Mei Wang(王元美)[1], Jun-Gang Li(李军刚)[1,†], Jian Zou(邹健)[1], and Bao-Ming Xu(徐宝明)[2]

[1] *School of Physics, Beijing Institute of Technology, Beijing* 100081, *China*

[2] *School of Physics, Qufu Normal University, Qufu* 273165, *China*

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In quantum metrology we usually extract information from the reduced probe system but ignore the information lost inevitably into the environment. However, K. Mølmer [*Phys. Rev. Lett.* **114**, 040401 (2015)] showed that the information lost into the environment has an important effect on improving the successful probability of quantum process discrimination. Here we reconsider the model of a driven atom coupled to an environment and distinguish which of two candidate Hamiltonians governs the dynamics of the whole system. We mainly discuss two measurement methods, one of which obtains only the information from the reduced atom state and the other obtains the information from both the atom and its environment. Interestingly, for the two methods the optimal initial states of the atom, used to improve the successful probability of the process discrimination, are different. By comparing the two methods we find that the partial information from the environment is very useful for the discriminations.

Keywords: quantum process discrimination, information in the environment, overlap

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1. Introduction

The overlap between quantum states is a fundamental quantity in quantum mechanics.[1–6] The overlap between two nonorthogonal states $|\psi_0\rangle$ and $|\psi_1\rangle$, $O = \langle\psi_0|\psi_1\rangle$, is useful for a variety of physical phenomena and quantum metrology tasks. For example, it can be used to obtain an upper bound on the probability of successful cloning of the quantum state.[7–9] Another important application of the overlap is that it can be considered as the essential quantity restricting the effectiveness of discrimination between two states. There are two common approaches for state discrimination.[10,11] The first approach is known as the minimum error state discrimination[1] which gives the minimum error probability, $P^{\rm err} = \frac{1}{2}(1-\sqrt{1-4p_0p_1|O|^2})$, of making an error when two states $|\psi_0\rangle$ and $|\psi_1\rangle$ are discriminated, with priori probabilities p_0 and $p_1 = 1-p_0$. The second one is the unambiguous state discrimination,[12–15] in which the observer is not allowed to give an error instead an inconclusive result. This method can be further optimized by minimizing the probability $P^{\rm inc}$ of occurrence of the inconclusive result and the minimum probability is $P^{\rm inc} = 2\sqrt{p_0p_1}|O|$. So it is obvious that the overlap of two states plays a dominating role in both methods mentioned above.

It is worth stressing that the overlap between two quantum states has an intrinsic property that is preserved when the two states experience the same unitary transformation simultaneously. However, if the states undergo different quantum operations, the overlap between the states will be changed. This property can be used to discriminate quantum operations.[16–21] It has been shown that the entanglement of input states[17] or quantum-memory[20] can improve the operation discrimination. Experimental results concerning quantum operations discrimination have been reported.[22,23] These results motivate people to consider the question of whether the overlap can be used to discriminate quantum dynamic processes. The answer is yes. Very recently, reference [24] provided a quantum circuit model to determine which of two candidate Hamiltonians governs the evolution of a quantum system.

We note that realistic quantum systems used in the quantum task are open to essentially uncontrollable environments that act as sources of decoherence and dissipation.[25–27] Then the information about an open quantum system will continuously leak out into its environment. According to traditional open quantum system theory,[25–27] we always make use of the information that is effectively contained in the system alone and ignore the information that has leaked out into the environment. In some cases, however, it can be possible to extract more information about the state from the environment.[24,28,29] It has been demonstrated that the information obtained by observing an environment that leads to spontaneous decay of a sensing qubit can be used to allow for error correction.[28] Reference [29] modeled noisy channels by coupling the Markov output to "environment" ancillas, and considered the scenario where the environment is monitored to increase the quantum Fisher information of the output. In the quantum process discrimination, it has been shown that

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†Corresponding author. E-mail: jungl@bit.edu.cn

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additional environment information may supplement what has been obtained from the measurements performed on the reduced system and may therefore improve the successful possibility of distinguishing the processes.[24]

In this paper, we reconsider the process discrimination given in Ref. [24]. In this model the input state of a quantum probe system can be chosen freely and thus we will focus on the question of which initial state of the probe system is the optimal one for the discrimination. For this purpose, we first give a general Hamiltonian of the system driven by a laser field, which can also be used in the non-resonant case. We show that the information lost into the environment has an important effect on improving the successful probability of the discrimination. We also show that when we monitor both the environment and the probe system the excited state is the optimal initial state of the probe system for both resonant and non-resonant cases. However, when we monitor only the probe system, its optimal initial state is always the ground state.

The rest of this paper is organized as follows. In Section 2 we give two measurement methods, one of which obtains only the information from the reduced system state and the other obtains the information from both the system and its environment. In Section 3 we describe in some detail the physical model employed throughout the paper. Section 4 contains our analytical results in the case of resonance and the results of our numerical simulations in the case of non-resonance with our physical explanation about them. Section 5 closes the paper with concluding remarks.

2. General formalism

We suppose that a quantum process is governed by one of two known Hamiltonians $H_0(t)$ and $H_1(t)$ with prior probabilities p_0 and $p_1 = 1 - p_0$, respectively. But we do not know which one it is. To discriminate which of the two Hamiltonians is given, we prepare a probe system in a special initial state and then let it access the process. In the dynamic process, the probe system is affected by the environment. Finally, we perform a measurement to determine the process. Here we mainly consider two different measurement methods illustrated in Fig. 1.

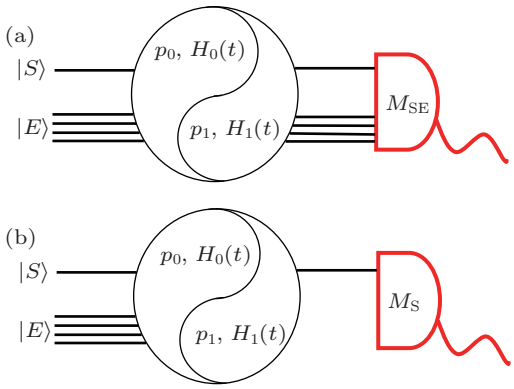


Fig. 1. (color online) Illustrations of the evolution of a probe system S and the environment E governed by a Hamiltonian $H_0(t)$ or $H_1(t)$ with a prior probability p_0 or p_1. In panel (a), we perform an effective measurement on both probe system and environment. While in panel (b) we perform the measurement purely on probe system S.

Method I is shown in Fig. 1(a). In this method we monitor both the environment and the probe system. Suppose that the probe system and its environment are prepared in a state $|\psi^{\rm SE}(0)\rangle$, and evolves into state $|\psi_0^{\rm SE}(t)\rangle$ or $|\psi_1^{\rm SE}(t)\rangle$ under the Hamiltonian $H_0(t)$ or $H_1(t)$, then at the final time we can determine which of the candidate Hamiltonians is correct by performing an associate effective measurement to gather the information from both the probe system and the environment as shown in Fig. 1(a). Obviously, the overlap

$$O_{\rm SE} = \langle\psi_1^{\rm SE}(t)|\psi_0^{\rm SE}(t)\rangle \tag{1}$$

determines our ability to discriminate the candidate Hamiltonians $H_0(t)$ and $H_1(t)$. Note that when we calculate the overlap $O_{\rm SE}$, the states $|\psi_0^{\rm SE}(t)\rangle$ and $|\psi_1^{\rm SE}(t)\rangle$ must be in the same interaction frame.[26] Besides, the overlap can be optimized by choosing an optimal initial state. It is prohibitively complicated to calculate the joint quantum state of the system and environment as it requires one to deal with an infinite number of degrees of freedom. The overlap $O_{\rm SE}$, however, can be obtained directly without necessarily calculating the whole states. We assume that $\{|SE\rangle\}$ is a set of complete bases of the Hilbert space of the probe system and environment. Substituting the complete condition $\sum_{\rm SE}|SE\rangle\langle SE| = I$ into Eq. (1), we can obtain

$$\begin{aligned} O_{\rm SE} &= \sum_{\rm SE}\langle\psi_1^{\rm SE}(t)|SE\rangle\langle SE|\psi_0^{\rm SE}(t)\rangle \\ &= {\rm Tr}_{\rm S}[\widetilde{\rho}(t)], \end{aligned} \tag{2}$$

where

$$\widetilde{\rho}(t) = {\rm Tr}_{\rm E}(|\psi_0^{\rm SE}(t)\rangle\langle\psi_1^{\rm SE}(t)|). \tag{3}$$

We note that $\widetilde{\rho}(t)$ is in the reduced Hilbert space of the probe system and we can obtain the dynamics of the overlap $O_{\rm SE}$ from the time evolution of the $\widetilde{\rho}(t)$. So in what follows, we will give the time evolution of $\widetilde{\rho}(t)$ through the formulation of specific measurement schemes in terms of a quantum operation. We assume that the Markov approximation is used, then $\widetilde{\rho}(t)$ has a Kraus-like representation[30]

$$\widetilde{\rho}(t+dt) = \sum_m M_m(H_0^{\rm S}, {\rm d}t)\widetilde{\rho}(t)M_m^\dagger(H_1^{\rm S}, {\rm d}t). \tag{4}$$

Here, the Kraus operators $\{M_m\}$ satisfy the normalization condition $\sum_m M_m^\dagger M_m = \mathbf{1}$. Suppose

$$\begin{aligned} M_0(H_\vartheta, {\rm d}t) &= \mathbb{I} + \left[-{\rm i}H_\vartheta - \frac{1}{2}\sum_{m>0}(\hat{c}_m^\vartheta)^\dagger\hat{c}_m^\vartheta\right]{\rm d}t, \\ M_m(H_\vartheta^{\rm S}, {\rm d}t) &= \sqrt{{\rm d}t}\hat{c}_m^\vartheta, \quad m = 1,2,3,\ldots \end{aligned} \tag{5}$$

Here, $H_\vartheta^{\rm S}$ ($\vartheta = 0$ or 1) denotes the system Hamiltonian and $\hat{c}_m^\vartheta$ denote the corresponding Lindblad operators. Substituting Eq. (5) into Eq. (4), expressing $\widetilde{\rho}(t+{\rm d}t) = \widetilde{\rho}(t) + {\rm d}t\dot{\widetilde{\rho}}(t)$ and

equating term of order $\mathrm{d}t$, we obtain the time evolution equation of $\widetilde{\rho}(t)$

$$\dot{\widetilde{\rho}} = -\mathrm{i}(H_0^{\mathrm{S}}\widetilde{\rho} - \widetilde{\rho}H_1^{\mathrm{S}}) + \sum_m \hat{c}_m^0 \widetilde{\rho}(\hat{c}_m^1)^\dagger - \frac{1}{2}[(\hat{c}_m^0)^\dagger \hat{c}_m^0 \widetilde{\rho} + \widetilde{\rho}(\hat{c}_m^1)^\dagger \hat{c}_m^1], \quad (6)$$

which was first given in Ref. [24] and its alternative derivation can be seen in Refs. [31]–[33]. Equation (6) is structured like the Lindblad master equation, but all operators multiplied with $\widetilde{\rho}$ from the left (right) pertain to H_0^{S} (H_1^{S}). If we choose $H_0^{\mathrm{S}} = H_1^{\mathrm{S}} = H^{\mathrm{S}}$, we can reduce Eq. (6) to a usual master equation which conserves the trace of the density matrix. Here $\widetilde{\rho}(t)$ is not a density operator and the trace of $\widetilde{\rho}(t)$ gives the overlap O_{SE} at time t. Then by solving Eq. (6) and taking the trace of $\widetilde{\rho}(t)$ we can obtain the time evolution of O_{SE}. Then the distinguishability of the two processes can be evaluated by the minimum error probability $P_{\mathrm{err}} = \frac{1}{2}(1 - \sqrt{1 - 4p_0 p_1 |O_{\mathrm{SE}}|^2})$.

There are two points that should be mentioned: firstly, the two candidate Hamiltonians in Eq. (6) must be in the same interaction frame. Secondly, there is freedom for us to choose an appropriate initial state of the probe system during the process in solving this linear differential equation.

Method II is a traditional one shown in Fig. 1(b). In this method only a measurement on the probe system is allowed at the end of the interaction time t. Then the process discrimination is reduced to distinguish the two density matrices ρ_0 and ρ_1 evolved by the following Lindblad master equation with the different Hamiltonians,

$$\dot{\rho_\vartheta} = -\mathrm{i}[H_\vartheta^{\mathrm{S}}, \rho_\vartheta] + \sum_m \hat{c}_m^\vartheta \rho_\vartheta (\hat{c}_m^\vartheta)^\dagger - \frac{1}{2}[(\hat{c}_m^\vartheta)^\dagger \hat{c}_m^\vartheta \rho_\vartheta + \rho_\vartheta (\hat{c}_m^\vartheta)^\dagger \hat{c}_m^\vartheta], \quad (7)$$

where $\vartheta = 0$ or 1. The minimal probability of making an assignment error is provided here by Helstrom,[1,26,34] $P_e^\rho = \frac{1}{2} + \sum_{\lambda_i \le 0} \lambda_i$, where the summation is over the negative eigenvalues of the operator $p_1\rho_1 - p_0\rho_0$.

3. Model

Now, we consider a two-level atom driven by a classically single mode laser field. In the Schrödinger picture, the interaction between the atom and the field can be described by the following Hamiltonian in the dipole approximation ($\hbar = 1$):

$$H_{a+L} = \omega_e |e\rangle\langle e| + \omega_g |g\rangle\langle g| - \wp\varepsilon_0(\cos\omega_L t + \varphi)(\sigma_+ + \sigma_-). \quad (8)$$

where $|g\rangle$ and $|e\rangle$ represent the ground and excited states of the atom, respectively; ω_e and ω_g are respectively the circular frequencies of the excited and ground states of the atom; ω_L and ε_0 are respectively the frequency and amplitude of the field; $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$ are the raising and lowering operators, respectively; $\wp$ is the matrix element of the electric dipole moment; φ is the phase of the dipole matrix element. Here, we assume that $\wp$ is a real value and $\varphi = 0$. In this model, our quantum process discrimination is essentially to detect which of the fields drives the atom. It is obvious that the field is determined by its frequency ω_L, amplitude ε_0 and $\wp$. In general, for convenient calculation, we often transform Hamiltonian Eq. (8) into the interaction frame.

Traditionally, as the first transformation method described in Appendix A, Hamiltonian Eq. (8) in the interaction frame becomes

$$H_{\mathrm{I}} = \delta|e\rangle\langle e| + \frac{\Omega}{2}(|e\rangle\langle g| + |g\rangle\langle e|), \quad (9)$$

where $\delta = \omega_e - \omega_g - \omega_L$ is the detuning of the atomic transition frequency and the driving field frequency. $\Omega = -\wp\varepsilon_0$ is the Rabi frequency and characterizes the coupling between the atomic dipole and the laser electric field. So in the interaction frame, the field can be described by a pair of δ and Ω. Equation (9) is a time-independent Hamiltonian and convenient for calculation.

Unfortunately, the unitary transformation operator from the Schrödinger picture to this interaction frame depends on the detuning δ (see the Appendix A). Then it will be inconvenient to calculate the overlap O_{SE} from Eq. (9) in the case that there is a nonzero detuning between the atom and the laser field. In order to avoid this problem, we use the second transformation method mentioned in Appendix A, for which the unitary transformation operator does not depend on δ, and obtain another equivalent form of Hamiltonian Eq. (8) in the interaction frame

$$H_{\mathrm{I}} = \frac{\Omega}{2}(\mathrm{e}^{\mathrm{i}\delta t}|e\rangle\langle g| + \mathrm{e}^{-\mathrm{i}\delta t}|g\rangle\langle e|). \quad (10)$$

Here, we note that this Hamiltonian is time-dependent.

Again, we should emphasize that the aim of this paper is to discriminate which of the candidate Hamiltonians governs the quantum process. For notation, we drop subscript I and introduce subscript $\vartheta = 0$ or 1 to represent different Hamiltonians with a prior probability p_0 or $p_1 = 1 - p_0$. Then equation (10) can be written as

$$H_\vartheta^{\mathrm{S}} = \frac{\Omega_\vartheta}{2}(\mathrm{e}^{\mathrm{i}\delta_\vartheta t}|e\rangle\langle g| + \mathrm{e}^{-\mathrm{i}\delta_\vartheta t}|g\rangle\langle e|). \quad (11)$$

Substitute Eq. (11) into Eq. (6) and choose a single Lindblad operator $\hat{c} = \sqrt{\kappa}|g\rangle\langle e|$, where κ is the decay rate of the atom excited state decays to its ground state by fluorescence emission, then we will be able to obtain

$$\dot{\widetilde{\rho}} = -\mathrm{i}(H_0^{\mathrm{S}}\widetilde{\rho} - \widetilde{\rho}H_1^{\mathrm{S}}) + \hat{c}\widetilde{\rho}\hat{c}^\dagger - \frac{1}{2}[\hat{c}^\dagger\hat{c}\widetilde{\rho} + \widetilde{\rho}\hat{c}^\dagger\hat{c}], \quad (12)$$

by which we can obtain $\widetilde{\rho}(t)$ and then the overlap O_{SE}. However, according to different laser field frequencies, the quantum process discrimination can be divided into resonance and non-resonant cases and we will discuss these two cases in detail later in this article.

4. Influence of initial state of the atom on the distinguishability

In this section we will discuss the influences of the initial state of the atom on the distinguishability of quantum processes in various cases. We suppose that the two candidate Hamiltonians H_0 and H_1 have the same prior probability, i.e., $p_0 = p_1 = 1/2$. We choose two kinds of initial states of the atom, one kind of initial states are pure states,

$$|\psi^{\rm S}(0)\rangle = \cos\theta|e\rangle + \sin\theta|g\rangle, \tag{13}$$

and from Eq. (13) we know that $\theta = 0$ or $\theta = \pi/2$ respectively corresponds to the initial state of the atom being the excited state or ground state. Another kind of initial states are mixed states,

$$\rho^{\rm S}(0) = p|e\rangle\langle e| + (1-p)|g\rangle\langle g|, \tag{14}$$

where p is the population of the excited states of the mixed state. It is obvious that the atomic initial state will be the maximally mixed state if $p = 1/2$. Due to the linearity of Eq. (12), we can deal with the mixed initial state. For this purpose we need to define the overlap of two mixed states as

$$O_{\rm SE}^{M} = pO_{\rm SE}^{(e)} + (1-p)O_{\rm SE}^{(g)}, \tag{15}$$

where

$$O_{\rm SE}^{(\varepsilon)} = \langle\psi_1^{\rm SE}(t)|\psi_0^{\rm SE}(t)\rangle_{\varepsilon},$$

and $\varepsilon = e, g$. $O_{\rm SE}^{(\varepsilon)}$ represents the overlap between the two states evolved from pure states, either the excited or the ground state.

4.1. Resonance cases

In this subsection we consider the resonance case in which the detuning $\delta_0 = \delta_1 = 0$. Firstly, we consider method I by solving Eq. (12) straightforwardly with $H_0^{\rm S}$ and $H_1^{\rm S}$, and then evaluate the modulus square of the overlap between states $|\psi_0^{\rm SE}(t)\rangle$ and $|\psi_1^{\rm SE}(t)\rangle$. The atom is initially prepared in the state Eq. (13) or mixed state Eq. (14). After some calculation we find that the analytic expression of the modulus square of the overlap can be written as

$$O_{\rm SE} = O'_{\rm SE} + 2{\rm i}\alpha\,{\rm e}^{-\kappa t/4}\frac{\Delta\Omega}{\varepsilon}\sinh\frac{\varepsilon t}{4}, \tag{16}$$

where

$$O'_{\rm SE} = {\rm e}^{-\kappa t/4}\left(\cosh\frac{\varepsilon t}{4} + \frac{1}{\varepsilon}\sinh\frac{\varepsilon t}{4}\right), \tag{17}$$

and $\varepsilon = \sqrt{\kappa^2 - 4(\Delta\Omega)^2}$, $\Delta\Omega = \Omega_1 - \Omega_0$, $\alpha = \rho_{eg} + \rho_{ge}$ with ρ_{ij} $(ij = eg$ or $ge)$ being the density matrix element of the initial state of the atom. In the limit $\Delta\Omega \to 0.5\kappa$, $\varepsilon \to 0$, equation (16) can be replaced by

$$O_{\rm SE} = \frac{1}{4}{\rm e}^{-\kappa t/4}[(4+\kappa t)\rho_0 + {\rm i}\alpha\kappa t]. \tag{18}$$

From Eq. (16), we can find that $O_{\rm SE}$ depends on the initial state of the atom. If the atom is initially prepared in the ground state, excited state or mixed state Eq. (14) , then $\alpha = 0$ and the overlap can be reduced to $O'_{\rm SE}$. We note that $O'_{\rm SE}$ is independent of the initial state of the atom and is always real no matter whether $\Delta\Omega > 0.5\kappa$ or $\Delta\Omega < 0.5\kappa$. From Eq. (16) we also find that the modulus square of $O_{\rm SE}$ achieves its optimal value $|O'_{\rm SE}|^2$ if the second term in Eq. (16) vanishes. This can be achieved by initially preparing the atom in the ground state, excited state or mixed state Eq. (14). Then we can come to the conclusion that these initial states are all the optimal initial states of the atom for the Hamiltonian discrimination. While for the initial superposition states (Eq. (13) with $\theta \neq 0$) the second term on the right-hand side of Eq. (16) will be present then the modulus square of $O_{\rm SE}$ will become larger. When $\Delta\Omega = 0.5\kappa$, we can still obtain the same conclusion that the optimal initial states of the atom are the ground state, excited state or mixed state.

In Fig. 2 we give the time evolutions of the modulus square of the overlap for $\Delta\Omega = 0.3\kappa$ (green triangle), $\Delta\Omega = 0.5\kappa$ (blue dot), and $\Delta\Omega = 3.0\kappa$ (red square). Here we have chosen the excited state as the initial state. While by using the ground state or mixed state mentioned above we can obtain the same results. Figure 2 shows that below the critical value $\Delta\Omega = 0.5\kappa$, the modulus square of the overlap has an exponential decay with time. However, it will vibrate periodically with time for $\Delta\Omega > 0.5\kappa$. From Fig. 2, we also find that the modulus square of the overlap decreases with the increase of $\Delta\Omega$. This is because for the resonant case, the difference between the two processes described respectively by H_0 and H_1 only depends on the Rabi frequency. The larger the $\Delta\Omega$, the more easily the H_0 and H_1 are distinguished from each other. Besides, we find that the modulus square of the overlap always achieves zero in the long time limit, which means that we can distinguish the two candidate Hamiltonians from each other perfectly. Furthermore, from Eq. (16) we can find that the modulus square of the overlap only depends on the value of $\Delta\Omega$ but is independent of the values of Ω_0 and Ω_1.

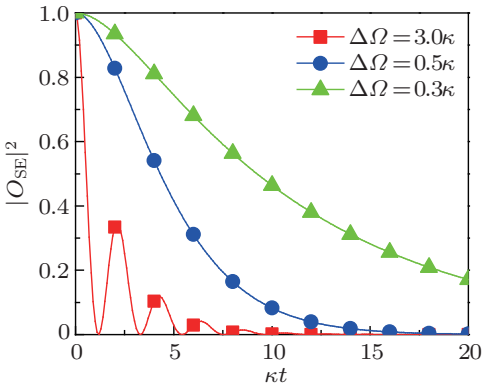


Fig. 2. (color online) Time evolutions of the modulus square of the overlap $|O_{\rm SE}|^2$ for $\Delta\Omega = 0.3\kappa$ (green triangle), $\Delta\Omega = 0.5\kappa$ (critical value, blue dot), and $\Delta\Omega = 3.0\kappa$ (red square).

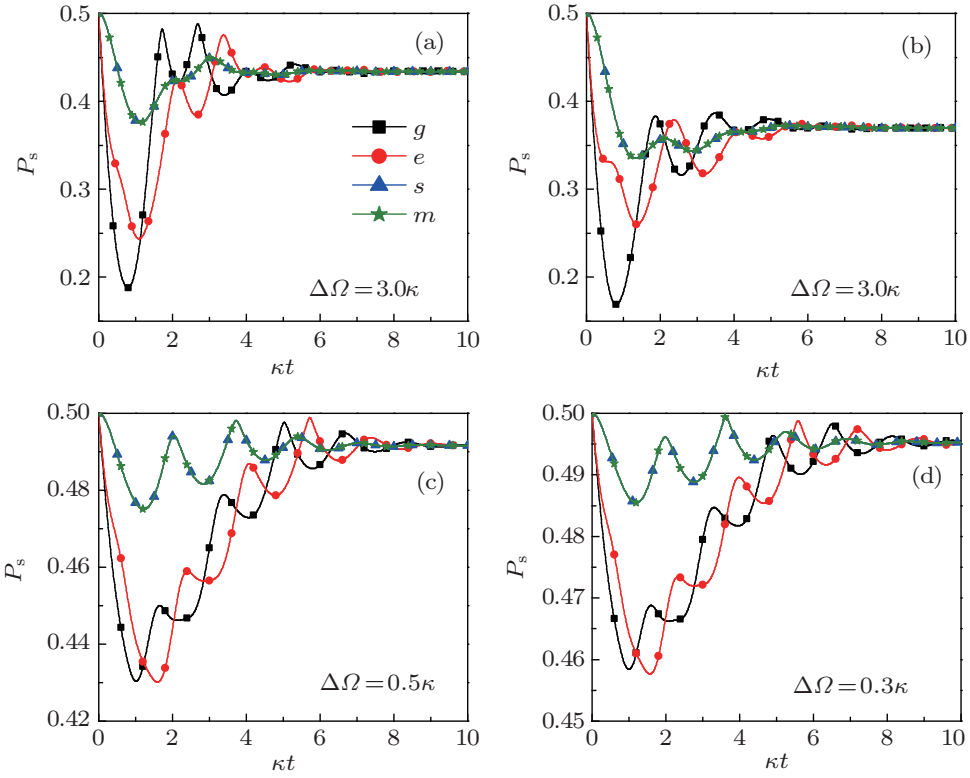


Fig. 3. (color online) Time evolutions of error probabilities P_s for four initial states of atom in resonance. The four initial states are ground state (g, black square), excited state (e, red dot), superposition state (s, blue triangle) and maximally mixed state (m, green star). The Rabi frequencies are different in the different panels: (a) $\Omega_0 = 2.0\kappa$, $\Omega_1 = 5.0\kappa$, (b) $\Omega_0 = 1.0\kappa$, $\Omega_1 = 4.0\kappa$, (c) $\Omega_0 = 3.5\kappa$, $\Omega_1 = 4.0\kappa$, and (d) $\Omega_0 = 3.7\kappa$, $\Omega_1 = 4.0\kappa$.

For method II, in which only a measurement on the probe system is allowed, it is not easy for us to obtain the analytic results and thus we give numerical simulations. In Fig. 3 we give the time evolutions of the error probability P_s for different values of Ω_0 and Ω_1 and different initial states of the atom. The parameters are chosen as follows: in Fig. 3(a) $\Omega_0 = 2.0\kappa$, $\Omega_1 = 5.0\kappa$; in Fig. 3(b) $\Omega_0 = 1.0\kappa$, $\Omega_1 = 4.0\kappa$; in Fig. 3(c) $\Omega_0 = 3.5\kappa$, $\Omega_1 = 4.0\kappa$; in Fig. 3(d) $\Omega_0 = 3.7\kappa$, $\Omega_1 = 4.0\kappa$. We choose the ground (g) state, excited (e) state, superposition (s, $\theta = \pi/4$) state and maximally mixed (m) state to be the initial states of the atom. The results of numerical calculation show that the error probability P_s depends not only on $\Delta\Omega$ but also on the concrete values of Ω_0 and Ω_1, which are illustrated in Figs. 3(a) and 3(b). Besides, we also find that the ground state is the optimal initial state of the atom for large $\Delta\Omega$, which is displayed from Figs. 3(b) to 3(d). While when $\Delta\Omega$ decreases, P_s will increase and the advantage of preparing the atom initially in its ground state will no longer be obvious. We can see that the two curves, which describe the behaviors of P_s for the atom being initially in the maximally mixed state or the superposition state, coincide, which means that the effects that these two states are regarded as the initial states are the same. Another different point is that the error probability obtained by method II does not approach to zero in the long time limit.

These differences between the two methods can be explained as follows. For method I, we consider the information from both the atom and environment while for method II we only consider the information from the reduced state of atom. So the information obtained from the environment is responsible for the improvement of the successful probability of the discrimination.

4.2. Non-resonant cases

In what follows, we will discuss the non-resonant case, i.e., $\delta \neq 0$. Generally, the quantum process discrimination about our model is to detect which of the laser fields is present and the candidate Hamiltonians $H_0^S \neq H_1^S$. As we are unable to give an analytical solution, in this case, we will use a numerical calculation to illustrate our results. We suppose that the laser field is present in only one of the two channels, that is to say, the candidate Hamiltonians $H_0^S = 0$, and H_1^S has a given Rabi frequency $\Omega_1 = 4.0\kappa$ and adjustable detuning.

Firstly, consider method I. In Fig. 4, we show the modulus squares of the overlap between the two final states for four different initial states of the atom and different values of $\delta_1 = 0.1\kappa$, 0.5κ, 1.5κ, 3.5κ. We find that for small δ_1, the dynamic behavior of $|O_{SE}|^2$ for the atom initially in the ground state is similar to that for the atom initially in the excited state or the maximally mixed state. All of them are better than choosing the superposition state as the initial state for most of the time. However, this property mentioned above will not apply to those conditions in which δ_1 increases and

the advantage of choosing the excited state of the atom as its initial state becomes more obvious. We still note that the maximally mixed state is better than the ground state when δ_1 is larger although it is not always the best initial state. Comparing Fig. 4(d) with Figs. 4(b) and 4(c), we find that $|O_{\rm SE}|^2$ will become large when δ_1 increases to a certain value. This can be illustrated as follows. The parameter δ_1 is the detuning between the atom transition frequency and the frequency of the laser. When the value of the detuning δ_1 increases, the effective coupling between the atom and laser decreases, and thus the influence of the driven field will become smaller. This will increase the difficulty in discriminating the process.

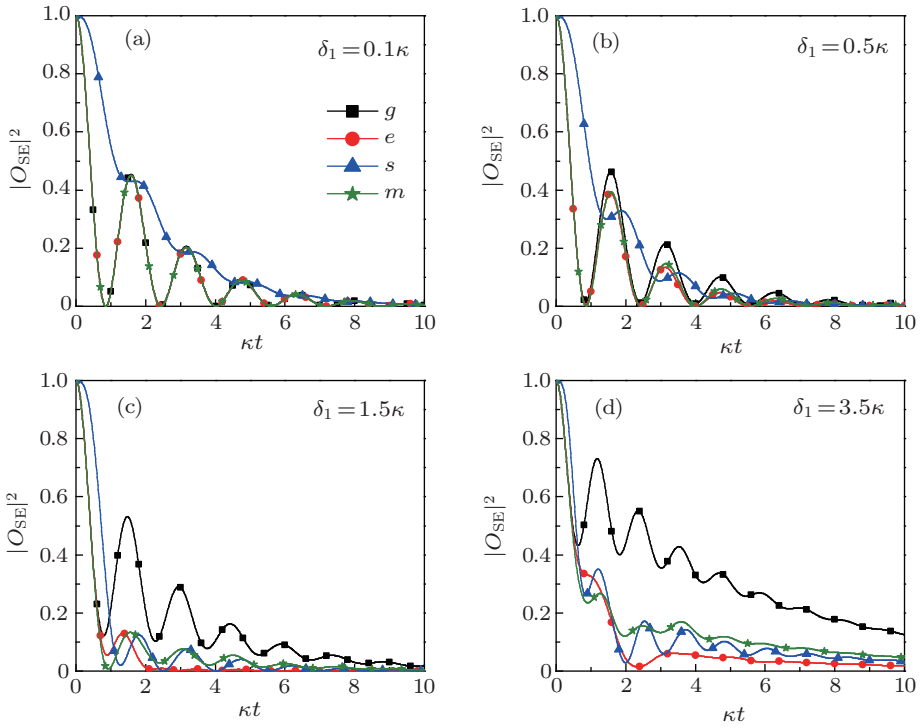


Fig. 4. (color online) Modulus squares of the overlap for the four initial states of atom when we choose $\delta_0=0$, $\Omega_0=0$, $\Omega_1=4.0\kappa$ and different values of δ_1: (a) $\delta_1=0.1\kappa$, (b) $\delta_1=0.5\kappa$, (c) $\delta_1=1.5\kappa$, (d) $\delta_1=3.5\kappa$. The initial states are ground (g, black square), excited (e, red dot), superposition (s, blue triangle), and maximally mixed (m, green star) states.

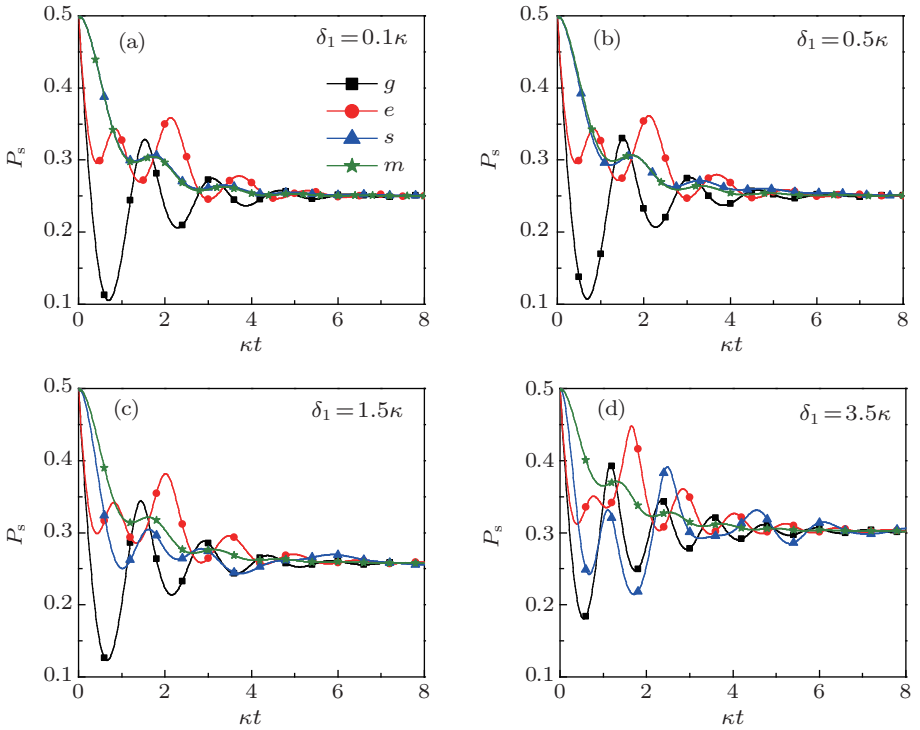


Fig. 5. (color online) The values of error probability $P_{\rm s}$ of distinguishing whether a two-level atom is evolved by $H_0^{\rm S}=0$ or $H_1^{\rm S}\neq 0$ initially in the ground (g, black square), excited (e, red dot), superposition (s, blue triangle), and maximally mixed (m, green star) states for $\Omega_1=4.0\kappa$ and (a) $\delta_1=0.1\kappa$, (b) $\delta_1=0.5\kappa$, (c) $\delta_1=1.5\kappa$, and (d) $\delta_1=3.5\kappa$.

Secondly, we consider method II so as to offer a vivid comparison with our previous method. The results are shown in Fig. 5 which depicts the time evolutions of P_s for different values of δ_1. Due to the interaction with the environment, the oscillations are damped over time, and it is intuitively clear that it is no longer optimal to choose the excited state as the initial state of the atom. On the contrary, the ground state is the optimal choice. For large δ_1, the minimal value of P_s is increased and the reason for this is the same as that given in the end of the previous paragraph.

Comparing Fig. 4 with Fig. 5, we find that the optimal initial states of the atom are different for the two different methods. The optimal initial state is the ground state for method II, while for method I it should be the excited state. Why is it like this? We can explain it as follows. First we consider the case without energy dissipation, i.e., the decay rate $\kappa = 0$. Suppose that the atom is initially in its ground or excited state, following the Schrödinger equation, it will evolve into $|\psi_0^S(t)\rangle$ or $|\psi_1^S(t)\rangle$ after a time t and the analytic expression of the overlap between these two states is

$$O_\pm = \cos\beta\cos\phi + \eta\sin\beta\sin\phi \pm \mathrm{i}(\eta\cos\beta\sin\phi - \sin\beta\cos\phi), \quad (19)$$

where $\beta = \delta_1 t/2$, $\phi = \sqrt{\delta_1^2+\Omega_1^2}\,t/2$, and $\eta = \delta_1/\sqrt{\delta_1^2+\Omega_1^2}$. The symbol "+" in the above equation corresponds to the atomic initial state that is the ground state while "−" corresponds to the excited state. So it is easy to prove that the distinguishability of the candidate Hamiltonians H_0 and H_1 are identical no matter whether the atom is initially in the ground state or excited state for the same parameters.

Then a further study is to take the environment into account. In this case, the atom will evolve following the master equation (7). Here we want to reveal the influence of a single quantum channel on the atom state evolution rather than discriminate two quantum processes. So we describe this channel by Hamiltonian H^S in this paragraph. In Fig. 6(a), the atom is initially in the ground state. The population of the excited states will be zero if it evolves under $H^S = 0$ and there is no information leaking into the environment. While the population of the excited states will suffer periodical oscillations with damped amplitudes if the atom goes through a channel described by the Hamiltonian $H^S \neq 0$. At this point, a partial information about the atom will transfer into its environment as the effect of the driven field. Figure 6(b) shows the other cases that the atom is initially in the excited state. We note that the population of the excited states will decay exponentially even when the Hamiltonian $H^S = 0$ and its environment will have some information about the atom. Comparing these two cases, we find that the advantage of the excited state is precisely due to the information extracted from the environment.

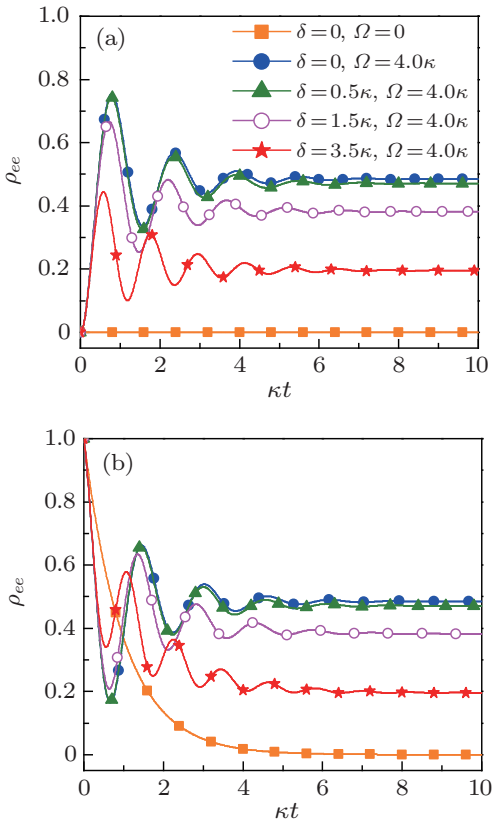


Fig. 6. (color online) The time evolutions of the population of the excited states of a driven atom coupled to the environment. The initial states of the atom are respectively in (a) ground state and (b) excited state. We consider two cases that the parameters: $H^S = 0$ (orange square), and $H^S \neq 0$, for $\delta = 0$ (blue dot), $\delta = 0.5\kappa$ (green triangle), $\delta = 1.5\kappa$ (pink circle), $\delta = 3.5\kappa$ (red star), and $\Omega = 4.0\kappa$.

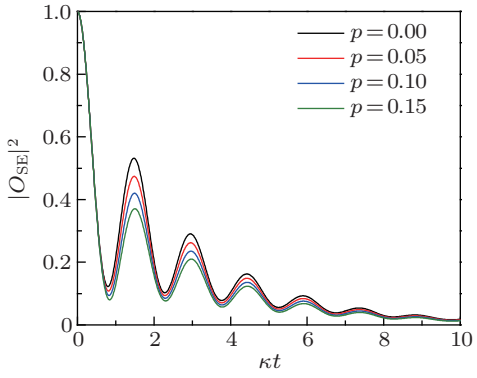


Fig. 7. (color online) Modulus squares of the overlap of different initial states, with the initial population of the excited states $p = 0$ (black), 0.05 (red), 0.1 (blue), and 0.15 (green) (from top to bottom) for $\Omega_0 = 0$, $\delta_0 = 0$, $\delta_1 = 1.5\kappa$, and $\Omega_1 = 4.0\kappa$.

The above discussion shows that the excited state is always better than the ground state in process discrimination for method I in the non-resonant case. This stimulates us to consider the influence of the population of the excited states of the atomic initial state on the overlap. We compare the ground state with the mixed state as the initial state of the atom for $\Omega_0 = 0$, $\delta_0 = 0$, $\delta_1 = 1.5\kappa$, and $\Omega_1 = 4.0\kappa$. The results are

illustrated in Fig. 7, where p is the initial population of the excited states of the mixed state. We find that as long as the mixed state in Eq. (14) has a nonzero population of the excited states, the modulus square of the overlap compared with the ground state can be reduced. It is because the atom will also leak information into the environment if it initially has a nonzero population of the excited states even though there is no external driven field. It is this piece of information that determines the importance of the population of the excited states in the initial state of the atom. This is a very useful and practical result in an experiment as it is easier for us to prepare such a mixed state than some pure state accurately.

5. Conclusions

In this work, we reconsider the problem of process discrimination by using the dynamics of the overlap and show that retrieving the information lost in the environment can improve the distinguishability. Specifically, we analyze the optimal initial state of the atom for process discrimination. We mainly discuss two measurement methods. In method I, we monitor the probe and its environment while in method II we monitor only the probe. The difference between these two methods lies in whether the information leaking into the environment is taken into account. For method I we find that in the resonance case, the ground state, excited state or mixed state with zero off-diagonal density matrix element are all the optimal initial states of the atom. However, the optimal state is the excited state in the non-resonance case. Besides we also find that the mixed state mentioned above is better than the ground state and the overlap will decrease as the initial population of the excited states increases. On the contrary, in the case of method II, the optimal initial state of the atom is always the ground state in both the resonance case and the non-resonance case. According to the comparative analyses of the two methods, we can conclude that the information lost into the environment not only contributes to improving the successful probability of process discrimination, but also has an important effect upon the selection of the initial state of the probe system. We hope that our results can stimulate further research on the discrimination of quantum processes.

Appendix A: Hamiltonians in different interaction frames

In this appendix we will simply review how to transform the Hamiltonian from the Schrödinger picture into different interaction frames.

In the Schrödinger picture, the interaction between an atom and a classically single mode laser field with frequency $\omega_{\rm L}$ can be described by the following Hamiltonian in the dipole approximation ($\hbar = 1$):

$$H_{a+{\rm L}} = \omega_e|e\rangle\langle e| + \omega_g|g\rangle\langle g| - \wp\varepsilon_0(\cos\omega_{\rm L}t + \varphi)(\sigma_+ + \sigma_-), \tag{A1}$$

where ω_e and ω_g are respectively the circular frequencies of the excited and ground states of the atom, ε_0 is the amplitude of the field, $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$ are the raising and lowering operators, $\wp$ is the matrix element of the electric dipole moment, and φ is the phase of the dipole matrix element. Here, we assume that $\wp$ is real and $\varphi = 0$. For convenient calculation, we often transform this Hamiltonian into the interaction frame. Traditionally, people always split the total Hamiltonian (A1) into two parts

$$H_{a+{\rm L}} = H_A + H_B, \tag{A2}$$

with free Hamiltonian

$$H_A = (\omega_g + \omega_{\rm L})|e\rangle\langle e| + \omega_g|g\rangle\langle g|, \tag{A3}$$

and interaction Hamiltonian

$$H_B = \delta|e\rangle\langle e| - \wp\varepsilon_0\cos(\omega_{\rm L}t + \varphi)(\sigma_+ + \sigma_-). \tag{A4}$$

$\delta = \omega_e - \omega_g - \omega_{\rm L}$ is the detuning between the laser and atomic frequencies. Using the unitary transformation operator $U(H_A,t) = \exp({\rm i}H_At)$ and rotating-wave approximation we can transform H_B into the interaction frame as

$$H_{\rm I} = \delta|e\rangle\langle e| + \frac{\Omega}{2}(|e\rangle\langle g| + |g\rangle\langle e|), \tag{A5}$$

where $\Omega = -\wp\varepsilon_0$ is the Rabi frequency and characterizes the coupling between the atomic dipole and the laser electric field.

It is worth emphasizing that the transformation operator $U(H_A,t)$ depends on the value of $\omega_{\rm L}$ and then depends on the detunning δ. Therefore, if the values of detunning δ are different, the interaction frames we obtain by utilizing the transformation operators are not the same. Under this circumstance, we cannot calculate directly the overlap between two states evolving by the Hamiltonian (A5) with different detunings. Now, the question is how to deal with it? We provide another transformation process as follows. We choose the free Hamiltonian and interaction Hamiltonian as follows:

$$H_{A'} = \omega_e|e\rangle\langle e| + \omega_g|g\rangle\langle g|, \tag{A6}$$

and

$$H_{B'} = -\wp\varepsilon_0\cos(\omega_{\rm L}t + \varphi)(\sigma_+ + \sigma_-). \tag{A7}$$

Using the unitary transformation operator $U(H_{A'},t) = \exp({\rm i}H_{A'}t)$ and rotating-wave approximation we can obtain the Hamiltonian in the interaction frame as

$$H_{\rm I} = \frac{\Omega}{2}({\rm e}^{{\rm i}\delta t}|e\rangle\langle g| + {\rm e}^{-{\rm i}\delta t}|g\rangle\langle e|). \tag{A8}$$

In the process of transformation, unlike the previous method, the transformation operator is not relevant to the detuning. Hence the interaction frames are identical even if the detunings have different values. That means that we can calculate the overlap between two states directly after the two states have evolved under the Hamiltonian described by Eq. (A8) with different detuning values.

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