# New approach for anti-normally and normally ordering bosonic-operator functions in quantum optics*

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In this paper, we provide a new kind of operator formula for anti-normally and normally ordering bosonic-operator functions in quantum optics, which can help us arrange a bosonic-operator function $f(\lambda\hat{Q}+\nu\hat{P})$ in its anti-normal and normal ordering conveniently. Furthermore, mutual transformation formulas between anti-normal ordering and normal ordering, which have good universality, are derived too. Based on these operator formulas, some new differential relations and some useful mathematical integral formulas are easily derived without really performing these integrations.

**Keywords:** Baker–Hausdorff formula, operator differentiation, mutual transformation

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## 1. Introduction

The operator ordering problem,[1,2] which bothers people from time to time, plays an important role in constructing quantum mechanical operators[3] and describing the phase space of quantum mechanics.[4] The operator ordering problems are often encountered in quantizing a classical function and in calculating various matrix elements of operators. In quantum optics, operator ordering problems arise in converting operator master equations into $c$-number equations in phase space. As is well known, the anti-normally ordered expansion of an operator is very useful in calculating the $P$-representation of the density operator, because the $P$-representation of a density operator is[5]

$$\rho=\int\frac{\mathrm{d}^2z}{\pi}P(z)\left|z\right\rangle\left\langle z\right|, \tag{1}$$

where $\left|z\right\rangle=\exp(-\frac{1}{2}zz^*+z\hat{a}^\dagger)\left|0\right\rangle$ is the coherent state,[6–8] $\hat{a}^\dagger$ and $\hat{a}$ are bosonic creation and annihilation operators respectively, obeying $[\hat{a},\hat{a}^\dagger]=1$. The inverse transformation of Eq. (1) reads

$$P(z)=\mathrm{e}^{zz^*}\int\frac{\mathrm{d}^2\beta}{\pi}\left\langle-\beta\right|\rho\left|\beta\right\rangle\exp(\beta\beta^*+\beta^*z-\beta z^*), \tag{2}$$

which is the Mehta formula.[9] By substituting Eq. (2) into Eq. (1) we have

$$\rho=\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\left\langle-\beta\right|\rho\left|\beta\right\rangle\exp(\beta\beta^*+\beta^*\hat{a}-\beta\hat{a}^\dagger+\hat{a}^\dagger\hat{a})\vdots, \tag{3}$$

where the symbol $\vdots\ \vdots$ represents the anti-normal ordering (all $\hat{a}$ stand on the left of all $\hat{a}^\dagger$). Equations (2) and (3) imply that its $P$-representation can be obtained immediately once the anti-normal ordering form of a quantum operator is given. The normally ordered form of an operator is closely related to handling miscellaneous calculations of expectation values of observables in the coherent states, because the matrix elements of the normally ordered operator function $:F(\hat{a}^\dagger,\hat{a}):$ in coherent states directly yields

$$\left\langle z\right|:F(\hat{a}^\dagger,\hat{a}):\left|z'\right\rangle=F(z^*,z')\left\langle z/z'\right\rangle, \tag{4}$$

where the symbol $:\ :$ denotes the normal ordering (all $\hat{a}^\dagger$ stand on the left of all $\hat{a}$). Therefore, it is an important topic to obtain the anti-normal and normal ordering form of the operators in quantum optics.

To the best of our knowledge, there are three main methods of handling operator ordering problems in most published reports, such as the Lie algebra method,[10] Louisell's differential operation method via the coherent state representation,[11] and the newly developed technique of integration within ordered product (IWOP) of operators.[12–15] However, from the above mentioned reports, we notice that when the authors derived some operator normal ordering forms by using the IWOP technique, they had to first know the completeness relation of the eigenvector of the operator to be ordered as a pure Gaussian integration in the normal ordering form, and then make use of some complex mathematical integral formula. In this work, in order to avoid these inconveniences, we report several new formulas for anti-normally and normally ordering bosonic-operator functions. Namely, we directly utilize the Baker–Hausdorff formula and differential method of operators

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to derive new and neat formulas for anti-normally and normally ordering bosonic-operator functions. Using these formulas we can arrange a bosonic-operator function $f(\lambda\hat{Q}+\nu\hat{P})$ in its anti-normal and normal ordering conveniently, where $\hat{Q}=(\hat{a}^{\dagger}+\hat{a})/\sqrt{2}$ and $\hat{P}=(\hat{a}-\hat{a}^{\dagger})/\mathrm{i}\sqrt{2}$ are the coordinate operator and momentum operator respectively. Furthermore, the mutual transformation formulas between anti-normal ordering and normal ordering are derived too. Based on these operator formulas, some new differential relations and some useful mathematical integral formulas are easily derived without really performing these integrations.

## 2. The differential operation for transforming $f(\lambda\hat{Q}+\nu\hat{P})$ to $\vdots f(\lambda\hat{Q}+\nu\hat{P})\vdots$ and $:f(\lambda\hat{Q}+\nu\hat{P}):$

In this section we discuss how an operation can transform $f(\lambda\hat{Q}+\nu\hat{P})$ to $\vdots f(\lambda\hat{Q}+\nu\hat{P})\vdots$ and $:f(\lambda\hat{Q}+\nu\hat{P}):$. To begin with, we study the exponential operator $\mathrm{e}^{t(\lambda\hat{Q}+\nu\hat{P})}$. By virtue of the Baker–Hausdorff formula

$$\mathrm{e}^{\hat{A}+\hat{B}}=\mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}}\mathrm{e}^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

($[\hat{A},[\hat{A},\hat{B}]]=[\hat{B},[\hat{A},\hat{B}]]=0$), we can obtain the anti-normal ordering form of $\mathrm{e}^{t(\lambda\hat{Q}+\nu\hat{P})}$ as follows:

$$\begin{aligned}\mathrm{e}^{t(\lambda\hat{Q}+\nu\hat{P})} &= \vdots\exp\left[t(\lambda\hat{Q}+\nu\hat{P})-\frac{\lambda^2+\nu^2}{4}t^2\right]\vdots\\ &= \vdots\exp\left[-\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)\right]\exp[t(\lambda\hat{Q}+\nu\hat{P})]\vdots\\ &= \vdots\sum_{n=0}^{\infty}\frac{t^n}{n!}\exp\left[-\frac{\lambda^2+\nu^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right]\\ &\quad\times(\lambda\hat{Q}+\nu\hat{P})^n\vdots. \end{aligned}\tag{5}$$

With considering the expansion of

$$\mathrm{e}^{t(\lambda\hat{Q}+\nu\hat{P})}=\sum_{n=0}^{\infty}\frac{t^n}{n!}(\lambda\hat{Q}+\nu\hat{P})^n$$

and comparing the terms with the same power of $t$ on both sides of Eq. (5), we easily obtain a neat formula as follows:

$$\begin{aligned}&(\lambda\hat{Q}+\nu\hat{P})^n\\ &=\vdots\exp\left[-\frac{\lambda^2+\nu^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right](\lambda\hat{Q}+\nu\hat{P})^n\vdots. \end{aligned}\tag{6}$$

So, for any function $f(\lambda\hat{Q}+\nu\hat{P})$ which can be expanded as a power series of $(\lambda\hat{Q}+\nu\hat{P})^n$, it follows that

$$\begin{aligned}&f(\lambda\hat{Q}+\nu\hat{P})\\ &=\vdots\exp\left(-\frac{\lambda^2+\nu^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right)f(\lambda\hat{Q}+\nu\hat{P})\vdots, \end{aligned}\tag{7}$$

which is just the formula for converting bosonic-operator function $f(\lambda\hat{Q}+\nu\hat{P})$ into its anti-normal ordering. Noticing that $\partial/\partial\hat{Q}$, $\partial/\partial\hat{P}$, $\partial/\partial(\lambda\hat{Q}+\partial\nu\hat{P})$ are merely differential operations, not quantum mechanical operators. Their consequences are permutable and they may enter into or go out of the symbols $:\ :$ and $\vdots\ \vdots$ freely.

Following Eq. (7), we can obtain

$$\begin{aligned}&\vdots f(\lambda\hat{Q}+\nu\hat{P})\vdots\\ &=\exp\left(\frac{\lambda^2+\nu^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right)f(\lambda\hat{Q}+\nu\hat{P}), \end{aligned}\tag{8}$$

which is the new approach to eliminating the symbol $\vdots\ \vdots$ from $\vdots f(\lambda\hat{Q}+\nu\hat{P})\vdots$.

Similarly, we may obtain that

$$\begin{aligned}&f(\lambda\hat{Q}+\nu\hat{P})\\ &=:\exp\left(\frac{\lambda^2+\nu^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right)f(\lambda\hat{Q}+\nu\hat{P}):, \end{aligned}\tag{9}$$

which is just the differential formula for converting bosonic-operator function $f(\lambda\hat{Q}+\nu\hat{P})$ into its normal ordering. Equation (9) yields that

$$\begin{aligned}&:f(\lambda\hat{Q}+\nu\hat{P}):\\ &=\exp\left(-\frac{\lambda^2+\nu^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right)f(\lambda\hat{Q}+\nu\hat{P}), \end{aligned}\tag{10}$$

which is the new approach to eliminating the symbol $:\ :$ from $:f(\lambda\hat{Q}+\nu\hat{P}):$.

Moreover, by introducing $\sigma=\lambda-\mathrm{i}\nu/\sqrt{2}$ and $\tau=\lambda+\mathrm{i}\nu/\sqrt{2}$, we can rewrite Eqs. (7) and (9) as

$$\begin{aligned}&f(\sigma\hat{a}+\tau\hat{a}^{\dagger})\\ &=\vdots\exp\left(-\frac{\sigma\tau}{2}\frac{\partial^2}{\partial(\sigma\hat{a}+\tau\hat{a}^{\dagger})^2}\right)f(\sigma\hat{a}+\tau\hat{a}^{\dagger})\vdots, \end{aligned}\tag{11}$$

$$\begin{aligned}&f(\sigma\hat{a}+\tau\hat{a}^{\dagger})\\ &=:\exp\left(\frac{\sigma\tau}{2}\frac{\partial^2}{\partial(\sigma\hat{a}+\tau\hat{a}^{\dagger})^2}\right)f(\sigma\hat{a}+\tau\hat{a}^{\dagger}):. \end{aligned}\tag{12}$$

Especially, when $\lambda=1$ and $\nu=0$, we immediately obtain

$$\begin{aligned}f(\hat{Q})&=\vdots\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)f(\hat{Q})\vdots,\\ f(\hat{Q})&=:\exp\left(\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)f(\hat{Q}):, \end{aligned}\tag{13}$$

and when $\lambda=0$ and $\nu=1$, we have

$$\begin{aligned}f(\hat{P})&=\vdots\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\hat{P}^2}\right)f(\hat{P})\vdots,\\ f(\hat{P})&=:\exp\left(\frac{1}{4}\frac{\partial^2}{\partial\hat{P}^2}\right)f(\hat{P}):. \end{aligned}\tag{14}$$

In Refs. [16] and [17], with the help of some complex mathematical integral formulas, the IWOP technique and intermediate coordinate-momentum representation are used. In

Ref. [18], their results are merely the ordered form of some operators such as $\hat{Q}^n$, $\hat{P}^n$, and $(\lambda\hat{Q}+\nu\hat{P})^n$. Without doubt, we provide a new kind of operator formulas for anti-normally and normally ordering bosonic-operator functions including $f(\hat{Q})$, $f(\hat{P})$, and $f(\lambda\hat{Q}+\nu\hat{P})$, and our approach shows its features of simpleness, neatness and directness.

## 3. Differential operation for transforming $\hat{Q}^m\hat{P}^n$ ($\hat{P}^m\hat{Q}^n$) to its anti-normal and normal ordering

Now we discuss how to transform $\hat{Q}^m\hat{P}^n$ ($\hat{P}^m\hat{Q}^n$) into its anti-normal and normal ordering. Considering the exponential operator $\mathrm{e}^{\lambda\hat{Q}}\mathrm{e}^{\nu\hat{P}}$ and by virtue of $[\hat{Q},\hat{P}]=\mathrm{i}$, $[\hat{a},\hat{a}^\dagger]=1$, and the Baker–Hausdorff formula, we have

$$\mathrm{e}^{\lambda\hat{Q}}\mathrm{e}^{\nu\hat{P}}=\vdots\exp\left(\lambda\hat{Q}+\nu\hat{P}-\frac{\lambda^2+\nu^2}{4}+\frac{\mathrm{i}}{2}\lambda\nu\right)\vdots=\vdots\sum_{m,n=0}^{\infty}\frac{\lambda^m\nu^n}{m!n!}\exp\left[-\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)+\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial\hat{Q}\partial\hat{P}}\right]\hat{Q}^m\hat{P}^n\vdots. \quad (15)$$

With considering the expansion of

$$\mathrm{e}^{\lambda\hat{Q}}\mathrm{e}^{\nu\hat{P}}=\sum_{m,n=0}^{\infty}\frac{\lambda^m\nu^n}{m!n!}\hat{Q}^m\hat{P}^n$$

and comparing the terms with the same powers of $\lambda$ and $\nu$ on both sides of Eq. (15), we easily obtain the neat formula

$$\hat{Q}^m\hat{P}^n=\vdots\exp\left[-\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)+\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial\hat{Q}\partial\hat{P}}\right]\hat{Q}^m\hat{P}^n\vdots, \quad (16)$$

which is just the differential formula for converting bosonic-operator $\hat{Q}^m\hat{P}^n$ into its anti-normal ordering. Following it, we have

$$\vdots\hat{Q}^m\hat{P}^n\vdots=\exp\left[\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)-\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial\hat{Q}\partial\hat{P}}\right]\hat{Q}^m\hat{P}^n, \quad (17)$$

which is the new approach to eliminate the symbol $\vdots\ \vdots$ from $\vdots\hat{Q}^m\hat{P}^n\vdots$. In a similar way, the normal ordering expansion of $\hat{Q}^m\hat{P}^n$, and the anti-normal and normal ordering expansions of $\hat{P}^m\hat{Q}^n$ are described respectively as

$$\begin{aligned}
\hat{Q}^m\hat{P}^n&=:\exp\left[\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)+\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial\hat{Q}\partial\hat{P}}\right]\hat{Q}^m\hat{P}^n:,\\
\hat{P}^m\hat{Q}^n&=\vdots\exp\left[-\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)-\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial\hat{Q}\partial\hat{P}}\right]\hat{P}^m\hat{Q}^n\vdots,\\
\hat{P}^m\hat{Q}^n&=:\exp\left[\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)-\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial\hat{Q}\partial\hat{P}}\right]\hat{P}^m\hat{Q}^n:. \quad (18)
\end{aligned}$$

For example

$$\begin{aligned}
\hat{Q}^2\hat{P}&=\vdots\exp\left[-\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)+\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial\hat{Q}\partial\hat{P}}\right]\hat{Q}^2\hat{P}\vdots\\
&=\vdots\hat{Q}^2\hat{P}\vdots-\frac{1}{2}\hat{P}+\mathrm{i}\hat{Q}. \quad (19)
\end{aligned}$$

Moreover, using the Hermite polynomial

$$\begin{aligned}
\mathrm{H}_n(x)&=\frac{\partial^n}{\partial t^n}\mathrm{e}^{-t^2+2xt}\Big|_{t=0}=\mathrm{e}^{-(1/4)\partial^2/\partial x^2}(2x)^n\\
&=\sum_{l=0}^{[n/2]}\frac{(-1)^l n!}{l!(n-2l)!}(2x)^{n-2l}
\end{aligned}$$

and the two-variable Hermite polynomial[19,20]

$$\begin{aligned}
\mathrm{H}_{m,n}(x,y)&=\frac{\partial^{m+n}}{\partial t_1^m\partial t_2^n}\mathrm{e}^{-t_1t_2+t_1x+t_2y}\Big|_{t_1,t_2=0}\\
&=\exp\left(-\frac{\partial^2}{\partial x\partial y}\right)x^m y^n\\
&=\sum_{j=0}^{\min(m,n)}\frac{(-1)^j m!n!}{j!(m-j)!(n-j)!}x^{m-j}y^{n-j}, \quad (20)
\end{aligned}$$

we may derive that

$$\begin{aligned}
\hat{Q}^m\hat{P}^n&=\vdots\exp\left[-\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)+\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial\hat{Q}\partial\hat{P}}\right]\hat{Q}^m\hat{P}^n\vdots\\
&=\frac{1}{2^m\mathrm{i}^n}\vdots\exp\left[-\frac{1}{4}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)\right]\mathrm{H}_{m,n}(2\hat{Q},\mathrm{i}\hat{P})\vdots\\
&=\sum_{j=0}^{\min(m,n)}\frac{\mathrm{i}^j m!n!}{2^{m+n-j}j!(m-j)!(n-j)!}\\
&\quad\vdots\mathrm{H}_{m-j}(\hat{Q})\cdot\mathrm{H}_{n-j}(\hat{P})\vdots. \quad (21)
\end{aligned}$$

which is the anti-normally ordered expansion of $\hat{Q}^m\hat{P}^n$. In fact, equations (16)–(18) provide the mutual transformation formulas between the quantum mechanical operators whose all $\hat{Q}$ (or $\hat{P}$) stand on the left hand of all $\hat{P}$ (or $\hat{Q}$) and their anti-normal and normal ordering.

## 4. Mutual transformation formulas between anti-normal and normal ordering

From Eqs. (7) and (9) we can derive the mutual transformation between anti-normal and normal ordering, i.e.,

$$\begin{aligned}
&:f(\lambda\hat{Q}+\nu\hat{P}):\\
&=\vdots\exp\left(-\frac{\lambda^2+\nu^2}{2}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right)f(\lambda\hat{Q}+\nu\hat{P})\vdots. \quad (22)
\end{aligned}$$

Since $\partial^2/\partial\hat{Q}^2+\partial^2/\partial\hat{P}^2=(\lambda^2+\nu^2)\partial^2/\partial(\lambda\hat{Q}+\nu\hat{P})^2$, the above equation can be rewritten as

$$:f(\hat{Q},\hat{P}):=\vdots\exp\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\hat{Q}^2}+\frac{\partial^2}{\partial\hat{P}^2}\right)\right]f(\hat{Q},\hat{P})\vdots, \quad (23)$$

which is the transformation formula for converting a normally ordered operator function into its anti-normally ordered form.

Here, we have replaced $f(\lambda\hat{Q}+\nu\hat{P})$ with an arbitrary operator function $f(\hat{Q},\hat{P})$ because it is in an ordered product symbol. Using $\hat{Q}=(\hat{a}^\dagger+\hat{a})/\sqrt{2}$ and $\hat{P}=(\hat{a}-\hat{a}^\dagger)/\mathrm{i}\sqrt{2}$, equation (23) may be represented as

$$:f(\hat{a},\hat{a}^\dagger):=\vdots\exp\left(-\frac{\partial^2}{\partial\hat{a}\partial\hat{a}^\dagger}\right)f(\hat{a},\hat{a}^\dagger)\vdots. \tag{24}$$

For example, $\hat{a}^{\dagger 2}\hat{a}$, which is normal ordering, then

$$\begin{aligned}\hat{a}^{\dagger 2}\hat{a} &= \vdots\exp\left(-\frac{\partial^2}{\partial\hat{a}\partial\hat{a}^\dagger}\right)\hat{a}^{\dagger 2}\hat{a}\vdots=\vdots\hat{a}^{\dagger 2}\hat{a}-2\hat{a}^\dagger\vdots\\ &= \hat{a}\hat{a}^{\dagger 2}-2\hat{a}^\dagger.\end{aligned} \tag{25}$$

The reciprocal transformation of Eq. (24) reads

$$\vdots f(\hat{a},\hat{a}^\dagger)\vdots=:\exp\left(\frac{\partial^2}{\partial\hat{a}\partial\hat{a}^\dagger}\right)f(\hat{a},\hat{a}^\dagger):, \tag{26}$$

which is the transformation formula for converting an anti-normally ordered operator function into its normally ordered form. For example, $\hat{a}^m\hat{a}^{\dagger n}$ is anti-normal ordering, and its compact form of normal ordering can be obtained as

$$\begin{aligned}\hat{a}^m\hat{a}^{\dagger n} &= :\exp\left(\frac{\partial^2}{\partial\hat{a}\partial\hat{a}^\dagger}\right)\hat{a}^m\hat{a}^{\dagger n}:\\ &= (-\mathrm{i})^{m+n}:\exp\left(-\frac{\partial^2}{\partial(\mathrm{i}\hat{a})\partial(\mathrm{i}\hat{a}^\dagger)}\right)(\mathrm{i}\hat{a})^m(\mathrm{i}\hat{a}^\dagger)^n:\\ &= (-\mathrm{i})^{m+n}:\mathrm{H}_{m,n}(\mathrm{i}\hat{a},\mathrm{i}\hat{a}^\dagger):,\end{aligned} \tag{27}$$

where $\mathrm{i}=\sqrt{-1}$ and $\mathrm{H}_{m,n}(x,y)$ is the two-variable Hermite polynomial function.

## 5. Applications

We now give several examples of some applications of the above operator formulas as follows.

**Example 1** First of all, we consider the Hermite polynomial operator function $\mathrm{H}_n(\hat{Q})$, which is defined by

$$\mathrm{H}_n(\hat{Q})=\frac{\partial^n}{\partial t^n}\exp(-t^2+2\hat{Q}t)\bigg|_{t=0}. \tag{28}$$

By virtue of Eq. (13), we know that

$$\begin{aligned}\mathrm{H}_n(\hat{Q}) &= :\exp\left(\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\mathrm{H}_n(\hat{Q}):\\ &= :\exp\left(\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\frac{\partial^n}{\partial t^n}\exp(-t^2+2\hat{Q}t)\bigg|_{t=0}:\\ &= :\frac{\partial^n}{\partial t^n}\exp(2\hat{Q}t)\bigg|_{t=0}:=:(2\hat{Q})^n:,\end{aligned} \tag{29}$$

which is the normally ordered form of $\mathrm{H}_n(\hat{Q})$. In addition, equation (29) implies that

$$\begin{aligned}&\exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)\mathrm{H}_n(x)=(2x)^n,\\ &\mathrm{H}_n(x)=\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^n,\end{aligned} \tag{30}$$

which are two new differential formulas for the Hermite polynomial $\mathrm{H}_n(x)$, and will be quite useful in our work. Furthermore, the anti-normally ordered expansion of $\mathrm{H}_n(\hat{Q})$ can be obtained, i.e.,

$$\begin{aligned}\mathrm{H}_n(\hat{Q}) &= \vdots\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\mathrm{H}_n(\hat{Q})\vdots\\ &= \vdots\exp\left(-\frac{1}{2}\frac{\partial^2}{\partial\hat{Q}^2}\right)(2\hat{Q})^n\vdots\\ &= 2^{n/2}\vdots\mathrm{H}_n(\hat{Q}/\sqrt{2})\vdots,\end{aligned} \tag{31}$$

where we have used Eq. (30) repeatedly.

**Example 2** Following Eqs. (7) and (30) we have

$$\begin{aligned}&\mathrm{H}_n(\lambda\hat{Q}+\nu\hat{P})\\ &= \vdots\exp\left(-\frac{\lambda^2+\nu^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right)\mathrm{H}_n(\lambda\hat{Q}+\nu\hat{P})\vdots\\ &= \vdots\exp\left(-\frac{1+\lambda^2+\nu^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right)[2(\lambda\hat{Q}+\nu\hat{P})]^n\vdots\\ &= (\sqrt{1+\lambda^2+\nu^2})^n\vdots\mathrm{H}_n\left(\frac{\lambda\hat{Q}+\nu\hat{P}}{\sqrt{1+\lambda^2+\nu^2}}\right)\vdots,\end{aligned} \tag{32}$$

which is the anti-normal ordering form of $\mathrm{H}_n(\lambda\hat{Q}+\nu\hat{P})$. In a similar way, the normal ordering expansion of $\mathrm{H}_n(\lambda\hat{Q}+\nu\hat{P})$ is described as

$$\begin{aligned}&\mathrm{H}_n(\lambda\hat{Q}+\nu\hat{P})\\ &= (\sqrt{1-\lambda^2-\nu^2})^n:\mathrm{H}_n\left(\frac{\lambda\hat{Q}+\nu\hat{P}}{\sqrt{1-\lambda^2-\nu^2}}\right):.\end{aligned} \tag{33}$$

**Example 3** By using Eqs. (7) and (30), we may obtain the anti-normal ordered form of $(\lambda\hat{Q}+\nu\hat{P})^n$ as follows:

$$\begin{aligned}(\lambda\hat{Q}+\nu\hat{P})^n &= \vdots\exp\left(-\frac{\lambda^2+\nu^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+\nu\hat{P})^2}\right)(\lambda\hat{Q}+\nu\hat{P})^n\vdots\\ &= \left(\frac{\sqrt{\lambda^2+\nu^2}}{2}\right)^n\vdots\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\left(\frac{\lambda\hat{Q}+\nu\hat{P}}{\sqrt{\lambda^2+\nu^2}}\right)^2}\right)\left(2\frac{\lambda\hat{Q}+\nu\hat{P}}{\sqrt{\lambda^2+\nu^2}}\right)^n\vdots=\left(\frac{\sqrt{\lambda^2+\nu^2}}{2}\right)^n\vdots\mathrm{H}_n\left(\frac{\lambda\hat{Q}+\nu\hat{P}}{\sqrt{\lambda^2+\nu^2}}\right)\vdots.\end{aligned} \tag{34}$$

Similarly, by virtue of Eqs. (9) and (30), the normal ordered form of $(\lambda\hat{Q}+v\hat{P})^n$ can be obtained as

$$(\lambda\hat{Q}+v\hat{P})^n = :\exp\left(\frac{\lambda^2+v^2}{4}\frac{\partial^2}{\partial(\lambda\hat{Q}+v\hat{P})^2}\right)(\lambda\hat{Q}+v\hat{P})^n: = \left(\frac{\sqrt{\lambda^2+v^2}}{2\mathrm{i}}\right)^n :\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\left(\mathrm{i}\frac{\lambda\hat{Q}+v\hat{P}}{\sqrt{\lambda^2+v^2}}\right)^2}\right)\left(2\mathrm{i}\frac{\lambda\hat{Q}+v\hat{P}}{\sqrt{\lambda^2+v^2}}\right)^n: = \left(\frac{\sqrt{\lambda^2+v^2}}{2\mathrm{i}}\right)^n :\mathrm{H}_n\left(\mathrm{i}\frac{\lambda\hat{Q}+v\hat{P}}{\sqrt{\lambda^2+v^2}}\right):. \quad (35)$$

This result is Eq. (11) of Ref. [17] and Eq. (11) of Ref. [18]. By comparison, our approach is simple and concise.

**Example 4** Using Eq. (29), we may derive the matrix elements of $\mathrm{H}_n(\hat{Q})$ in coherent states directly as

$$\begin{aligned}&\langle z|\,\mathrm{H}_n(\hat{Q})\,|z'\rangle = 2^n\langle z|:\hat{Q}^n:|z'\rangle\\ &= 2^{n/2}(z^*+z')^n\langle z\,|\,z'\rangle\\ &= 2^{n/2}(z^*+z')^n\,\mathrm{e}^{(-zz^*-z'z'^*+2z^*z')/2}.\end{aligned} \quad (36)$$

Here, our method is simple and convenient with utilizing neither the completeness relation of coordinate representation $\int_{-\infty}^{\infty}\mathrm{d}q\,|q\rangle\langle q| = 1$ nor any complex mathematical integral formula.

**Example 5** Substituting $\delta(q-\hat{Q})$ for $f(\hat{Q})$ in Eq. (8), it follows that

$$\exp\left(\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\delta(q-\hat{Q}) = :\delta(q-\hat{Q}):. \quad (37)$$

On the other hand

$$\begin{aligned}&\exp\left(\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\delta(q-\hat{Q})\\ &= \exp\left(\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}u(q-\hat{Q})}\mathrm{d}u\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{4}u^2+\mathrm{i}u(q-\hat{Q})\right]\mathrm{d}u\\ &= \frac{1}{\sqrt{\pi}}\exp[-(q-\hat{Q})^2],\end{aligned} \quad (38)$$

so

$$:\delta(q-\hat{Q}): = \frac{1}{\sqrt{\pi}}\exp[-(q-\hat{Q})^2]. \quad (39)$$

By using Eq. (7) once again, we have

$$\begin{aligned}&\frac{1}{\sqrt{\pi}}\exp[-(q-\hat{Q})^2]\\ &= \frac{1}{\sqrt{\pi}}:\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\exp[-(q-\hat{Q})^2]:.\end{aligned} \quad (40)$$

Comparing Eq. (39) with Eq. (40), we can obtain

$$\frac{1}{\sqrt{\pi}}\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial q^2}\right)\exp[-(q-q')^2] = \delta(q-q'), \quad (41)$$

which is a new differential form of Dirac's delta function. The reciprocal transformation of Eq. (41) reads

$$\exp\left(\frac{1}{4}\frac{\partial^2}{\partial q^2}\right)\delta(q-q') = \frac{1}{\sqrt{\pi}}\exp[-(q-q')^2], \quad (42)$$

which will be useful in operator-ordering theory and special function theory.

**Example 6** Replacing $f(\hat{Q})$ with $\mathrm{e}^{-\lambda\hat{Q}^2}$ in Eq. (8), we have

$$\begin{aligned}\mathrm{e}^{-\lambda\hat{Q}^2} &= :\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\mathrm{e}^{-\lambda\hat{Q}^2}:\\ &= :\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\hat{Q}^{2n}:\\ &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{2^{2n}n!}:\mathrm{H}_{2n}(\hat{Q}):,\end{aligned} \quad (43)$$

which is the anti-normal ordering form of $\mathrm{e}^{-\lambda\hat{Q}^2}$ for an arbitrary parameter $\lambda$.

Imposing the condition $1>\lambda>0$ yields that

$$\begin{aligned}\mathrm{e}^{-\lambda\hat{Q}^2} &= :\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\mathrm{e}^{-\lambda\hat{Q}^2}:\\ &= :\exp\left(\frac{1-\lambda}{4\lambda}\frac{\partial^2}{\partial\hat{Q}^2}\right)\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial(\sqrt{\lambda}\hat{Q})^2}\right)\mathrm{e}^{-\lambda\hat{Q}^2}:\\ &= \sqrt{\pi}:\exp\left(\frac{1-\lambda}{4\lambda}\frac{\partial^2}{\partial\hat{Q}^2}\right)\delta(\sqrt{\lambda}\hat{Q}):\\ &= \frac{1}{\sqrt{1-\lambda}}:\exp\left(-\frac{\lambda}{1-\lambda}\hat{Q}^2\right):.\end{aligned} \quad (44)$$

In the above calculation we have used Eqs. (41) and (42) repeatedly. Obviously, when $\lambda=0$, then $\mathrm{e}^{-\lambda\hat{Q}^2}=1$, and when $\lambda=1$, then $\mathrm{e}^{-\lambda\hat{Q}^2} = \sqrt{\pi}:\delta(\hat{Q}):$.

**Example 7** Now we prove the relation between the Laguerre polynomial and the Hermite polynomial. By virtue of Eq. (7) we know

$$L_n(\hat{Q}) = :\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)L_n(\hat{Q}):, \quad (45)$$

where the definition of the Laguerre polynomial is $L_n(x) = \sum_{l=0}^{n}(-1)^l C_n^l n!x^l/l!$. On the other hand,

$$\begin{aligned}L_n(\hat{Q}) &= \sum_{l=0}^{n}(-1)^l C_n^l\frac{n!}{l!}:\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\hat{Q}^2}\right)\hat{Q}^l:\\ &= \sum_{l=0}^{n}(-1)^l C_n^l\frac{n!}{2^l l!}:\mathrm{H}_l(\hat{Q}):.\end{aligned} \quad (46)$$

Comparing Eq. (45) with Eq. (46), we can obtain

$$\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)L_n(x) = \sum_{l=0}^{n}(-1)^l C_n^l\frac{n!}{2^l l!}\mathrm{H}_l(x), \quad (47)$$

the above equation shows that the Laguerre polynomial and the Hermite polynomial are related by the differential operator $\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)$.

**Example 8** Now we come to deduce a useful mathematical integral formula. By using the completeness relation of coordinate representation, we may have

$$f(\hat{Q}) = \int_{-\infty}^{\infty} \mathrm{d}q f(q) |q\rangle \langle q| = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}q f(q) : \mathrm{e}^{-(q-Q)^2} : . \tag{48}$$

On the other hand, by virtue of Eq. (7) we have

$$f(\hat{Q}) = : \exp\left(\frac{1}{4}\frac{\partial^2}{\partial \hat{Q}^2}\right) f(\hat{Q}) : .$$

By comparison, the mathematical integral formula is given as

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x f(x) \mathrm{e}^{-(x-y)^2} = \exp\left(\frac{1}{4}\frac{\partial^2}{\partial y^2}\right) f(y), \tag{49}$$

which holds true for any function $f(x)$. Especially, when $f(x) = \mathrm{H}_n(x)$, equation (49) turns into

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{H}_n(x) \mathrm{e}^{-(x-y)^2} \mathrm{d}x = \exp\left(\frac{1}{4}\frac{\partial^2}{\partial y^2}\right) \mathrm{H}_n(y) = 2^n y^n, \tag{50}$$

which is easily derived, without really performing the integration. By the way, Eq. (49) can be seen as a mathematical or physical transformation.

**Example 9** As a final example, using the over-completeness relation of coherent state

$$\int \frac{\mathrm{d}^2 z}{\pi} |z\rangle \langle z| = 1$$

and $\hat{a}|z\rangle = z|z\rangle$, we have

$$\vdots f(\hat{a}, \hat{a}^\dagger) \vdots = \int \frac{\mathrm{d}^2 z}{\pi} f(z, z^*) |z\rangle \langle z| = \int \frac{\mathrm{d}^2 z}{\pi} f(z, z^*) : \exp(-zz^* + z\hat{a}^\dagger + z^* \hat{a} - \hat{a}^\dagger \hat{a}) : . \tag{51}$$

On the other hand, it follows from Eq. (26) that

$$\vdots f(\hat{a}, \hat{a}^\dagger) \vdots = : \exp\left(\frac{\partial^2}{\partial \hat{a} \partial \hat{a}^\dagger}\right) f(\hat{a}, \hat{a}^\dagger) : .$$

Since in the normal ordering, symbols $\hat{a}$ and $\hat{a}^\dagger$ are commutative, they may be treated as a $c$-number. Setting $\hat{a}^\dagger \to \xi$ and $\hat{a} \to \eta$, we easily obtain

$$\mathrm{e}^{-\eta\xi} \int \frac{\mathrm{d}^2 z}{\pi} f(z, z^*) \exp(-zz^* + z\xi + z^*\eta) = \exp\left(\frac{\partial^2}{\partial \xi \partial \eta}\right) f(\eta, \xi). \tag{52}$$

This integral transformation formula holds true for any function $f(z, z^*)$. Especially, when $f(z, z^*) = z^n z^{*m}$, we can derive the following identity:

$$\mathrm{e}^{-\eta\xi} \int \frac{\mathrm{d}^2 z}{\pi} z^n z^{*m} \exp(-zz^* + z\xi + z^*\eta) = \exp\left(\frac{\partial^2}{\partial \xi \partial \eta}\right) \eta^n \xi^m = (-\mathrm{i})^{n+m} \mathrm{H}_{n,m}(\mathrm{i}\eta, \mathrm{i}\xi), \tag{53}$$

which is often used in some published reports.

## 6. Conclusions

In this work, by directly employing the Baker–Hausdorff formula and the operator differential method, we find a new kind of operator formula for anti-normally and normally ordering bosonic-operator functions, which can help us transform some operators such as $f(\hat{Q})$, $f(\hat{P})$, $f(\lambda\hat{Q} + \nu\hat{P})$, $\hat{Q}^m\hat{P}^n$, and $\hat{P}^m\hat{Q}^n$ into their anti-normal and normal ordering conveniently. As their important byproducts, the mutual transformation formulas between anti-normal ordering and normal ordering are derived too, which have the good universality. Furthermore, based on these operator formulas, some new differential relations and some useful mathematical integral formulas are easily derived without really performing these integrations. These new formulas are useful in operator-ordering theory and special function theory. By comparison, our method is simple and neat.

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