Dynamics of two arbitrary qubits strongly coupled to a quantum oscillator*

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Using adiabatic approximation, a two arbitrary qubits Rabi model has been studied in ultra-strong coupling. The analytical expressions of the eigenvalues and the eigenvalues are obtained. They are in accordance with the numerical determined results. The dynamical behavior of the system and the evolution of entanglement have also been discussed. The collapse and revival phenomena has garnered particular attention. The influence of inconsistent coupling strength on them is studied. These results will be applied in quantum information processing.

Keywords: Rabi model, ultra-strong coupling, adiabatic approximation

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1. Introduction

The quantum Rabi model (QRM) is a significant model in quantum optics. It describes the system of a two-level atom (qubit) interacting with a single mode of a quantum harmonic oscillator. This model was initially presented in 1936 by Rabi in order to study the nuclear magnetic spin resonance.[1,2] The QRM is of great interest in circuit quantum electrodynamics (QED)[3–6] and cavity QED,[7–13] both experimentally and theoretically. Though the Rabi model has been studied for over 80 years, analytical solutions for the eigenvalues and eigenfunctions was recently presented by Braak[14,15] and Chen[16] due to the mathematical difficulty. However, there are still many difficulties in the application due to its complicated form.[17] So a number of approximations have been developed to solve QRM. The rotating wave approximation(RWA)[18,19] is the most widely used. However, the RWA relies upon the assumption of near resonance and weak coupling between qubits and oscillator. With the development of circuit QED and cavity QED, it has achieved so-called ultra-strong coupling between qubits and oscillator. Motivated by those experimental developments, another approximation was investigated by Irish *et al.*, called the adiabatic approximation (AA).[20–23] This approximation was shown to work best at Quasi-degenerate qubit and ultra-strong coupling.

The QRM with two qubits has received more attention as it can be used to implement quantum entanglement and quantum information processing (QIP).[24–34] The QRM with two qubits can be realized in several solid devices.[35,36] There have been some previous theoretical discussions to solve the QRM with two qubits.[21,37–44] Chen *et al.* presented a concise analytic solution to the QRM with two qubits using the method of extended coherent states.[44] However, in some cases,[21,43] the two qubits are not arbitrary qubits, instead identical qubits and the same coupling with the quantum harmonic oscillator. In other cases, the dynamical properties and the evolution of entanglement have not been sufficiently studied. Accordingly, it is very necessary to study of the dynamical properties and the evolution of entanglement between two arbitrary qubits in the ultra-strong coupling strength range.

The organization of this paper is as follows. In the first part, we use the AA to calculate the eigenvalues and eigenfunctions of the QRM with two arbitrary qubits. In the second part, we discuss the evolution of this system. The evolution of entanglement is given in last part.

2. The Spectrum of model Hamiltonian

The Hamiltonian of the QRM with two arbitrary qubits can be written as[38–40]

$$\hat{H} = \hbar\frac{\omega_1}{2}\hat{\sigma}_{z,1} + \hbar\frac{\omega_2}{2}\hat{\sigma}_{z,2} + \hbar\omega\hat{a}^{\dagger}\hat{a} + \hbar\omega\frac{\beta_1}{2}(\hat{a}+\hat{a}^{\dagger})\hat{\sigma}_{x,1} + \hbar\omega\frac{\beta_2}{2}(\hat{a}+\hat{a}^{+})\hat{\sigma}_{x,2}, \quad (1)$$

where $\hat{\sigma}_{z,i}$ and $\hat{\sigma}_{x,i}$ $(i = 1,2)$ are the usual Pauli matrices in the Hilbert space of the i-th qubit, and $\hat{a}^{\dagger}$ and $\hat{a}$ refer to the creation and the annihilation operators of an interacting mode of a harmonic oscillator, ω_i and ω is the frequency of the i-th qubit and quantum harmonic oscillator respectively. β_i is the coupling parameter between the i-th qubit and the quantum harmonic oscillator.

The first task is to establish a new set of basis states. The self-energy of two qubits $\hbar(\omega_1/2)\hat{\sigma}_{z,1} + \hbar(\omega_2/2)\hat{\sigma}_{z,2}$ can be treated as the perturbation when $\omega_i \ll \omega$. Then the Hamiltonian can be written as

$$\hat{H}_0 = \hbar\omega\hat{a}^{\dagger}\hat{a} + \hbar\omega\frac{\beta_1}{2}(\hat{a}+\hat{a}^{\dagger})\hat{\sigma}_{x,1} + \hbar\omega\frac{\beta_2}{2}(\hat{a}+\hat{a}^{\dagger})\hat{\sigma}_{x,2}, \quad (2)$$

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The eigenvalue equation of $\hat{H}_0$ is

$$\hat{H}_0 = \hbar\omega\left[\hat{a}^\dagger\hat{a} + \frac{\beta_1}{2}(\hat{a}+\hat{a}^\dagger)\hat{\sigma}_{x,1} + \frac{\beta_2}{2}(\hat{a}+\hat{a}^\dagger)\hat{\sigma}_{x,2}\right]|\Phi\rangle = E|\Phi\rangle. \tag{3}$$

The eigenstate form of $|\Phi\rangle$ is

$$|\Phi\rangle = |m\rangle|\phi_m\rangle,$$

where $|m\rangle$ denotes the eigenstate of $(\beta_1/2)\hat{\sigma}_{x,1} + (\beta_2/2)\hat{\sigma}_{x,2}$, and $|\phi_m\rangle$ are the oscillator eigenstates of $\hat{H}_0$ by replacing $(\beta_1/2)\hat{\sigma}_{x,1} + (\beta_2/2)\hat{\sigma}_{x,2}$ to $|m\rangle$.

The eigenvalues of $(\beta_1/2)\hat{\sigma}_{x,1} + (\beta_2/2)\hat{\sigma}_{x,2}$ are m:

$$m_1 = (\beta_1+\beta_2)/2, \quad m_2 = (\beta_2-\beta_1)/2,$$
$$m_3 = (\beta_1-\beta_2)/2, \quad m_4 = -(\beta_1+\beta_2)/2.$$

In the matrix picture, $|m\rangle$, the eigenstate of $(\beta_1/2)\hat{\sigma}_{x,1} + (\beta_2/2)\hat{\sigma}_{x,2}$ can be written as:

$$|m_1\rangle = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad |m_2\rangle = \frac{1}{2}\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix},$$
$$|m_3\rangle = \frac{1}{2}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}, \quad |m_4\rangle = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}. \tag{4}$$

And $|\phi_m\rangle$ satisfies the eigenvalue equation

$$\hbar\omega[\hat{a}^\dagger\hat{a} + m(\hat{a}+\hat{a}^\dagger)]|\phi_m\rangle = E|\phi_m\rangle. \tag{5}$$

Applying the displacement operator $\hat{D}^\dagger(\lambda) = \exp[-\lambda(\hat{a}^\dagger - \hat{a})]$ (for real λ) and the inverse operator of displacement to Eq. (5), we obtain

$$[\hat{a}^\dagger\hat{a} + \lambda(\hat{a}^\dagger+\hat{a}) + \lambda^2 + m(\hat{a}^\dagger+\hat{a}) + 2m\lambda][\hat{D}^\dagger(\lambda)|\phi_m\rangle] = \frac{E}{\hbar\omega}[\hat{D}^\dagger(\lambda)|\phi_m\rangle]. \tag{6}$$

Setting $\lambda = -m$, we can obtain

$$\hat{a}^\dagger\hat{a}[\hat{D}^\dagger(-m)|\phi_m\rangle] = \left[\frac{E}{\hbar\omega} + m^2\right][\hat{D}^\dagger(-m)|\phi_m\rangle] = N[\hat{D}^\dagger(-m)|\phi_m\rangle]. \tag{7}$$

where $N = E/\hbar\omega + m^2$. According to the eigenvalue equation of the number operator, we can obtain

$$\hat{D}^\dagger(-m)|\phi_m\rangle = |N\rangle, or|\phi_m\rangle = \hat{D}(-m)|N\rangle \equiv |N_m\rangle. \tag{8}$$

Thus, we obtain the eigenstates of $\hat{H}_0$

$$|\Phi\rangle \longrightarrow |\Phi_{m,N}\rangle = |m\rangle|N_m\rangle, \tag{9}$$

and eigenvalue

$$E_{N,m} = \hbar\omega(N - m^2). \tag{10}$$

These are so-called displaced oscillator basis states.[20,21]

The Hamiltonian can be rewritten in the basis of the displaced oscillator basis states. The transition matrix element between different N is

$$H_{N,N'} = \hbar\omega_1|\langle m_i|\frac{\sigma_{z,1}}{2}|m_i\rangle\langle N_{m_i}|N'_{m_i}\rangle| + \hbar\omega_2|\langle m_i|\frac{\sigma_{z,2}}{2}|m_i\rangle\langle N_{m_i}|N'_{m_i}\rangle|. \tag{11}$$

If $H_{N,N'}$ is much smaller than the energy gap between different N:

$$\hbar\omega_1|\langle m_i|\frac{\sigma_{z,1}}{2}|m_i\rangle\langle N_{m_i}|N'_{m_i}\rangle| + \hbar\omega_2|\langle m_i|\frac{\sigma_{z,2}}{2}|m_i\rangle\langle N_{m_i}|N'_{m_i}\rangle| \ll \omega|N-N'|, \tag{12}$$

the transitions between different N can be ignored. This approximation is reasonable when $\omega_i \ll \omega$. Accordingly, we only preserve the transitions between states with the coincident N. So we can use basis $|m\rangle|N_m\rangle$ to obtain the block diagonal form of the Hamiltonian. This approximation was used by Irish *et al.*[20] and Agarwal *et al.*[21] to study a single qubit case or identical qubits case. For each N, in the order of $|m_1\rangle|N_{m_1}\rangle$, $|m_2\rangle|N_{m_2}\rangle$, $|m_3\rangle|N_{m_3}\rangle$, and $|m_4\rangle|N_{m_4}\rangle$ for rows and columns, the Hamiltonian has the form

$$\hat{H}_N = \hbar\omega\begin{pmatrix} N-m_1^2 & \Omega_{N,1} & \Omega_{N,2} & 0 \\ \Omega_{N,1} & N-m_2^2 & 0 & \Omega_{N,2} \\ \Omega_{N,2} & 0 & N-m_3^2 & \Omega_{N,1} \\ 0 & \Omega_{N,2} & \Omega_{N,1} & N-m_4^2 \end{pmatrix}. \tag{13}$$

The off-diagonal terms $\Omega_{N,1}$ and $\Omega_{N,2}$ are given by

$$\Omega_{N,1} = -\sqrt{\frac{1}{2}}\frac{\omega_1}{\omega}\,\mathrm{e}^{-\beta_1^2/2}\,\mathrm{L}_N(\beta_1^2), \qquad \Omega_{N,2} = -\sqrt{\frac{1}{2}}\frac{\omega_2}{\omega}\,\mathrm{e}^{-\beta_2^2/2}\,\mathrm{L}_N(\beta_2^2), \tag{14}$$

where $\mathrm{L}_N(x)$ is the Laguerre polynomial. The normalized eigenfunctions and eigenvalues of $\hat{H}_N$ are

$$|E_N^1\rangle = \frac{1}{l_{N,1}}\begin{pmatrix}1\\ y_{N,1}\\ y_{N,1}\\ 1\end{pmatrix}, \quad |E_N^2\rangle = \frac{1}{l_{N,2}}\begin{pmatrix}1\\ y_{N,2}\\ y_{N,2}\\ 1\end{pmatrix},$$

$$|E_N^3\rangle = \frac{1}{l_{N,3}}\begin{pmatrix}-1\\ y_{N,3}\\ -y_{N,3}\\ 1\end{pmatrix}, \quad |E_N^4\rangle = \frac{1}{l_{N,4}}\begin{pmatrix}-1\\ y_{N,4}\\ -y_{N,4}\\ 1\end{pmatrix},$$

$$y_{N,1} = \frac{-2(\Omega_{N,1}+\Omega_{N,2})}{\beta_1\beta_2 - \sqrt{\beta_1^2\beta_2^2+4(\Omega_{N,1}+\Omega_{N,2})^2}}, \quad y_{N,2} = \frac{-2(\Omega_{N,1}+\Omega_{N,2})}{\beta_1\beta_2 + \sqrt{\beta_1^2\beta_2^2+4(\Omega_{N,1}+\Omega_{N,2})^2}},$$

$$y_{N,3} = \frac{-2(\Omega_{N,2}-\Omega_{N,1})}{\beta_1\beta_2 - \sqrt{\beta_1^2\beta_2^2+4(\Omega_{N,1}-\Omega_{N,2})^2}}, \quad y_{N,4} = \frac{-2(\Omega_{N,2}-\Omega_{N,1})}{\beta_1\beta_2 + \sqrt{\beta_1^2\beta_2^2+4(\Omega_{N,1}-\Omega_{N,2})^2}},$$

$$l_{N,i}^2 = 2+y_{N,i}^2,$$

$$E_N^1 = \frac{\hbar\omega}{4}\left[4N-(\beta_1^2+\beta_2^2)+2\sqrt{\beta_1^2\beta_2^2+4(\Omega_{N,1}+\Omega_{N,2})^2}\right],$$

$$E_N^2 = \frac{\hbar\omega}{4}\left[4N-(\beta_1^2+\beta_2^2)-2\sqrt{\beta_1^2\beta_2^2+4(\Omega_{N,1}+\Omega_{N,2})^2}\right],$$

$$E_N^3 = \frac{\hbar\omega}{4}\left[4N-(\beta_1^2+\beta_2^2)+2\sqrt{\beta_1^2\beta_2^2+4(\Omega_{N,1}-\Omega_{N,2})^2}\right],$$

$$E_N^4 = \frac{\hbar\omega}{4}\left[4N-(\beta_1^2+\beta_2^2)-2\sqrt{\beta_1^2\beta_2^2+4(\Omega_{N,1}-\Omega_{N,2})^2}\right]. \tag{15}$$

We calculated the eigenvalue E_4^i from Eq. (15) numerically and plot the results in Fig. 1. As shown in Fig. 1, our result can describe the system correctly in full values of β as long as $\omega_i \ll \omega$.

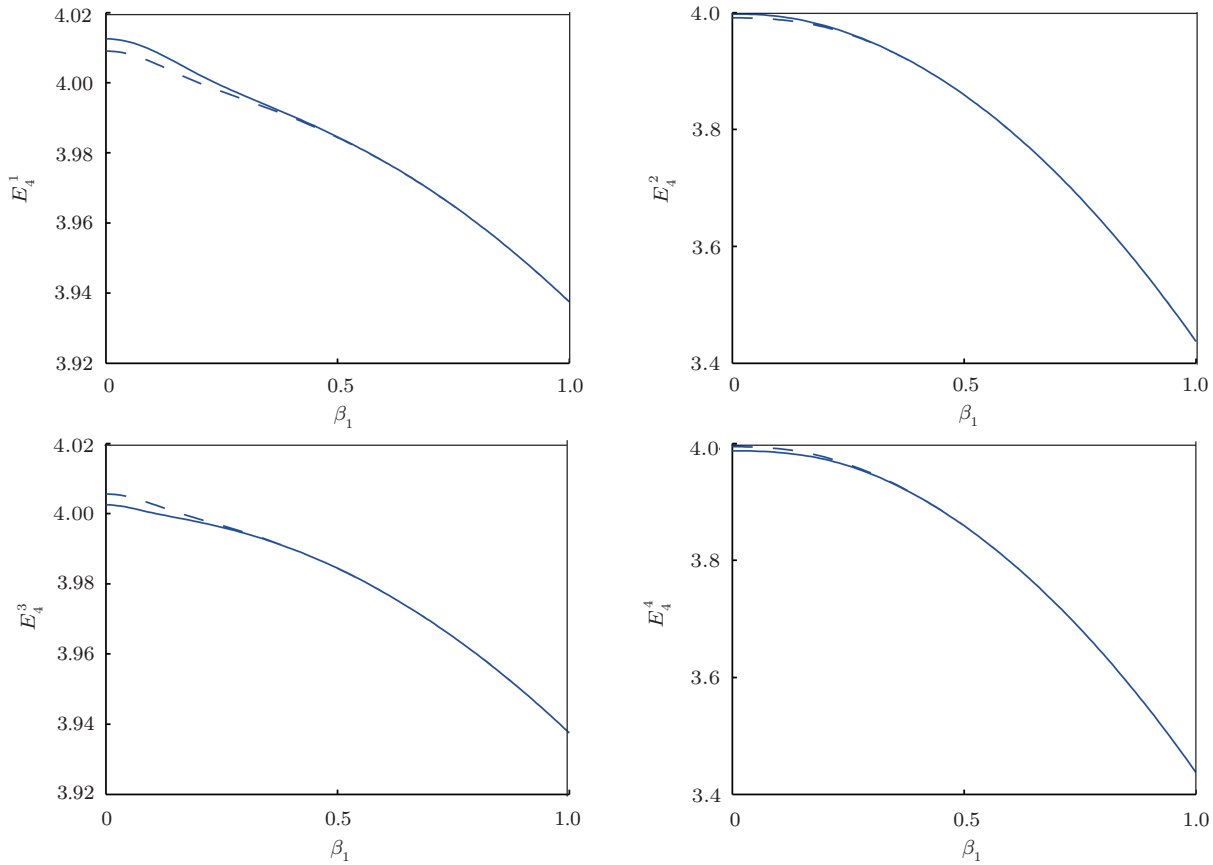


Fig. 1. (color online) The eigenvalue E_N^i from adiabatic approximation (dash line) and numerically (solid line) for $\omega_1 = 0.15\omega$, $\omega_2 = 0.075\omega$, and $\beta_2 = 0.15$.

From Eq. (15), we can obtain $E_N^3 = \hbar\omega N$ if $\omega_1 = \omega_2$ and $\beta_1 = \beta_2$. These conditions are the same conditions for the dark or trapping states[45] by Peng *et al.*[40] The qubits are maximally entangled, even in the system with dissipation.[45] So our results may be applied in QIP.

3. Dynamical behavior

In order to make sure our results can truly be applied to the actual projects, we need to study the dynamical behavior of the QRM with two arbitrary qubits. Analysis of the dynamical properties of two equivalent qubits can be found in Ref. [21]. In this section we extend the results in Ref. [21] to the case of two arbitrary qubits. We will show the effect of different couplings for the i-th qubit with a quantum harmonic oscillator in the ultra-strong coupling regime. The Fock state and the coherent state will be treated as the initial state for a harmonic oscillator.

First, we consider the initial states in a displaced Fock state, such that $|m_1> |N_{m_1}>$. According to Eq. (15), we obtain

$$\begin{aligned}|\Psi_{m_1}(0)\rangle &= |m_1\rangle|N_{M_1}\rangle \\ &= \frac{1}{l_{N,1}}|E_N^1\rangle + \frac{1}{l_{N,2}}|E_N^2\rangle - \frac{1}{l_{N,3}}|E_N^3\rangle - \frac{1}{l_{N,4}}|E_N^4\rangle. \quad (16)\end{aligned}$$

Applying the time-evolution operator to act on $|\Psi_{m_1}(0)\rangle$

$$|\Psi_{m_1}(t)\rangle = \exp\left(\frac{\mathrm{i}\hat{H}t}{\hbar}\right)|\Psi_{m_1}(0)\rangle$$

$$\begin{aligned}&= \frac{\exp\left(\mathrm{i}E_N^1 t/\hbar\right)}{l_{N,1}}|E_N^1\rangle + \frac{\exp\left(\mathrm{i}E_N^2 t/\hbar\right)}{l_{N,2}}|E_N^2\rangle \\ &\quad - \frac{\exp\left(\mathrm{i}E_N^3 t/\hbar\right)}{l_{N,3}}|E_N^3\rangle - \frac{\exp\left(\mathrm{i}E_N^4 t/\hbar\right)}{l_{N,4}}|E_N^4\rangle. \quad (17)\end{aligned}$$

Using Eq. (15), it is easy to find the probability for the qubits to recur in the initial state:

$$\begin{aligned}P_{m_1}(N,t) &= |\langle\Psi_{m_1}(0)|\Psi_{m_1}(t)\rangle|^2 \\ &= \frac{1}{l_{N,1}^4} + \frac{1}{l_{N,2}^4} + \frac{1}{l_{N,3}^4} + \frac{1}{l_{N,4}^4} \\ &\quad + \frac{2}{l_{N,1}^2 l_{N,2}^2}\cos\omega_{1,2}t - \frac{2}{l_{N,1}^2 l_{N,3}^2}\cos\omega_{1,3}t \\ &\quad - \frac{2}{l_{N,1}^2 l_{N,4}^2}\cos\omega_{1,4}t - \frac{2}{l_{N,2}^2 l_{N,3}^2}\cos\omega_{2,3}t \\ &\quad - \frac{2}{l_{N,2}^2 l_{N,4}^2}\cos\omega_{2,4}t + \frac{2}{l_{N,3}^2 l_{N,4}^2}\cos\omega_{3,4}t, \quad (18)\end{aligned}$$

where

$$\omega_{i,j} = \frac{E_n^i - E_n^j}{\hbar}. \quad (19)$$

Since four basis states are included, the probability contains six frequencies. This is in contrast to the equivalent qubits case where only three Rabi frequencies control the dynamical behavior.[21] Using the same method for $P_{m_1}(N,t)$, we obtain

$$\begin{aligned}P_{m_2}(N,t) &= \frac{y_{N,1}^4}{l_{N,1}^4} + \frac{y_{N,2}^4}{l_{N,2}^4} + \frac{y_{N,3}^4}{l_{N,3}^4} + \frac{y_{N,4}^4}{l_{N,4}^4} + \frac{2y_{N,1}^2 y_{N,2}^2}{l_{N,1}^2 l_{N,2}^2}\cos\omega_{1,2}t + \frac{2y_{N,1}^2 y_{N,3}^2}{l_{N,1}^2 l_{N,3}^2}\cos\omega_{1,3}t \\ &\quad + \frac{2y_{N,1}^2 y_{N,4}^2}{l_{N,1}^2 l_{N,4}^2}\cos\omega_{1,4}t + \frac{2y_{N,2}^2 y_{N,3}^2}{l_{N,2}^2 l_{N,3}^2}\cos\omega_{2,3}t + \frac{2y_{N,2}^2 y_{N,4}^2}{l_{N,2}^2 l_{N,4}^2}\cos\omega_{2,4}t + \frac{2y_{N,3}^2 y_{N,4}^2}{l_{N,3}^2 l_{N,4}^2}\cos\omega_{3,4}t, \quad (20)\\ P_{m_3}(N,t) &= \frac{y_{N,1}^4}{l_{N,1}^4} + \frac{y_{N,2}^4}{l_{N,2}^4} + \frac{y_{N,3}^4}{l_{N,3}^4} + \frac{y_{N,4}^4}{l_{N,4}^4} + \frac{2y_{N,1}^2 y_{N,2}^2}{l_{N,1}^2 l_{N,2}^2}\cos\omega_{1,2}t - \frac{2y_{N,1}^2 y_{N,3}^2}{l_{N,1}^2 l_{N,3}^2}\cos\omega_{1,3}t \\ &\quad - \frac{2y_{N,1}^2 y_{N,4}^2}{l_{N,1}^2 l_{N,4}^2}\cos\omega_{1,4}t - \frac{2y_{N,2}^2 y_{N,3}^2}{l_{N,2}^2 l_{N,3}^2}\cos\omega_{2,3}t - \frac{2y_{N,2}^2 y_{N,4}^2}{l_{N,2}^2 l_{N,4}^2}\cos\omega_{2,4}t + \frac{2y_{N,3}^2 y_{N,4}^2}{l_{N,3}^2 l_{N,4}^2}\cos\omega_{3,4}t, \quad (21)\\ P_{m_4}(N,t) &= \frac{1}{l_{N,1}^4} + \frac{1}{l_{N,2}^4} + \frac{1}{l_{N,3}^4} + \frac{1}{l_{N,4}^4} + \frac{2}{l_{N,1}^2 l_{N,2}^2}\cos\omega_{1,2}t + \frac{2}{l_{N,1}^2 l_{N,3}^2}\cos\omega_{1,3}t \\ &\quad + \frac{2}{l_{N,1}^2 l_{N,4}^2}\cos\omega_{1,4}t + \frac{2}{l_{N,2}^2 l_{N,3}^2}\cos\omega_{2,3}t + \frac{2}{l_{N,2}^2 l_{N,4}^2}\cos\omega_{2,4}t + \frac{2}{l_{N,3}^2 l_{N,4}^2}\cos\omega_{3,4}t. \quad (22)\end{aligned}$$

Normally, the number state is not an appropriate state for describing the quantum harmonic oscillator in the experiment. A coherent state, by contrast, is desirable to be able to describe the quantum harmonic oscillator. More importantly, the collapse and revival behaviors can be expected.

For convenience of calculations, we make the following reasonable assumptions

$$\Omega_{N,i} \gg \beta_i. \quad (23)$$

So the expressions for Eq. (15) are simplified to

$$|E_N^1\rangle = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad |E_N^2\rangle = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix},$$

$$|E_N^3\rangle = \frac{1}{2}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}, \quad |E_N^4\rangle = \frac{1}{l_{N,4}}\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix},$$

$$E_N^1 = \hbar\omega[N+\Omega_{N,1}+\Omega_{N,2}],$$
$$E_N^2 = \hbar\omega[N-\Omega_{N,1}+\Omega_{N,2}],$$
$$E_N^3 = \hbar\omega[N+\Omega_{N,1}-\Omega_{N,2}],$$
$$E_N^4 = \hbar\omega[N-\Omega_{N,1}+\Omega_{N,2}]. \quad (24)$$

Now we let a coherent-state $|\alpha\rangle$ as the initial state for harmonic oscillator

$$|\Psi_{m_2}(0)\rangle = |m_2\rangle|\alpha_{m_2}\rangle = |m_2\rangle|\hat{D}(-m_2)\alpha\rangle. \quad (25)$$

By the same calculation as that of $P_{m_2}(N,t)$, we obtain

$$P_{m_2}(\alpha,t) = \frac{1}{4}+\sum_N \frac{1}{4}\frac{e^{-|\alpha|^2}|\alpha|^{2N}}{N!}\left[\frac{1}{2}\cos 2\omega(\Omega_{N,1}+\Omega_{N,2})t + \frac{1}{2}\cos 2\omega(\Omega_{N,1}-\Omega_{N,2})t+\cos 2\omega(\Omega_{N,1})t + \cos 2\omega(\Omega_{N,2})t\right]. \quad (26)$$

The $P_{m_2}(\alpha,t)$ from Eq. (26) and numerical calculations are plotted in Fig. 2. The revival time, cycle, and the heights of the $P_{m_2}(\alpha,t)$ with AA are consistent with the numerical calculations. This is evident in Figs. 2(a) and 2(b). From Figs. 2(b) and 2(c), we can see that there are different revival times, cycles and the heights of the $P_{m_4}(\alpha,t)$ in different conditions, $\beta_1=\beta_2$ or $\beta_1\neq\beta_2$. There are four revival sequences for the $P_{m_2}(\alpha,t)$ because it has four Rabi frequencies in the two arbitrary qubits. This is in contrast to the equivalent qubits case where only two revival sequences.[21]

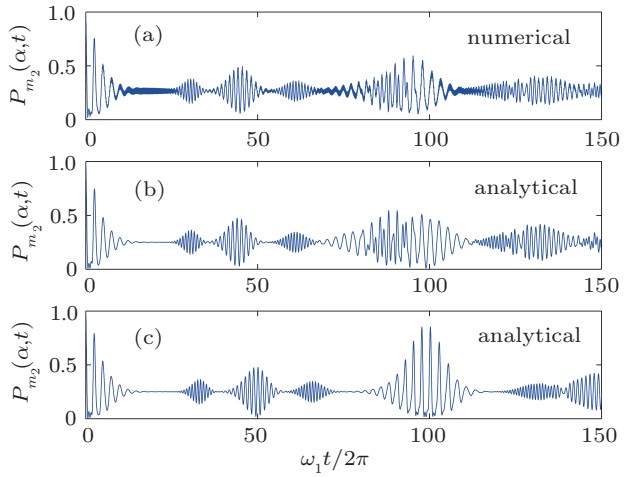


Fig. 2. (color online) Collapse and revival dynamics for $P_{m_4}(\alpha,t)$ with $\omega_1=0.15\omega$, $\omega_2=0.075\omega$, $\alpha=3$, $\beta_1=0.16$ and $\beta_2=0.14$ [(a) and (b)]; $\omega_1=0.15\omega$, $\omega_2=0.075\omega$, $\alpha=3$, and $\beta_1=\beta_2=0.14$ [(c)].

4. Entanglement dynamics

In this section, the entanglement dynamics of two qubits will be investigated. The influence of $\beta_1\neq\beta_2$ has drawn particular attention.

We consider the case that the qubits and the oscillator are prepared initially in one of the $\sigma_x(\sigma_x|\pm\rangle=\pm|\pm\rangle)$ Bell states and in an undisplaced coherent state respectively

$$|\xi(0)\rangle = \frac{1}{\sqrt{2}}(|+,+\rangle+|-,-\rangle)|\alpha\rangle = \frac{1}{\sqrt{2}}(|m_1\rangle+|m_4\rangle)|\alpha\rangle. \quad (27)$$

When $N\gg\beta_i$, we can obtain

$$|N\rangle \approx \hat{D}(-m_i)|N\rangle = |N_{m_i}\rangle. \quad (28)$$

So equation (27) becomes

$$|\xi(0)\rangle = \frac{1}{\sqrt{2}}(|m_1\rangle+|m_4\rangle)\sum_{N=0}^{\infty}\frac{e^{-|\alpha|^2/2}\alpha^N}{\sqrt{N!}}|N\rangle = \frac{1}{\sqrt{2}}\left(|m_1\rangle\sum_{N=0}^{\infty}\frac{e^{-|\alpha|^2/2}\alpha^N}{\sqrt{N!}}|N_{m_1}\rangle + |m_4\rangle\sum_{N=0}^{\infty}\frac{e^{-|\alpha|^2/2}\alpha^N}{\sqrt{N!}}|N_{m_4}\rangle\right) = \frac{1}{\sqrt{2}}(|m_1\rangle|\alpha_{m_1}\rangle+|m_4\rangle|\alpha_{m_1}\rangle). \quad (29)$$

According to Eq. (24), we have

$$|\xi(0)\rangle = \sum_{N=0}^{\infty}\frac{e^{-|\alpha|^2/2}\alpha^N}{\sqrt{2}\sqrt{N!}}(|E_N^1\rangle+|E_N^2\rangle). \quad (30)$$

Using Eqs. (24) and (28), we can obtain

$$|\xi(t)\rangle = \sum_{N=0}^{\infty}\frac{e^{-|\alpha|^2/2}\alpha^N}{\sqrt{2}\sqrt{N!}}[(e^{-iE_N^1t/\hbar}+e^{-iE_N^2t/\hbar})(|m_1\rangle+|m_4\rangle) + (e^{-iE_N^1t/\hbar}-e^{-iE_N^2t/\hbar})(|m_2\rangle+|m_3\rangle)]|N\rangle. \quad (31)$$

Then we use the eigenbasis σ_z to compact in Eq. (31)

$$|\xi(t)\rangle = \sum_{N=0}^{\infty}\frac{e^{-|\alpha|^2/2}\alpha^N}{\sqrt{2}\sqrt{N!}} \times\left[(e^{-iE_N^1t/\hbar}|ee\rangle+e^{-iE_N^2t/\hbar}|gg\rangle)\right]|N\rangle. \quad (32)$$

The reduced density matrix for the qubits is obtained by tracing out the oscillator states

$$\rho^{1,2}(t) = \sum_N\langle N|\xi(t)\rangle\langle\xi(t)|N\rangle = \frac{1}{2}(|ee\rangle\langle ee|+|gg\rangle\langle gg|) + \frac{1}{2}\left(\sum_{N=0}^{\infty}S(t)|gg\rangle\langle ee|+\text{H.c.}\right), \quad (33)$$

where

$$S(t) = \sum_N p_{\text{coh}}\cos(\omega 2\sqrt{2}\Omega_N t)$$
$$p_{\text{coh}} = \frac{e^{-|\alpha|^2}|\alpha|^{2N}}{N!}. \quad (34)$$

The concurrence[46] for Eq. (33) is

$$C(t) = \left|\sum_{k=0}^{\infty}S(t)\right|. \quad (35)$$

If $|\alpha|^2 \gg 1$ we can calculate Eq. (34) approximately (see Appendix A) to obtain

$$S(t) = \mathrm{Re}\left[\sum_{k=0}^{\infty} \overline{S}(t)\right], \tag{36}$$

where

$$\mathrm{Re}\,[\overline{S}(t)] = \exp\left(\frac{-(t-t_k)^2|\alpha|^2 d^2}{2(1+(\pi k(f_1+f_2))^2)}\right) \times \frac{\cos(\Phi_{\mathrm{Im}})}{(1+(\pi k(f_1+f_2))^2)^{1/4}}. \tag{37}$$

In Eq. (37) we have defined

$$\begin{aligned} &x_i = \beta_i^2, \quad \tau_i = \omega_i t e^{-x_i/2}, \quad f_i = |\alpha|^2 x_i, \\ &\tau_k = 2\pi k\left[1+\frac{f_1+f_2}{2}\right]\Big/ d, \\ &d = \omega_1 x_1\, \mathrm{e}^{-x_1/2} + \omega_2 x_2\, \mathrm{e}^{-x_2/2} \end{aligned} \tag{38}$$

and Φ_{Im} is given in Appendix A.

If we just considered the envelope of $S(t)$, we can obtain the concurrence for $\rho^{1,2}(t)$ as

$$C(t) \approx \sum_{k=0}^{\infty} \exp\left(\frac{-(t-t_k)^2|\alpha|^2 d^2}{2(1+(\pi k(f_1+f_2))^2)}\right) \times \frac{1}{(1+(\pi k(f_1+f_2))^2)^{1/4}}. \tag{39}$$

From Eqs. (36) and (37), we can clearly see that $S(t)$ manifests collapse and revival with $\overline{S}(t)$. That means the entanglement collapse and revival phenomena may also appear. The exponential part in Eq. (39) dictates the envelope of the revival entanglement. The denumerator term dominates the height of entanglement for each revival. The revival time and the height of $C(t)$ are

$$t_k^{rev} = \frac{2\pi k}{\omega_1\beta_1^2 + \omega_2\beta_2^2}\left[1+\frac{|\alpha|^2}{2}(\beta_1^2+\beta_2^2)\right], \tag{40}$$

$$h_k = \frac{1}{(1+k^2\pi^2|\alpha|^4(\beta_1^2+\beta_2^2)^2)^{1/4}}. \tag{41}$$

From Eq. (39), we can see that the width of the primary revival ($k=0$) is

$$\delta\tau_0 = \frac{1}{|\alpha| d}, \tag{42}$$

and the width of the k-th revival is

$$\delta\tau_k = \delta\tau_0\sqrt{1+1+k^2\pi^2|\alpha|^4(\beta_1^2+\beta_2^2)^2}. \tag{43}$$

Now, we discuss the effect of $\beta_1 \neq \beta_2$ on the collapse and revival phenomena for $C(t)$. From Eqs. (41) and Eq. (43), we need to compare the value of $\beta_1^2+\beta_2^2$ in different situations, $\beta_1=\beta_2$ or $\beta_1\neq\beta_2$. The situation becomes more complicated, with no conditionality for β_i. When $\beta_1\neq\beta_2$, the value of $\beta_1^2+\beta_2^2$ is bigger if we have an additional restrictive condition for $\beta_1+\beta_2=$ constant. That means the width of the i-th revival will increase, and the height of the i-th revival will decrease for $\beta_1\neq\beta_2$. These are evidently shown in Fig. 3.

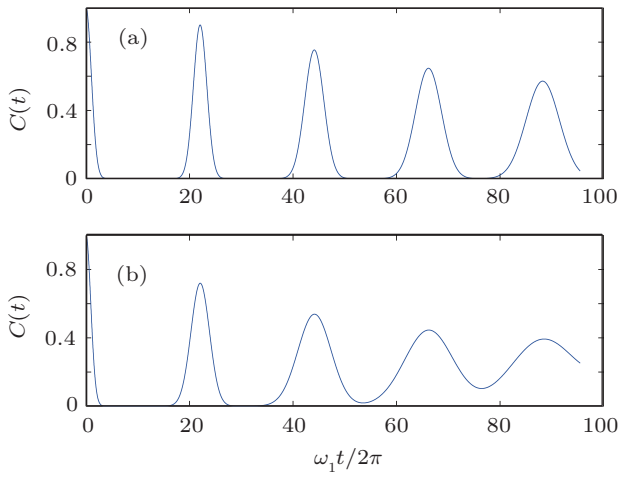


Fig. 3. (color online) The entanglement dynamics between the two qubits for $\omega_i = 0.15\omega$, $\alpha = 3$. Here (a) $\beta_1 = \beta_2 = 0.16$ and (b) $\beta_1 = 0.1$, $\beta_2 = 0.22$.

The conclusion mentioned above can be applied in areas of QIP. According to the specific demands, we can extend the duration of revival or enhance the high degree of revival by selecting a suitable nonlinear coupling parameter.

5. Conclusion

In this report we studied the QRM with two arbitrary qubits. The main conclusions are as follows.

(i) Using the AA, we obtain the analytical expression for the eigenstates and eigenvalues.

(ii) The analysis of the dynamical properties is obtained.

(iii) The evolution of entanglement is researched. The expressions for the width and peak of revival are obtained, and we show the influence of $\beta_1 \neq \beta_2$ on the collapse and revival phenomena. With the progress of circuit and cavity QED, we would like to expect that our results will find more applications in QIP.

Appendix A: Calculation of $S(t)$

If $|\alpha|$ is big enough, the Poisson distribution p_{coh} can be replaced with a Gaussian distribution

$$p_{\mathrm{coh}} = \frac{\mathrm{e}^{-|\alpha|^2}|\alpha|^{2N}}{N!} \longrightarrow \frac{1}{\sqrt{2\pi|\alpha|^2}}\exp\left[-\frac{(N-|\alpha|^2)^2}{2|\alpha|^2}\right]. \tag{A1}$$

Then if $|\beta_i| \ll 1$, we just retain the first three terms of the Laguerre polynomial in Eq. (34)

$$\begin{aligned} \Omega_{N,1} &\approx -\frac{1}{2}\frac{\omega_1}{\omega}\,\mathrm{e}^{-x_1/2}\left(1-Nx_1+\frac{N(N-1)}{4}x_1^2\right), \\ \Omega_{N,2} &\approx -\frac{1}{2}\frac{\omega_2}{\omega}\,\mathrm{e}^{-x_2/2}\left(1-Nx_2+\frac{N(N-1)}{4}x_2^2\right), \end{aligned} \tag{A2}$$

where $x_i = \beta_i^2$. According to the above approximation, equation (34) can be expressed as

$$S(t) = \mathrm{Re}\sum_N p_{\mathrm{coh}}\exp\left[\mathrm{i}(\tau_1)\left(1-Nx_1+\frac{N(N-1)}{4}x_1^2\right)\right.$$

$$+\mathrm{i}(\tau_2)\left(1-Nx_2+\frac{N(N-1)}{4}x_2^2\right)\Bigg], \tag{A3}$$

where $\tau_i=\omega_i t\,\mathrm{e}^{-x_i/2}$. According to the Poisson sum formula, we have

$$S(t)=\mathrm{Re}\left[\sum_{k=-\infty}^{\infty}\overline{S}(t)+\frac{1}{2}p_{\mathrm{coh}}(0)\exp(\tau_1+\tau_2)\right], \tag{A4}$$

where

$$\overline{S}(t)=\int_0^{\infty}\mathrm{d}n p_{\mathrm{coh}}\,\mathrm{e}^{2\mathrm{i}\pi kn}\exp\left[\mathrm{i}(\tau_1)\left(1-nx_1+\frac{n(n-1)}{4}x_1^2\right)\right.$$
$$\left.+\mathrm{i}(\tau_2)\left(1-nx_2+\frac{n(n-1)}{4}x_2^2\right)\right]. \tag{A5}$$

Using the Gaussian integral, we obtain

$$\overline{S}(t)=\frac{\exp(\Phi_{\mathrm{Re}}+\mathrm{i}\Phi_{\mathrm{Im}})}{(1+(y_1f_1/2+y_2f_2/2)^2)^{1/4}}, \tag{A6}$$

where

$$\Phi_{\mathrm{Re}}=\frac{|\alpha|^2}{2(1+(y_1f_1/2+y_2f_2/2)^2)}$$
$$\times\left[1-g^2+(y_1f_1/2+y_2f_2/2)g\right]-\frac{|\alpha|^2}{2}, \tag{A7}$$

$$\Phi_{\mathrm{Im}}=\frac{|\alpha|^2}{2(1+(y_1f_1/2+y_2f_2/2))}$$
$$\times\left\{\left[1-g^2\right]\left(\frac{y_1f_1}{2}+\frac{y_2f_2}{2}\right)-2g\right\}$$
$$-\frac{\theta}{2}+\tau_1+\tau_2, \tag{A8}$$

and we have defined

$$y_i=\tau_i x_i,\quad f_i=|\alpha|^2x_i,$$
$$\theta=\tan^{-1}\left(y_1f_1/2+y_2f_2/2\right)^2,$$
$$g=y_1+y_2+y_1x_1/4++y_2x_2/4-2\pi k. \tag{A9}$$

So we can see that $\overline{S}(t)$ will be maximum when the time is around $y_i=2k\pi$. With this observation, we can simplify the expression for Φ_{Re} by neglecting the terms that are of the order of yx, $(y_i-2\pi k)f_i^2$, f_i^3, $(y_i-2\pi k)^2f_i$, and higher powers of these terms to obtain

$$\Phi_{\mathrm{Re}}=-\frac{|\alpha|^2}{2(1+(\pi k(f_1+f_2))^2)}$$
$$\times\left[y_1+y_2-2\pi k\left(1+\frac{f_1+f_2}{2}\right)\right]^2. \tag{A10}$$

In the same way, we can obtain

$$\Phi_{\mathrm{Im}}=-\frac{\tan^{-1}(\pi k(f_1+f_2))^2}{2}$$
$$+\frac{1}{x_1}\left[2\pi kf_1-y_1(1+f_1)\right]$$
$$+\frac{1}{x_2}\left[2\pi kf_2-y_2(1+f_2)\right]. \tag{A11}$$

Thus, the form of $\overline{S}(t)$ is

$$\overline{S}(t)=\exp\left(\frac{-(t-t_k)^2|\,\alpha|^2d^2}{2(1+(\pi k(f_1+f_2))^2)}\right)$$
$$\times\frac{\cos(\Phi_{\mathrm{Im}})}{(1+(\pi k(f_1+f_2))^2)^{1/4}}, \tag{A12}$$

where

$$\tau_k=2\pi k\left[1+\frac{f_1+f_2}{2}\right]\Big/d,$$
$$d=\omega_1x_1\,\mathrm{e}^{-x_1/2}+\omega_2x_2\,\mathrm{e}^{-x_2/2}. \tag{A13}$$

Obviously, the amplitude of $\overline{S}(t)$ is much larger than $P_{\mathrm{coh}}(0)$. Thus, we can neglect the term $(1/2)P_{\mathrm{coh}}(0)\exp(\mathrm{i}(\tau_1+\tau_2))$ in Eq. (A4). We just need the positive time. So $k\geq 0$ in Eq. (A4). Finally, we simplify the form of $S(t)$ to

$$S(t)=\mathrm{Re}\left[\sum_{k=0}^{\infty}\overline{S}(t)\right]. \tag{A14}$$

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