Generalized Birkhoffian representation of nonholonomic systems and its discrete variational algorithm*

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By using the discrete variational method, we study the numerical method of the general nonholonomic system in the generalized Birkhoffian framework, and construct a numerical method of generalized Birkhoffian equations called a self-adjoint-preserving algorithm. Numerical results show that it is reasonable to study the nonholonomic system by the structure-preserving algorithm in the generalized Birkhoffian framework.

Keywords: nonholonomic system, generalized Birkhoffian system, conditions of self-adjointness, self-adjoint-preserving algorithm

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1. Introduction

Nonholonomic systems are mechanical systems subject to unintegrable differential constraints in the sense of Forbenius theorem, which have been attracting a great deal of attention in mechanics, physics, and engineering.[1–3] Recently, the study of nonholonomic mechanics focuses on two aspects: one is European scholars who developed the theoretical framework of geometrodynamics for nonholonomic systems,[4–8] and the other is North American scholars whose achievements mainly embody the symmetry reduction of the nonholonomic system and its application in motion planning and control of nonlinearity.[9–12] Modern differential geometry, such as fibre bundle, symplectic manifold, and Poisson manifold, has been fully used in many aspects.[13,14] People turned their attention to the geometric numerical integration for nonholonomic systems driven by the researches of reachability and controllability of nonholonomic systems, motion and route planning, optimum control, path tracking, stabilization of dots and sets.[15–23] With the development of theoretical verification and application of nonholonomic systems, geometric numerical integration is not only a powerful supplement for geometrodynamics of nonholonomic dynamics, but also a connection between the theoretical research and practical application.[24] However the works of geometric numerical integration of nonholonomic systems are limited[18–26] and mainly focused on extension and transformation of the existing structure-preserving algorithms, such as symplectic-preserving algorithm and Lie group algorithm. The discrete variational methods have been adopted to discrete directly the d'Alembert–Lagrange principle, and obtain the discrete Euler–Lagrange equations that can preserve the momentum or energy conservation.[18–22,26] Moreover, these methods are only suitable for nonholonomic systems subject to linear constraints, and the energy is still a conserved quantity for the system that can be transformed into Hamilton or Lagrange systems, but it is useless for the system subject to the general nonholonomic constraints.

In nonholonomic Lagrangian or Hamiltonian mechanics, the constraint submanifold of system is not a symplectic manifold, and the symplectic form constructed from the Lagrangian or Hamiltonian is no longer preserved as in the unconstrained case. Moreover, in the case of a nonholonomic system with symmetry, the momentum map is not conserved in general, due to the presence of the constraint force.[19] So the geometric numerical integrator for nonholonomic systems is not a symplectic-preserving algorithm. For some special nonholonomic system satisfying the self-adjoint conditions, we can transform the integral problems of nonholonomic motion equations into integral problems of conditional holonomic equations. The symplectic algorithm can be used to solve the special nonholonomic problems.[24,26] But for most of general nonholonomic systems, the Lagrangian or Hamiltonian formulation for a dynamical system is not directly universal if the physical variables remain without using Darboux transformations, and the equations of motion are not self-adjointness. Based on the Cauchy–Kowalevski theorem of integrability conditions for partial differential equation, we can reduce the equations of motion of nonholonomic systems to a first-order

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form, which satisfy the conditions of self-adjointness, such as the Helmholtz's conditions.[27] Because of the existing nonholonomic constraints, the first-order form is a generalized Birkhoffian system whose regular conditions are broken.[27,28] Then we can construct the structure-preserving algorithms for universal nonholonomic systems in the generalized Birkhoffian frameworks. The systems whose equations of motion are represented by the first-order Birkhoffian equations on a presymplectic or a contact manifold spanned by the physical variables are called generalized Birkhoffian systems, and they are self-adjointness. So the new algorithm for Birkhoffian systems can be named as a self-adjoint-preserving algorithm.

In Section 2, the generalized Birkhoffian system and its geometric properties are introduced, and the generalized Birkhoffian realization of nonholonomic systems is given. Section 3 constructs the discrete variational algorithm for generalized Birkhoffian systems and gives out the generalized discrete Birkhoffian integrator. In Section 4, an example is given to demonstrate the effectiveness of the numerical algorithm proposed. Conclusions are drawn in Section 5. For convenience, if there is no special illustration, Einstein's summation convention is used throughout this paper and α, $\beta = 1,2,\ldots,l; i,j,k = 1,2,\ldots,n;$ $I,J = 1,2,\ldots;$ $m = 2k+l;$ $\mu,\nu = 1,2,\ldots,n-l$.

2. Nonholonomic system and its generalized Birkhoffian realization

2.1. The conditions of self-adjointness of classical dynamical systems

The Newtonian equations of motion for classical dynamical systems can be written in configuration space according to the fundamental form

$$\begin{aligned} &A_{ki}(t,q,\dot{q})\ddot{q}^i + B_k(t,q,\dot{q}) = 0,\\ &A_{ki}, B_k \in \ell^2(R^{2n+1}),\\ &|A_{ki}|\,(R^{2n+1}) \neq 0. \end{aligned} \tag{1}$$

The independent variations of the system (1) satisfy the conditions of self-adjointness[29,30]

$$\begin{aligned} &A_{ik} = A_{ki},\\ &\frac{\partial A_{ik}}{\partial \dot{q}^j} = \frac{\partial A_{jk}}{\partial \dot{q}^i},\\ &\frac{\partial B_i}{\partial q^j} + \frac{\partial B_j}{\partial q^i} = 2\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\} A_{ij},\\ &\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right). \end{aligned} \tag{2}$$

The Newtonian system (1) can be transformed into a system of ordinary first-order differential equations in configuration space

$$X_{ki}(t,q)\dot{q}^i + Y_k(t,q) = 0. \tag{3}$$

Then the conditions of self-adjointness (2) can be simplified as[28]

$$\begin{aligned} &X_{ij} + X_{ji} = 0,\\ &\frac{\partial X_{ij}}{\partial q^k} + \frac{\partial X_{jk}}{\partial q^i} + \frac{\partial X_{ki}}{\partial q^j} = 0,\\ &\frac{\partial X_{ij}}{\partial t} = \frac{\partial Y_i}{\partial q^j} - \frac{\partial Y_j}{\partial q^i}. \end{aligned} \tag{4}$$

2.2. The nature of self-adjointness for nonholonomic systems

Consider the mechanical system on the contact manifold $R \times TQ$ with local coordinates $\{t, q^i, \dot{q}^i\}$. Denote the Lagrangian of the system by $L(t,q^i,\dot{q}^i)$ and suppose that the system is subject to the nonholonomic constraints

$$\dot{q}^\alpha = \varphi^\alpha\left(t, q^i, \dot{q}^\mu\right). \tag{5}$$

Then the dynamical equations of the nonholonomic system can be represented by a set of mixed second- and first-order ordinary differential equations[27]

$$\begin{aligned} &\ddot{q}^\mu = f^\mu\left(t, q^\nu, \dot{q}^\nu, q^\alpha\right),\\ &\dot{q}^\alpha = \varphi^\alpha\left(t, q^\nu, \dot{q}^\nu, q^\beta\right). \end{aligned} \tag{6}$$

Because of the nonholonomic constraints, the symplectic structure of the dynamical system is destroyed, and the conditions of self-adjointness (2) are not satisfied by Eq. (6).[13]

Introduce l regular coordinates $\{x^\mu\}$

$$\begin{aligned} &x^\mu = \xi^\mu(t, q^\nu, \dot{q}^\nu, q^\alpha),\\ &\det(\partial \xi^\mu / \partial \dot{q}^\nu) \neq 0, \end{aligned} \tag{7}$$

whose inverse transformation is

$$\dot{q}^\mu = \zeta^\mu(t, q^\nu, x^\nu, q^\alpha). \tag{8}$$

Substituting Eq. (8) into Eq. (7), we can get the first-order equations

$$\begin{aligned} &\dot{q}^\mu = \zeta^\mu(t, q^\nu, x^\nu, q^\alpha),\\ &\dot{x}^\mu = \psi^\mu(t, q^\nu, x^\nu, q^\alpha),\\ &\dot{q}^\alpha = \phi^\alpha(t, q^\nu, x^\nu, q^\beta). \end{aligned} \tag{9}$$

If we directly choose x^μ to be the generalized velocity $\dot{q}^\mu$ or the generalized momentum p_μ, the $m = 2k+l$ local coordinates $\{q^\nu, x^\nu, q^\alpha\}$ on constraint manifold M can be denoted by $\{a^I\}$. Then equation (9) can be reformulated by

$$\dot{a}^I = \Xi^I(t, a^J). \tag{10}$$

If functions $\Xi^I(t,a^J)$ are local, analytic and regular, equation (10) admits a generalized Birkhoffian formulation

$$\left[\frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}\right]\dot{a}^J - \left(\frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}\right) = 0, \tag{11}$$

where the total energy of the system can be taken as the Birkhoffian $B(t,a)$ and the functions $R_I(t,a)$ are related with the nonholonomic constraint forces.

Remark The regularity of the Hessian matrix for the original nonholonomic mechanical system does not assure the regularity of the matrix determined by the integrability of the constraints or the nonholomicity of an odd number of constraints. The system is a Birkhoffian system if l is even, and Eq. (11) is the generalized Birkhoffian when l is odd.

Taking

$$X_{IJ}=\frac{\partial R_J}{\partial a^I}-\frac{\partial R_I}{\partial a^J},\ \ Y_I=\frac{\partial B}{\partial a^I}+\frac{\partial R_I}{\partial t}, \tag{12}$$

equation (11) can be represented into Eq. (3), which satisfies the conditions of self-adjointness (4).

Define a closed 2-form

$$\begin{aligned}\Omega &= \mathrm{d}\left(R_I\,\mathrm{d}a^I-B\mathrm{d}t\right)\\ &= \frac{1}{2}X_{IJ}\,\mathrm{d}a^I\wedge\mathrm{d}a^J+Y_I\,\mathrm{d}t\wedge\mathrm{d}a^I.\end{aligned} \tag{13}$$

If the closed 2-form Ω is regular, the system (10) is reduced to the Birkhoffian system. On the manifold M, the closed 2-form (13) defines a pre-symplectic structure, which gives out the geometric structure of generalized Birkhoffian systems. Thus we have a global formulation of Eq. (11)

$$i_X\Omega=0,\ \ i_X\,\mathrm{d}t=1, \tag{14}$$

where

$$X=\frac{\partial}{\partial t}+\Xi^I(t,a)\frac{\partial}{\partial a^I}$$

is the dynamical vector field.[27] It is easy to verify the equivalence relation between the closure of the 2-form Ω and the self-adjointness of Eq. (11). When the equations of nonholonomic motion have been transformed into the generalized Birkhoffian system, the nonholonomic system is a self-adjoint system. So we can apply the structure-preserving algorithms of the nonholonomic systems in the generalized Birkhoffian framework.

3. Discrete variational algorithm for generalized Birkhoffian systems

Consider the generalized Birkhoffian system on the m-dimension smooth manifold M with local coordinates $\{t,a\}$. Because the value of m is not always even, the manifold M is a pre-symplectic manifold, and the configuration space of the generalized Birkhoffian system is a pre-symplectic space. Denoting the generalized Birkhoffian $B:M\to R$ and functions $R_I:M\to R$, the generalized Pfaffian is

$$P(t,a,\dot{a})=R_I(t,a)\,\dot{a}^I-B(t,a). \tag{15}$$

3.1. Generalized Discrete Birkhoffian equation and its geometric properties

In order to study numerically the generalized Birkhoffian system, the pre-symplectic space is discretized firstly. With discrete coordinate $\{(t_k,a_k)\}_{k=0}^N$, define the discrete configuration space as $M\times M$ and the generalized discrete Pfaffian $P_\mathrm{d}:M\times M\to R$ as

$$P_\mathrm{d}=P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1}), \tag{16}$$

where k is the discrete points, t_k is the discrete time coordinates, and a_k is the discrete space coordinates. Take $c_\mathrm{d}\in C_\mathrm{d}$ as the discrete curves in discrete space. The discrete curves are sets $(t_k,a_k)\in M$ of discrete points $(t_k,a_k)\in M$. Define the discrete path space as

$$C_\mathrm{d}(M)=C_\mathrm{d}\left(\{t_k\}_{k=0}^N,M\right)=\left\{a_\mathrm{d}^v:\{t_k\}_{k=0}^N\to M\right\}.$$

Definition 1 Generalized discrete action functional is a map $A_\mathrm{d}:C_\mathrm{d}\to R$ with coordinates

$$A_\mathrm{d}=\sum_{k=1}^{N-1}P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1}). \tag{17}$$

Then we present the generalized discrete Pfaff–Birkhoff principle.

Theorem 1 There exist a unique $C^{k-1}(k>1)$ mapping $D_\mathrm{B}:M\times M\to T^*M$ and unique one-forms $\Theta_{P_\mathrm{d}}^+$ and $\Theta_{P_\mathrm{d}}^-$ for the C^k generalized discrete Pfaffian $P_\mathrm{d}:M\times M\to R$ in the discrete state space $M\times M$. For the discrete variation of all curves $\delta c_\mathrm{d}\in TC_\mathrm{d}$, we have

$$\begin{aligned}&\mathrm{d}A_\mathrm{d}(c_\mathrm{d})\cdot\delta c_\mathrm{d}\\ =&\sum_{k=1}^{N-1}D_\mathrm{B}P_\mathrm{d}(t_{k-1},t_k,t_{k+1},a_{k-1},a_k,a_{k+1})\cdot(\delta t_k,\delta a_k)\\ &+\Theta_{P_\mathrm{d}}^+(t_{N-1},t_N,a_{N-1},a_N)\cdot(\delta t_{N-1},\delta t_N,\delta a_{N-1},\delta a_N)\\ &-\Theta_{P_\mathrm{d}}^-(t_0,t_1,a_0,a_1)\cdot(\delta t_0,\delta t_0,\delta a_0,\delta a_1),\end{aligned} \tag{18}$$

where map D_B is called the generalized discrete Birkhoffian map and one-forms $\Theta_{P_\mathrm{d}}^+$ and $\Theta_{P_\mathrm{d}}^-$ are the generalized discrete Birkhoffian one-forms, which have coordinate expression

$$\begin{aligned}&D_\mathrm{B}P_\mathrm{d}(t_{k-1},t_k,t_{k+1},a_{k-1},a_k,a_{k+1})\\ =&\left[D_1P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1})+D_2P_\mathrm{d}(t_{k-1},t_k,a_{k-1},a_k)\right]\mathrm{d}t_k\\ &+\left[D_3P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1})\right.\\ &\left.+D_4P_\mathrm{d}(t_{k-1},t_k,a_{k-1},a_k)\right]\mathrm{d}a_k,\end{aligned} \tag{19}$$

$$\begin{aligned}&\Theta_{P_\mathrm{d}}^+(t_k,t_{k+1},a_k,a_{k+1})\\ =&D_2P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1})\,\mathrm{d}t_{k+1}\\ &+D_4P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1})\,\mathrm{d}a_{k+1},\end{aligned} \tag{20}$$

$$\begin{aligned}&\Theta_{P_\mathrm{d}}^-(t_k,t_{k+1},a_k,a_{k+1})\\ =&-D_1P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1})\,\mathrm{d}t_k\\ &-D_3P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1})\,\mathrm{d}a_k.\end{aligned} \tag{21}$$

The theorem can be proved by using the contemporaneous variations methods.

It is clear from Eq. (18) that $c_{\rm d}$ is a solution if and only if $\mathrm{d}A_{\rm d}(c_{\rm d})\cdot\delta c_{\rm d}=0$ at all points expect the endpoints 0 and N. This statement at k reads

$$D_{\rm B}P_{\rm d}(t_{k-1},t_k,t_{k+1},a_{k-1},a_k,a_{k+1})=0. \tag{22}$$

Equation (22) is named as the generalized discrete Brikhoff equation, which can be separated into the time and configuration components

$$D_1P_{\rm d}(t_k,t_{k+1},a_k,a_{k+1})+D_2P_{\rm d}(t_{k-1},t_k,a_{k-1},a_k)=0, \tag{23}$$
$$D_3P_{\rm d}(t_k,t_{k+1},a_k,a_{k+1})+D_4P_{\rm d}(t_{k-1},t_k,a_{k-1},a_k)=0. \tag{24}$$

Because the generalized discrete action is a functional of the entire paths, not just some discrete associated curves, the discrete paths satisfying Eq. (23) do not automatically satisfy Eq. (24). That is, both Eqs. (23) and (24) are necessary in solving the generalized Birkhoffian equations numerically and contribute restrictions on the space of solutions.

Equation (22) defines a generalized discrete Birkhoffian map $F_{P_{\rm d}}:M\times M\to M\times M$ in coordinate

$$F_{P_{\rm d}}(t_{k-1},t_k,a_{k-1},a_k)=(t_k,t_{k+1},a_k,a_{k+1}). \tag{25}$$

If the points $\{(t_k,a_k)\}_{k=0}^{N}$ satisfy map (25), the discrete curve $c_{\rm d}\in C_{\rm d}$ is a solution of the generalized discrete Birkhoffian equation (22) for each $k=1,\cdots,N-1$. There are two one-forms (20) and (21) which arise from the boundary terms in the discrete case. By using $d^2=0$, we have

$$\mathrm{d}\Theta^{+}_{P_{\rm d}}=\mathrm{d}\Theta^{-}_{P_{\rm d}}. \tag{26}$$

Defining generalized discrete action functional (17) as a map $A_{\rm d}:M\times M\to R$ in coordinates $A_{\rm d}(t_0,t_1,a_0,a_1)=A_{\rm d}(c_{\rm d})$ with the generalized discrete Pfaffian (16), the discrete solution space $C_{P_{\rm d}}$ can be taken as the initial condition space of system $M\times M$. Here $c_{\rm d}\in C_{P_{\rm d}}$ is the solution of generalized discrete Birkhoffian equations with the initial conditions $c(0)=(t_0,a_0)$ and $c(1)=(t_1,a_1)$. From the variation (18), we can prove that the one-forms (20) and (21) satisfy

$$\mathrm{d}A_{\rm d}=\left(F^{N-1}_{P_{\rm d}}\right)^{*}\left(\Theta^{+}_{P_{\rm d}}\right)-\Theta^{-}_{P_{\rm d}}.$$

With the fact that $\mathrm{d}^2A_{P_{\rm d}}=0$, we obtain the conservation law

$$\left(F^{N-1}_{P_{\rm d}}\right)^{*}(\Omega_{P_{\rm d}})=\Omega_{P_{\rm d}},$$

where $\Omega_{P_{\rm d}}=-\mathrm{d}\Theta^{+}_{P_{\rm d}}=-\mathrm{d}\Theta^{-}_{P_{\rm d}}$ are the generalized discrete Birkhoffian two-forms as well as the discrete Birkhoffian pre-symplectic forms. Take an arbitrary subinterval of the discrete sequence $0,1,\cdots,N$ and perform the same procedure, we can obtain the conservation law in the discrete interval. So the generalized discrete Birkhoffian map $F_{P_{\rm d}}$ is a generalized discrete pre-symplectic map.

3.2. Generalized discrete Birkhoffian integrator

We can regard the generalized discrete Birkhoffian equation (22) as an integrator named the generalized discrete Birkhoffian integrator. From Section 3.1, we can see that the discrete Birkhoffian map $\hat{F}_{P_{\rm d}}$ can preserve the pre-symplectic structure, so it is a generalized symplectic integrator, and the generalized discrete Pfaffian $P_{\rm d}:\hat{M}\times\hat{M}\to R$ satisfies

$$\begin{aligned}&P_{\rm d}(t_k,t_{k+1},a_k,a_{k+1})\\ =&\int_{t_k}^{t_{k+1}}P(t,a(t),\dot{a}(t))\,\mathrm{d}t+O\left(|t_{k+1}-t_k|^{r+1}\right),\end{aligned}$$

where $a:[t_k,t_{k+1}]\to\hat{M}$ is any solution of the generalized Birkhoffian Eq. (11) for $P_{\rm d}$. Here r is known as the order of the discrete Pfaffian, and we require $r\geq 1$ for $P_{\rm d}$ to be consistent. To construct the integrators, we define the maps by using Eqs. (23) and (24)

$$\begin{aligned}&\hat{F}^{+}P_{\rm d}:(t_k,t_{k+1},a_k,a_{k+1})\\ \to\ &(t_{k+1},b_{k+1}=D_4P_{\rm d},a_{k+1},E_{k+1}=-D_2P_{\rm d}),\\ &\hat{F}^{-}P_{\rm d}:(t_k,t_{k+1},a_k,a_{k+1})\\ \to\ &(t_{k+1},b_k=-D_3P_{\rm d},a_{k+1},E_k=D_1P_{\rm d}),\end{aligned} \tag{27}$$

where b_k,b_{k+1},E_k,E_{k+1} are the intermediate variables, and map (27) satisfies

$$\hat{F}_{P_{\rm d}}=\hat{F}^{+}P_{\rm d}\circ(\hat{F}^{-}P_{\rm d})^{-1}.$$

Regarding $\hat{F}_{P_{\rm d}}$ as an integrator of the systems, it maps

$$\hat{F}_{P_{\rm d}}:(t_k,E_k,a_k,b_k)\to(t_{k+1},E_{k+1},a_{k+1},b_{k+1}),$$

and is defined by the relations

$$\begin{aligned}b_k&=-D_3P_{\rm d}(t_k,t_{k+1},a_k,a_{k+1}),\\ E_k&=D_1P_{\rm d}(t_k,t_{k+1},a_k,a_{k+1}),\\ b_{k+1}&=D_4P_{\rm d}(t_k,t_{k+1},a_k,a_{k+1}),\end{aligned} \tag{28}$$
$$E_{k+1}=-D_2P_{\rm d}(t_k,t_{k+1},a_k,a_{k+1}), \tag{29}$$

together with the requirement that $t_{k+1}>t_k$ for each point k. In computation, the implicit Eq. (28) is solved and advanced to t_{k+1} and a_{k+1} under the restriction $t_{k+1}>t_k$, then with t_{k+1} and a_{k+1} the explicit Eq. (29) is propagated to give b_{k+1} and E_{k+1}, thus the map $\hat{F}_{P_{\rm d}}:(t_k,E_k,a_k,b_k)\to(t_{k+1},E_{k+1},a_{k+1},b_{k+1})$ can be obtained. As has been stated, the method defined by Eq. (22) preserves a discrete generalized symplectic form and satisfies the discrete energy evolution equations. There are many methods to discrete the generalized Pfaffian, such as the Euler mid-point rule, Verlet scheme, R–K method,[31] and so on. In this paper, the Euler mid-point rule will be used.

4. Numerical simulations

Consider a system subjected to a nonholonomic constraint

$$\dot{q}^1 + t\dot{q}^2 - q^2 + t = 0. \tag{30}$$

The configuration of the system is denoted by $\{q^1, q^2\}$ and the Lagrangian is

$$L(q^1, q^2) = \frac{1}{2}\left(\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2\right). \tag{31}$$

Then the differential equations of motion for the system are

$$\begin{aligned} &\left(1+t^2\right)\ddot{q}^2 + 2t\dot{q}^2 + 2\dot{q}^1 - 2q^2 + 3t = 0, \\ &\dot{q}^1 + t\dot{q}^2 - q^2 + t = 0. \end{aligned} \tag{32}$$

Letting $a^1 = q^2, a^2 = \dot{q}^2, a^3 = q^1$, equation (32) can be transformed into the first-order differential equations

$$\dot{a}^1 = a^2,\ \dot{a}^2 = \frac{-t}{1+t^2},\ \dot{a}^3 = a^1 - ta^2 - t. \tag{33}$$

The equations of motion (33) can be written in the matrix form

$$\begin{pmatrix} \dot{a}^1 \\ \dot{a}^2 \\ \dot{a}^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -t & 0 \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} + \begin{pmatrix} 0 \\ -t/(1+t^2) \\ -t \end{pmatrix},$$

with three independent first integrals

$$\begin{aligned} I^1 &= a^3 - t\left(a^1 - ta^2 - t\right) - \frac{1}{2}\ln\left(1+t^2\right), \\ I^2 &= a^1 - ta^2 - t + \arctan t, \\ I^3 &= a^2 + \frac{1}{2}\ln\left(1+t^2\right). \end{aligned} \tag{34}$$

It can be verified that Eq. (32) of motion does not satisfy the conditions of self-adjointness (4), and cannot be expressed as the classical Lagrangian equations or Hamilton's equations. By using Hojman's method,[27,29] the Birkhoffian functions can be gotten

$$R_1 = -t(a^1 - ta^2 - t + \arctan t), \tag{35}$$

$$R_2 = t^2(a^1 - ta^2 - t + \arctan t) + a^2 + \frac{1}{2}\ln(1+t^2), \tag{36}$$

$$R_3 = a^1 - ta^2 - t + \arctan t, \tag{37}$$

$$\begin{aligned} B = &\left(2ta^2 + 2t - a^1 - \frac{t}{1+t^2}\right)(-a^1 + ta^2 + t - \arctan t) \\ &- \frac{t}{1+t^2}\left[a^2 + \frac{1}{2}\ln(1+t^2)\right]. \end{aligned} \tag{38}$$

So the equations of motion (33) can be represented in the generalized Birkhoffian equations

$$\Omega_{IJ}\dot{a}^J - \left(\frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}\right) = 0, \tag{39}$$

where

$$\Omega_{IJ} = \left(\frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}\right) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -t \\ -1 & t & 0 \end{pmatrix}$$

is the pre-symplectic tensor. It can be proved that Eq. (39) satisfies the conditions of self-adjointness (4).

If we take $a^1(0) = 1$, $a^2(0) = 1$, $a^3(0) = 0$ as the initial conditions, Eq. (33) has analytical solution

$$\begin{aligned} a^1 &= -\frac{1}{2}t\ln(1+t^2) + t - \arctan(t) + t + 1, \\ a^2 &= -\frac{1}{2}\ln(1+t^2) + 1, \\ a^3 &= -t\arctan(t) + \frac{1}{2}\ln(1+t^2) + t. \end{aligned} \tag{40}$$

Figure 1(a) gives out the analytic motion curve in the plane of $\left(a^1, a^3\right)$.

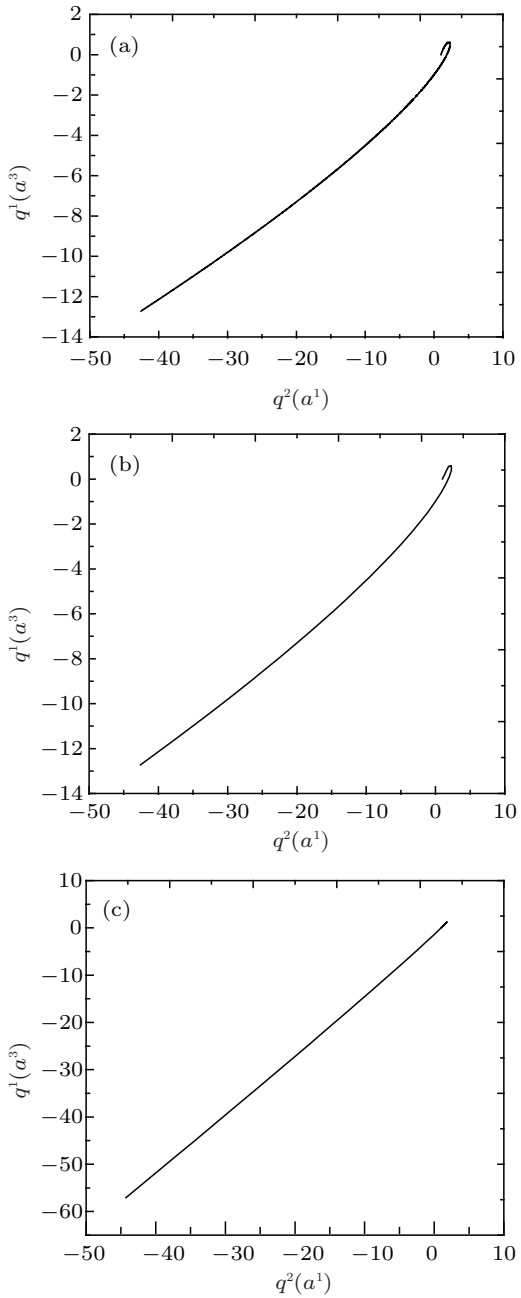


Fig. 1. The motion curve of system subjected to $\dot{q}^1 + t\dot{q}^2 - q^2 + t = 0$, obtained by analytic solution (a), Bikhoff's framework (b), and Runge–Kutta method (c).

Now we adopt the discrete variational method to solve numerically Eq. (39) in the generalized Birkhoffian framework. According to Eq. (15), the discrete Pfaffian can be obtained by using the Euler mid-point scheme[10]

$$
\begin{aligned}
P_{\mathrm{d}}^{1/2} = {} & (t_{k+1}-t_k)\Bigg[-\left(\frac{t_k+t_{k+1}}{2}\right)\left(\frac{a_k^1+a_{k+1}^1}{2}-\frac{t_k+t_{k+1}}{2}\frac{a_k^2+a_{k+1}^2}{2}-\arctan\left(\frac{t_k+t_{k+1}}{2}\right)\right)\frac{a_{k+1}^1-a_k^1}{t_{k+1}-t_k} \\
& +\left(\left(\frac{t_k+t_{k+1}}{2}\right)^2\left(\frac{a_k^1+a_{k+1}^1}{2}-\frac{t_k+t_{k+1}}{2}\frac{a_k^2+a_{k+1}^2}{2}-\arctan\left(\frac{t_k+t_{k+1}}{2}\right)\right)+\frac{a_k^2+a_{k+1}^2}{2}\right. \\
& \left.+\frac{1}{2}\ln\left(1+\left(\frac{t_k+t_{k+1}}{2}\right)^2\right)\right)\frac{a_{k+1}^2-a_k^2}{t_{k+1}-t_k}+\left(\frac{a_k^1+a_{k+1}^1}{2}-\frac{t_k+t_{k+1}}{2}\frac{a_k^2+a_{k+1}^2}{2}-\arctan\left(\frac{t_k+t_{k+1}}{2}\right)\right)\frac{a_{k+1}^3-a_k^3}{t_{k+1}-t_k} \\
& +\left(\frac{a_k^1+a_{k+1}^1}{2}-\frac{t_k+t_{k+1}}{2}\frac{a_k^2+a_{k+1}^2}{2}-\arctan\left(\frac{t_k+t_{k+1}}{2}\right)\right) \\
& \times\left((t_k+t_{k+1})\frac{a_k^2+a_{k+1}^2}{2}+(t_k+t_{k+1})-\frac{a_k^1+a_{k+1}^1}{2}-\frac{2(t_k+t_{k+1})}{4+(t_k+t_{k+1})^2}\right) \\
& +\left(\frac{a_k^2+a_{k+1}^2}{2}+\frac{1}{2}\ln\left(1+\left(\frac{t_k+t_{k+1}}{2}\right)^2\right)\right)\frac{2(t_k+t_{k+1})}{4+(t_k+t_{k+1})^2}\Bigg],
\end{aligned}
$$

and here we take $\alpha = 1/2$. Substituting it into Eqs. (23) and (24), we can get the discrete variational equations for the first-order nonholonomic system (33). Taking the time step $h = 0.0001$, the dynamic behavior of the nonholonomic system (33) in the plane can be obtained by solving the discrete variational equations, shown in Fig. 1(b). At the same time, we calculate numerically Eq. (33) by the 2-order Runge–Kutta method and the motion curve in the plane is shown in Fig. 1(c). We can see that the numerical results in Fig. 1(b) are better than those in Fig. 1(c). We also compute the relative error of $q^2(a^1)$ and $q^1(a^3)$ by the different methods as shown in Fig. 2. Numerical results show that the relative errors by the discrete variational method in the Birkhoffian framework are better than those by the Runge–Kutta method in the original equation's framework.

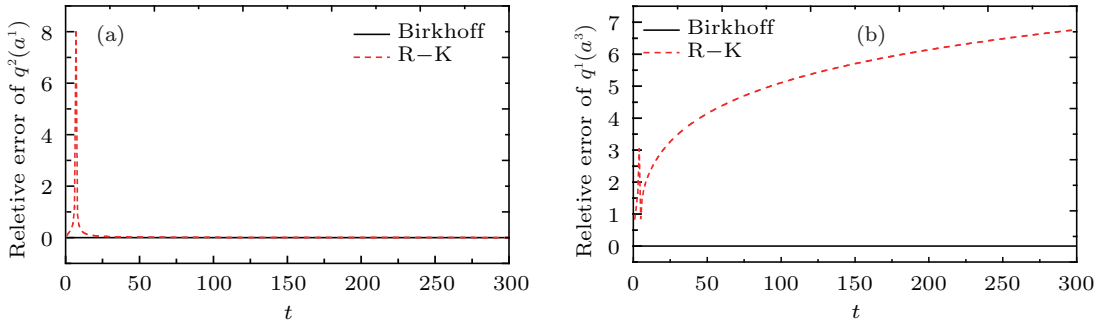


Fig. 2. (color online) Errors of (a) $q^2(a^1)$ and (b) $q^1(a^3)$ computed by discrete variation method (solid line) and Runge–Kutta method (dotted line).

5. Conclusion

In this paper, we study the numerical method of the general nonholonomic system in the generalized Birkhoffian framework. When the motion equations of nonholonomic system are transformed into the generalized Birkhoffian equations, the new equations satisfy the conditions of self-adjointness. By using the discrete variational method, we construct a numerical method for the generalized Birkhoffian equations called the self-adjoint-preserving algorithm. With the numerical example, we conclude that the discrete variational method in the generalized Birkhoffian framework can preserve perfectly the inherent quality of the original dynamical system, and relatively accurate numerical results have been obtained.

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