Cluster synchronization of community network with distributed time delays via impulsive control*

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Cluster synchronization is an important dynamical behavior in community networks and deserves further investigations. A community network with distributed time delays is investigated in this paper. For achieving cluster synchronization, an impulsive control scheme is introduced to design proper controllers and an adaptive strategy is adopted to make the impulsive controllers unified for different networks. Through taking advantage of the linear matrix inequality technique and constructing Lyapunov functions, some synchronization criteria with respect to the impulsive gains, instants, and system parameters without adaptive strategy are obtained and generalized to the adaptive case. Finally, numerical examples are presented to demonstrate the effectiveness of the theoretical results.

Keywords: cluster synchronization, community network, distributed time delay, impulsive control

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1. Introduction

In the real world, community structure has been discovered in many large-scale physical and social systems such as school friendship networks,[1] biological networks,[2] congressional cosponsorship networks,[3] protein–protein interaction networks,[4] World Wide Web,[5] and so on. They are usually modeled by a community network, in which the nodes in the same community have the identical function or property but in different communities have nonidentical ones. For example, in protein–protein interaction networks, communities correspond to functional groups/clusters, i.e., to proteins having the same or similar functions, which are expected to be involved in the same processes.[6] A great number of phenomena show that the current state of a node is affected by not only the current states but also the states in the previous period of its neighbor nodes. For example, in biological neural networks, it usually has a spatial extent due to the presence of parallel pathways with a variety of axon sizes and lengths.[7,8] Thus, there is a distribution of transmission delays and it cannot be modeled with discrete time delays. In such a case, a more appropriate way is to incorporate the distributed time delays in the models.[7,8] Until now, many valuable results about dynamical networks with distributed delays have been obtained.[9–11] However, community networks with distributed delays have seldom been considered, thus they deserve further investigations.

In recent years, synchronization has gained extensive attention from various fields on account of its broad practical applications, such as regulation of power grid, parallel image processing, the operation of no-man air vehicle, the realization of chain detonation, etc.[12,13] Cluster synchronization,[14–22] a special synchronization of complex dynamical networks, denotes that synchronization of the nodes in the same community can be achieved, but synchronization of the nodes in different communities cannot be achieved. That is, the nodes in the same or different communities have identical or nonidentical goals. Further, many studies show that cluster synchronization has increasing applications in different fields.[23,24] Kaneko pointed out that cluster synchronization is an important phenomenon in communication engineering and biological science.[23] Yoshioka showed that neurons are divided into several clusters in neurons networks interconnected with chemical synapses and synchronization of the neurons is achieved in each cluster. From the viewpoint of real life, cluster synchronization by inner adjustment of communities and influence of different communities without external control cannot be achieved. An effective method of achieving synchronization, i.e., adding proper controllers, has been found from lots of obtained results. Thus, how to design proper controllers is a key issue.

Impulsive control is a representative control strategy and its main characteristic is working on the nodes only at inconsecutive instants.[21,22,25–31] Due to the characteristic, researchers have been studying impulsive control deeply for a long time. Cai *et al.* considered the exponential cluster synchronization of hybrid-coupled delayed dynamical networks via impulsive control.[21] Fan *et al.* considered the cluster synchronisation in non-linearly coupled networks with non-identical nodes and time-varying delays via impulsive control.[22] As we know, the impulsive gains and intervals related to the cost and efficiency are important parameters of

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impulsive controllers. Therefore, designing proper impulsive controllers means choosing appropriate impulsive gains and intervals in terms of synchronization criteria. However, for different community networks with nonidentical system parameters, the needed impulsive gains and intervals need to be chosen repetitively, which undoubtedly increases workloads and reduces efficiency. In order to overcome the shortcoming, we introduce an adaptive strategy to design unified impulsive controllers. A community network combined with adaptive impulsive controllers can adjust impulsive gains or intervals adaptively and achieve cluster synchronization.

Motivated by the above discussion, the cluster synchronization of the community network with distributed time delays is investigated. On one hand, proper impulsive controllers are added to the community networks for achieving cluster synchronization, and simple criteria about the system parameters, the impulsive gains, and intervals are obtained on the basis of the Lyapunov function method and linear matrix inequality technique. On the other hand, proper controllers are improved by introducing an adaptive strategy and criteria are obtained as well. Noticeably, the adaptive law of the impulsive gain and the method for estimating the impulsive instants are provided in terms of synchronization criteria. In Section 2, the network model is introduced and some preliminaries are given. In Section 3, the impulsive controllers for achieving synchronization are designed and the synchronization criteria are provided. In Section 4, several numerical simulations are performed to verify the results. In Section 5, the conclusion for this paper is given.

2. Model and preliminaries

Consider a community network consisting of k ($2 \leq k \ll N$) communities and N nodes with distributed time delays. As we know, in many real networks with community structure, such as neural, social, biological, and communication networks, the individual nodes in the same community can be viewed as the identical functional units, while any pair of nodes in different communities are essentially different according to their functions.[21] Therefore, we assume that the node dynamics and the distributed delays in the same community are identical, while those in different communities are nonidentical. Further, the distributed delays between different pairs of communities are nonidentical as well. According to these settings, the community network can be described as

$$\begin{aligned}\dot{y}_i(t) &= f_{\pi_i}(y_i(t)) + \sum_{m=1}^{k}\sum_{j\in G_m} b_{ij}\mathcal{R} \\ &\quad \times \int_0^{\xi_{\pi_i\pi_j}} \phi_{\pi_i\pi_j}(\theta) y_j(t-\theta)\,\mathrm{d}\theta, \\ &\quad i = 1,2,\ldots,N, \end{aligned} \tag{1}$$

where $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^{\mathrm{T}} \in \mathbf{R}^n$ is the state variable of node i, and $f_{\pi_i} : \mathbf{R}^n \to \mathbf{R}^n$ is a vector function in the π_i-th community. In different communities, the node dynamics are supposed to be nonidentical, i.e., if $\pi_i \neq \pi_j$, then $f_{\pi_i} \neq f_{\pi_j}$. $\mathcal{R} = \mathrm{diag}(r_1, r_2, \ldots, r_n)$ is the inner coupling matrix, G_m $(m = 1,2,\ldots,k)$ are sets of all nodes belonging to the m-th community, if node $j \in G_m$, then $\pi_j = m$. $B = (b_{ij}) \in \mathbf{R}^{N\times N}$ is the zero-row-sum outer coupling matrix, which denotes the network topology and coupling strength and is defined as follows: if node j has influence on node i $(i \neq j)$, then $b_{ij} \neq 0$; otherwise, $b_{ij} = 0$. $\xi_{\pi_i\pi_j} > 0$ are time delays. $\phi_{\pi_i\pi_j} : [0, \tau_{\pi_i\pi_j}] \to [0, +\infty]$ is the continuous density function and satisfies $\int_0^{\xi_{\pi_i\pi_j}} \phi_{\pi_i\pi_j}(\theta)\,\mathrm{d}\theta = 1$.

Let $\eta_{\pi_i}(t)$ be a solution of an isolated node in the π_i-th community, i.e., $\dot{\eta}_{\pi_i}(t) = f(\eta_{\pi_i}(t))$. The objective here is to achieve synchronization of the network (1) by designing proper controllers, i.e.,

$$\lim_{t\to\infty} \|y_i(t) - \eta_{\pi_i}(t)\| = 0,$$

where $\|\cdot\|$ denotes the Euclidean norm.

For achieving the synchronization, proper impulsive controllers are designed and applied onto the network (1). The controlled network is

$$\begin{aligned}\dot{y}_i(t) &= f_{\pi_i}(y_i(t)) + \sum_{m=1}^{k}\sum_{j\in G_m} b_{ij}\mathcal{R} \\ &\quad \times \int_0^{\xi_{\pi_i\pi_j}} \phi_{\pi_i\pi_j}(\theta) y_j(t-\theta)\,\mathrm{d}\theta, \quad t \neq t_\upsilon, \\ y_i(t_\upsilon^+) &= y_i(t_\upsilon^-) + \varphi(t_\upsilon)\big(y_i(t_\upsilon^-) - \eta_{\pi_i}(t_\upsilon)\big), \quad t = t_\upsilon, \end{aligned} \tag{2}$$

where $i = 1,2,\ldots,N$, $\upsilon = 1,2,\ldots$, the impulsive time instants t_υ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_\upsilon < \cdots$, and $t_\upsilon \to \infty$ as $\upsilon \to \infty$. $y_i(t_\upsilon^+) = \lim_{t\to t_\upsilon^+} y_i(t)$, $y_i(t_\upsilon^-) = \lim_{t\to t_\upsilon^-} y_i(t)$. Any solution of Eq. (2) is assumed to be left continuous at each t_υ, i.e., $y_i(t_\upsilon^-) = y_i(t_\upsilon)$. $\varphi(t_\upsilon)$ is the impulsive gain at $t = t_\upsilon$ and $\varphi(t) = 0$ for $t \neq t_\upsilon$.

Let $e_i(t) = y_i(t) - \eta_{\pi_i}(t)$ be the synchronization errors. Then the error systems are

$$\begin{aligned}\dot{e}_i(t) &= f_{\pi_i}(y_i(t)) - f(\eta_{\pi_i}(t)) + \sum_{m=1}^{k}\sum_{j\in G_m} b_{ij}\mathcal{R} \\ &\quad \times \int_0^{\xi_{\pi_i\pi_j}} \phi_{\pi_i\pi_j}(\theta) e_j(t-\theta)\,\mathrm{d}\theta, \quad t \neq t_\upsilon, \\ e_i(t_\upsilon^+) &= (1+\varphi(t_\upsilon)) e_i(t_\upsilon^-), \quad t = t_\upsilon. \end{aligned} \tag{3}$$

Let $G_m = \{\gamma_{m-1}+1, \ldots, \gamma_m\}$ $(m = 1,2,\ldots,k)$ with $\gamma_0 = 0$ and $\gamma_k = N$. Suppose that $B \in \mathbf{R}^{N\times N}$ can be divided into the following block form:

$$\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \cdots & B_{kk} \end{bmatrix},$$

where $B_{pm} \in \mathbf{R}^{(\gamma_p - \gamma_{p-1})\times(\gamma_m - \gamma_{m-1})}$, $p,m = 1,2,\ldots,k$.

Definition 1 If B_{pp} is a zero-row-sum, symmetric, and irreducible matrix with non-negative off-diagonal elements, we say that B_{pp} belongs to class $\mathcal{C}_1$ and denote $B_{pp} \in \mathcal{C}_1$, $p = 1,2,\ldots,k$.

Definition 2 If B_{pm} is a zero-row-sum matrix, we say that B_{pm} belongs to class $\mathcal{C}_2$ and denote $B_{pm} \in \mathcal{C}_2$, $p,m = 1,2,\ldots,k$.

Definition 3 If $B_{pp} \in \mathcal{C}_1$, $B_{pm} \in \mathcal{C}_2$, $p,m = 1,2,\ldots,k$ $(p \neq m)$, we say that B belongs to class $\mathcal{C}_3$ and denote $B \in \mathcal{C}_3$.

Assumption 1 Suppose that there exists a positive constant M such that

$$(y(t)-x(t))^{\mathrm{T}}(f_p(y(t))-f_p(x(t))) \leq M(y(t)-x(t))^{\mathrm{T}}(y(t)-x(t))$$

holds for any $x(t), y(t) \in \mathbf{R}^n$, $p = 1,2,\ldots,k$, and $t > 0$.

Lemma 1 (see Ref. [7]) Let A be a symmetric $n \times n$ real matrix and $A > 0$. Then

$$\begin{aligned}&\int_0^{\xi} \phi(\theta) e_i^{\mathrm{T}}(t-\theta) A e_i(t-\theta)\,\mathrm{d}\theta \\ \geq\ & \int_0^{\xi} \phi(\theta) e_i^{\mathrm{T}}(t-\theta)\,\mathrm{d}\theta A \int_0^{\xi} \phi(\theta) e_i(t-\theta)\,\mathrm{d}\theta,\end{aligned}$$

where $\xi > 0$, $\phi(\theta) \geq 0$, and $\int_0^{\xi} \phi(\theta)\,\mathrm{d}\theta = 1$.

3. Main results

Let $E_m(t) = (e^{\mathrm{T}}_{\gamma_{m-1}+1}(t), e^{\mathrm{T}}_{\gamma_{m-1}+2}(t), \ldots, e^{\mathrm{T}}_{\gamma_m}(t))^{\mathrm{T}}$, $m = 1,2,\ldots,k$, $E(t) = (E_1^{\mathrm{T}}(t), E_2^{\mathrm{T}}(t), \ldots, E_k^{\mathrm{T}}(t))^{\mathrm{T}}$, $\rho_{\upsilon} = t_{\upsilon} - t_{\upsilon-1}$ be the impulsive intervals, $\mathcal{K}(t,v,\xi_{pm}) = \{\sigma |\ t-\theta < t_{\sigma} < t,\ t \in (t_{\upsilon-1}, t_{\upsilon}],\ \theta \in [0,\xi_{pm}]\}$, $\underline{\mathcal{K}}(t,v,\xi_{pm})$ and $\overline{\mathcal{K}}(t,v,\xi_{pm})$ be the minimum and the maximum of $\mathcal{K}(t,v,\xi_{pm})$, λ be the largest eigenvalue of $(2M+k)I + (B \otimes \mathcal{R})(B \otimes \mathcal{R})^{\mathrm{T}}$, $\mu(t) = (1+\varphi(t))^2$. From the definition of $\varphi(t)$, one has $\mu(t) = 1$ for $t \neq t_{\upsilon}$. The outer coupling matrix B considered in this paper is supposed to belong to $\mathcal{C}_3$.

Theorem 1 Suppose that Assumption 1 holds. If there exists a constant $\delta > 0$ such that

$$\ln \mu(t_{\upsilon}) + \delta + \lambda \rho_{\upsilon} < 0, \quad \upsilon = 1,2,\ldots, \tag{4}$$

hold, then the cluster synchronization of network (2) is achieved.

Proof Consider the following Lyapunov function candidate:

$$\begin{aligned}V(t) =\ & \sum_{p=1}^{k} \sum_{i \in G_p} e_i^{\mathrm{T}}(t) e_i(t) + \mu(t) \sum_{p=1}^{k} \sum_{m=1}^{k} \sum_{j \in G_m} \int_0^{\xi_{pm}} \phi_{pm}(\theta) \\ & \times \Big(\int_{t-\theta}^{t^-_{\underline{\mathcal{K}}(t,v,\xi_{pm})}} e_j^{\mathrm{T}}(s) e_j(s)\,\mathrm{d}s \\ & + \sum_{q=\underline{\mathcal{K}}(t,v,\xi_{pm})}^{\overline{\mathcal{K}}(t,v,\xi_{pm})-1} \int_{t_q^+}^{t^-_{q+1}} e_j^{\mathrm{T}}(s) e_j(s)\,\mathrm{d}s \\ & + \int_{t^+_{\overline{\mathcal{K}}(t,v,\xi_{pm})}}^{t} e_j^{\mathrm{T}}(s) e_j(s)\,\mathrm{d}s \Big)\,\mathrm{d}\theta,\end{aligned}$$

for $t \in (t_{\upsilon-1}, t_{\upsilon}]$, $\upsilon = 1,2,\ldots$

When $t \in (t_{\upsilon-1}, t_{\upsilon})$,

$$\begin{aligned}V(t) =\ & \sum_{p=1}^{k} \sum_{i \in G_p} e_i^{\mathrm{T}}(t) e_i(t) + \sum_{p=1}^{k} \sum_{m=1}^{k} \sum_{j \in G_m} \int_0^{\xi_{pm}} \phi_{pm}(\theta) \\ & \times \Big(\int_{t-\theta}^{t^-_{\underline{\mathcal{K}}(t,v,\xi_{pm})}} e_j^{\mathrm{T}}(s) e_j(s)\,\mathrm{d}s \\ & + \sum_{q=\underline{\mathcal{K}}(t,v,\xi_{pm})}^{\overline{\mathcal{K}}(t,v,\xi_{pm})-1} \int_{t_q^+}^{t^-_{q+1}} e_j^{\mathrm{T}}(s) e_j(s)\,\mathrm{d}s \\ & + \int_{t^+_{\overline{\mathcal{K}}(t,v,\xi_{pm})}}^{t} e_j^{\mathrm{T}}(s) e_j(s)\,\mathrm{d}s \Big)\,\mathrm{d}\theta,\end{aligned}$$

and the derivative of $V(t)$ is

$$\begin{aligned}\dot{V}(t) =\ & 2 \sum_{p=1}^{k} \sum_{i \in G_p} e_i^{\mathrm{T}}(t) \Big(f(y_i(t)) - f(\eta_{\pi_i}(t)) + \sum_{m=1}^{k} \sum_{j \in G_m} b_{ij} \mathrm{R} \int_0^{\xi_{\pi_i \pi_j}} \phi_{\pi_i \pi_j}(\theta) e_j(t-\theta)\,\mathrm{d}\theta \Big) \\ & + \sum_{p=1}^{k} \sum_{m=1}^{k} \sum_{j \in G_m} \int_0^{\xi_{pm}} \phi_{pm}(\theta) e_j^{\mathrm{T}}(t) e_j(t)\,\mathrm{d}\theta - \sum_{p=1}^{k} \sum_{m=1}^{k} \sum_{j \in G_m} \int_0^{\xi_{pm}} \phi_{pm}(\theta) e_j^{\mathrm{T}}(t-\theta) e_j(t-\theta)\,\mathrm{d}\theta \\ \leq\ & 2M \sum_{p=1}^{k} \sum_{i \in G_p} e_i^{\mathrm{T}}(t) e_i(t) + 2 \sum_{p=1}^{k} \sum_{m=1}^{k} \sum_{i \in G_p} \sum_{j \in G_m} b_{ij} e_i^{\mathrm{T}}(t) \mathrm{R} \int_0^{\xi_{pm}} \phi_{pm}(\theta) e_j(t-\theta)\,\mathrm{d}\theta \\ & + k \sum_{m=1}^{k} \sum_{j \in G_m} e_j^{\mathrm{T}}(t) e_j(t) - \sum_{p=1}^{k} \sum_{m=1}^{k} \sum_{j \in G_m} \int_0^{\xi_{pm}} \phi_{pm}(\theta) e_j^{\mathrm{T}}(t-\theta)\,\mathrm{d}\theta \int_0^{\xi_{pm}} \phi_{pm}(\theta) e_j(t-\theta)\,\mathrm{d}\theta \\ =\ & 2M E^{\mathrm{T}}(t) E(t) + 2 \sum_{p=1}^{k} \sum_{m=1}^{k} E_p^{\mathrm{T}}(t) (B_{pm} \otimes \mathcal{R}) \int_0^{\xi_{pm}} \phi_{pm}(\theta) E_m(t-\theta)\,\mathrm{d}\theta \\ & + k E^{\mathrm{T}}(t) E(t) - \sum_{p=1}^{k} \sum_{m=1}^{k} \int_0^{\xi_{pm}} \phi_{pm}(\theta) E_m^{\mathrm{T}}(t-\theta)\,\mathrm{d}\theta \int_0^{\xi_{pm}} \phi_{pm}(\theta) E_m(t-\theta)\,\mathrm{d}\theta \\ \leq\ & (2M+k) E^{\mathrm{T}}(t) E(t) + \sum_{p=1}^{k} \sum_{m=1}^{k} E_p^{\mathrm{T}}(t) (B_{pm} \otimes \mathcal{R}) (B_{pm} \otimes \mathcal{R})^{\mathrm{T}} E_p(t) \\ =\ & E^{\mathrm{T}}(t) \big((2M+k) I + (B \otimes \mathcal{R})(B \otimes \mathcal{R})^{\mathrm{T}} \big) E(t) \leq \lambda E^{\mathrm{T}}(t) E(t) \leq \lambda V(t),\end{aligned}$$

which gives

$$V(t) \le V(t_{\upsilon-1}^+)\exp\big(\lambda(t-t_{\upsilon-1})\big),\ t\in(t_{\upsilon-1},t_\upsilon). \tag{5}$$

When $t=t_\upsilon$,

$$\begin{aligned}
V(t_\upsilon^+) &= \sum_{p=1}^{k}\sum_{i\in G_p} e_i^{\mathrm{T}}(t_\upsilon^+)e_i(t_\upsilon^+)\\
&\quad+\mu(t_\upsilon)\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{pm}}\phi_{pm}(\theta)\\
&\quad\times\Big(\int_{t_\upsilon^+-\theta}^{t_{\underline{\mathcal{K}}(t,v,\xi_{pm})}^-} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s\\
&\quad+\sum_{q=\underline{\mathcal{K}}(t,v,\xi_{pm})}^{\overline{\mathcal{K}}(t,v,\xi_{pm})-1}\int_{t_q^+}^{t_{q+1}^-} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s\\
&\quad+\int_{t_{\overline{\mathcal{K}}(t,v,\xi_{pm})}^+}^{t_\upsilon^+} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s\Big)\mathrm{d}\theta\\
&= \big(1+\varphi(t_\upsilon)\big)^2\sum_{p=1}^{k}\sum_{i\in G_p} e_i^{\mathrm{T}}(t_\upsilon^-)e_i(t_\upsilon^-)\\
&\quad+\mu(t_\upsilon)\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{pm}}\phi_{pm}(\theta)\\
&\quad\times\Big(\int_{t_\upsilon^--\theta}^{t_{\underline{\mathcal{K}}(t,v,\xi_{pm})}^-} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s\\
&\quad+\sum_{q=\underline{\mathcal{K}}(t,v,\xi_{pm})}^{\overline{\mathcal{K}}(t,v,\xi_{pm})-1}\int_{t_q^+}^{t_{q+1}^-} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s\\
&\quad+\int_{t_{\overline{\mathcal{K}}(t,v,\xi_{pm})}^+}^{t_\upsilon^-} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s\Big)\mathrm{d}\theta\\
&= \mu(t_\upsilon)V(t_\upsilon^-).
\end{aligned} \tag{6}$$

When $\upsilon=1$, from inequalities (5) and (6),

$$\begin{aligned}
V(t_1^-) &\le V(t_0)\exp(\lambda\rho_1),\\
V(t_1^+) &\le \mu(t_1)V(t_1^-)\le\mu(t_1)V(t_0)\exp(\lambda\rho_1).
\end{aligned}$$

When $\upsilon=2$,

$$\begin{aligned}
V(t_2^-) &\le V(t_1^+)\exp(\lambda\rho_2)\\
&\le \mu(t_1)V(t_0)\exp\big(\lambda(\rho_1+\rho_2)\big),\\
V(t_2^+) &\le \mu(t_2)V(t_2^-)\\
&\le \mu(t_2)\mu(t_1)V(t_0)\exp\big(\lambda(\rho_1+\rho_2)\big)\\
&= V(t_0)\prod_{\varepsilon=1}^{2}\mu(t_\varepsilon)\exp(\lambda\rho_\varepsilon).
\end{aligned}$$

By mathematical induction, for any positive integer υ,

$$V(t_\upsilon^+)\le V(t_0)\prod_{\varepsilon=1}^{\upsilon}\mu(t_\varepsilon)\exp(\lambda\rho_\varepsilon).$$

If condition (4) holds,

$$\mu(t_\varepsilon)\exp(\lambda\rho_\varepsilon)\le\exp(-\delta),\ \varepsilon=1,2,\ldots,$$

$$V(t_\upsilon^+)\le V(t_0)\exp(-\upsilon\delta),$$

which implies $V(t_\upsilon^+)\to 0$ for $\upsilon\to\infty$.

Then, for $t\in(t_\upsilon,t_{\upsilon+1}]$,

$$V(t)\le V(t_\upsilon^+)\exp(\lambda(t-t_\upsilon)),$$

which implies $V(t)\to 0$ for $t\to\infty$, i.e., the cluster synchronization is achieved. This completes the proof.

Remark 1 In particular, if the set $\mathcal{K}(t,v,\xi_{pm})$ is empty set, the Lyapunov function is

$$\begin{aligned}
V(t) &= \sum_{p=1}^{k}\sum_{i\in G_p} e_i^{\mathrm{T}}(t)e_i(t)+\mu(t)\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{pm}}\phi_{pm}(\theta)\\
&\quad\times\int_{t-\theta}^{t} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s\mathrm{d}\theta;
\end{aligned}$$

if the set $\mathcal{K}(t,v,\xi_{pm})$ only has an element d, i.e., $\underline{\mathcal{K}}(t,v,\xi_{pm})=\overline{\mathcal{K}}(t,v,\xi_{pm})=d$,

$$\begin{aligned}
V(t) &= \sum_{p=1}^{k}\sum_{i\in G_p} e_i^{\mathrm{T}}(t)e_i(t)\\
&\quad+\mu(t)\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{pm}}\phi_{pm}(\theta)\\
&\quad\times\Big(\int_{t-\theta}^{t_d^-} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s+\int_{t_d^+}^{t} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s\Big)\mathrm{d}\theta.
\end{aligned}$$

Remark 2 From condition (4), for any given community network, the needed impulsive gains and intervals for achieving cluster synchronization can be estimated by some calculations. Similar to many existing results about the impulsive control, when the impulsive gains are fixed, the upper bound of the impulsive intervals should be smaller than a certain positive constant. On the other hand, for different community networks, the needed values are different. That is, condition (4) is conservative. For solving this problem, an adaptive strategy is adopted to design adaptive impulsive controllers.

Theorem 2 Suppose that Assumption 1 holds. If there exists a constant $\delta>0$ such that

$$\ln\mu(t_\upsilon)+\delta+\hat{\mathcal{Q}}(t_\upsilon)\rho_\upsilon<0,\ \upsilon=1,2,\ldots, \tag{7}$$

hold, where $\dot{\hat{\mathcal{Q}}}(t)=\omega\sum_{p=1}^{k}\sum_{i\in G_p}e_i^{\mathrm{T}}(t)e_i(t)$ and $\omega>0$ is an adaptive gain, then the synchronization of network (2) is achieved.

Proof Consider the following Lyapunov function candidate:

$$\begin{aligned}
V(t) &= \sum_{p=1}^{k}\sum_{i\in G_p} e_i^{\mathrm{T}}(t)e_i(t)+mu(t)\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{pm}}\phi_{pm}(\theta)\\
&\quad\times\Big(\int_{t-\theta}^{t_{\underline{\mathcal{K}}(t,v,\xi_{pm})}^-} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s\\
&\quad+\sum_{q=\underline{\mathcal{K}}(t,v,\xi_{pm})}^{\overline{\mathcal{K}}(t,v,\xi_{pm})-1}\int_{t_q^+}^{t_{q+1}^-} e_j^{\mathrm{T}}(s)e_j(s)\mathrm{d}s
\end{aligned}$$

$$+\int_{t^+_{\overline{\mathcal{K}}(t,v,\xi_{pm})}}^{t} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\Big)\,\mathrm{d}\theta+\frac{\mu(t)}{2\omega}(\hat{\mathcal{Q}}(t)-\lambda)^2,$$

for $t\in(t_{\upsilon-1},t_\upsilon]$, $\upsilon=1,2,\ldots$

When $t\in(t_{\upsilon-1},t_\upsilon)$,

$$\begin{aligned}V(t) &= \sum_{p=1}^{k}\sum_{i\in G_p} e_i^{\mathrm{T}}(t)e_i(t)+\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{pm}}\phi_{pm}(\theta)\\ &\quad\times\Big(\int_{t-\theta}^{t^-_{\underline{\mathcal{K}}(t,v,\xi_{pm})}} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\\ &\quad+\sum_{q=\underline{\mathcal{K}}(t,v,\xi_{pm})}^{\overline{\mathcal{K}}(t,v,\xi_{pm})-1}\int_{t_q^+}^{t^-_{q+1}} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\\ &\quad+\int_{t^+_{\overline{\mathcal{K}}(t,v,\xi_{pm})}}^{t} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\Big)\,\mathrm{d}\theta+\frac{1}{2\omega}(\hat{\mathcal{Q}}(t)-\lambda)^2,\end{aligned}$$

and the derivative of $V(t)$ is

$$\begin{aligned}\dot{V}(t) &= 2\sum_{p=1}^{k}\sum_{i\in G_p} e_i^{\mathrm{T}}(t)\Big(f(y_i(t))-f(\eta_{\pi_i}(t))\\ &\quad+\sum_{m=1}^{k}\sum_{j\in G_m} b_{ij}\mathrm{R}\int_0^{\xi_{\pi_i\pi_j}}\phi_{\pi_i\pi_j}(\theta)e_j(t-\theta)\,\mathrm{d}\theta\Big)\\ &\quad+\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{pm}}\phi_{pm}(\theta)e_j^{\mathrm{T}}(t)e_j(t)\,\mathrm{d}\theta\\ &\quad-\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{\pi_i\pi_j}}\phi_{\pi_i\pi_j}(\theta)e_j^{\mathrm{T}}(t-\theta)e_j(t-\theta)\,\mathrm{d}\theta\\ &\quad+\frac{1}{\omega}(\hat{\mathcal{Q}}(t)-\lambda)\dot{\hat{\mathcal{Q}}}(t)\\ &\le (2M+k)E^{\mathrm{T}}(t)E(t)\\ &\quad+\sum_{p=1}^{k}\sum_{m=1}^{k}E_p^{\mathrm{T}}(t)(B_{pm}\otimes\mathcal{R})(B_{pm}\otimes\mathcal{R})^{\mathrm{T}}E_p(t)\\ &\quad+(\hat{\mathcal{Q}}(t)-\lambda)E^{\mathrm{T}}(t)E(t)\\ &= E^{\mathrm{T}}(t)\big((2M+k)I+(B\otimes\mathcal{R})(B\otimes\mathcal{R})^{\mathrm{T}}\big)E(t)\\ &\quad+(\hat{\mathcal{Q}}(t)-\lambda)E^{\mathrm{T}}(t)E(t)\\ &\le \lambda E^{\mathrm{T}}(t)E(t)+(\hat{\mathcal{Q}}(t)-\lambda)E^{\mathrm{T}}(t)E(t)\\ &\le \hat{\mathcal{Q}}(t)V(t)\le\hat{\mathcal{Q}}(t_\upsilon)V(t),\end{aligned}$$

which gives

$$V(t)\le V(t^+_{\upsilon-1})\exp\big(\hat{\mathcal{Q}}(t)(t-t_{\upsilon-1})\big),\ t\in(t_{\upsilon-1},t_\upsilon).$$

When $t=t_\upsilon$,

$$\begin{aligned}V(t^+_\upsilon) &= \sum_{p=1}^{k}\sum_{i\in G_p} e_i^{\mathrm{T}}(t^+_\upsilon)e_i(t^+_\upsilon)\\ &\quad+\mu(t_\upsilon)\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{pm}}\phi_{pm}(\theta)\\ &\quad\times\Big(\int_{t^+_\upsilon-\theta}^{t^-_{\underline{\mathcal{K}}(t,v,\xi_{pm})}} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\\ &\quad+\sum_{q=\underline{\mathcal{K}}(t,v,\xi_{pm})}^{\overline{\mathcal{K}}(t,v,\xi_{pm})-1}\int_{t_q^+}^{t^-_{q+1}} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\\ &\quad+\int_{t^+_{\overline{\mathcal{K}}(t,v,\xi_{pm})}}^{t^+_\upsilon} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\Big)\,\mathrm{d}\theta\\ &\quad+\frac{\mu(t_\upsilon)}{2\omega}(\hat{\mathcal{Q}}(t^+_\upsilon)-\lambda)^2\\ &= \big(1+\varphi(t_\upsilon)\big)^2\sum_{p=1}^{k}\sum_{i\in G_p} e_i^{\mathrm{T}}(t^-_\upsilon)e_i(t^-_\upsilon)\\ &\quad+\mu(t_\upsilon)\sum_{p=1}^{k}\sum_{m=1}^{k}\sum_{j\in G_m}\int_0^{\xi_{pm}}\phi_{pm}(\theta)\\ &\quad\times\Big(\int_{t^-_\upsilon-\theta}^{t^-_{\underline{\mathcal{K}}(t,v,\xi_{pm})}} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\\ &\quad+\sum_{p=\underline{\mathcal{K}}(t,v,\xi_{pm})}^{\overline{\mathcal{K}}(t,v,\xi_{pm})-1}\int_{t_p^+}^{t^-_{p+1}} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\\ &\quad+\int_{t^+_{\overline{\mathcal{K}}(t,v,\xi_{pm})}}^{t^-_\upsilon} e_j^{\mathrm{T}}(s)e_j(s)\,\mathrm{d}s\Big)\,\mathrm{d}\theta+\frac{\mu(t_\upsilon)}{2\omega}(\hat{\mathcal{Q}}(t^-_\upsilon)-\lambda)^2\\ &= \mu(t_\upsilon)V(t^-_\upsilon).\end{aligned}$$

Thus, similar to the proof of Theorem 1, the proof is obtained.

Remark 3 Generally, for any given ρ_υ and δ, the conditions (7) hold when choosing

$$\begin{aligned}&-\exp\Big(-\frac{\delta+\hat{\mathcal{Q}}(t_\upsilon)\rho_\upsilon}{2}\Big)-1+\alpha\le\varphi(t_\upsilon)\\ &\le\exp\Big(-\frac{\delta+\hat{\mathcal{Q}}(t_\upsilon)\rho_\upsilon}{2}\Big)-1-\alpha,\end{aligned}$$

where $\alpha>0$ is an arbitrary small constant. In practical applications, we can choose $\varphi(t_\upsilon)=\exp(-\frac{\delta+\hat{\mathcal{Q}}(t_\upsilon)\rho_\upsilon}{2})-1-\alpha$ with small positive constants α and δ, and $\hat{\mathcal{Q}}(t_0)$ as a small value. The definition of $\hat{\mathcal{Q}}(t)$ implies that $\hat{\mathcal{Q}}(t)$ is a monotone increasing function. When the cluster synchronization is achieved, $\hat{\mathcal{Q}}(t)$ converges to a positive constant and the impulsive gain $\varphi(t_\upsilon)$ converges to a needed value.

Remark 4 In practical application, when $\varphi(t_\upsilon)$ and δ are fixed, the control intervals should be chosen to be as large as possible to reduce the cost. Thus, the impulsive instants t_υ can be chosen by finding the maximum value of t_υ subject to $t_\upsilon<t_{\upsilon-1}-\big(\ln\mu(t_\upsilon)+\delta\big)\hat{\mathcal{Q}}^{-1}(t_\upsilon)$ with $\upsilon=1,2,\ldots$.

Remark 5 From conditions (7), the constant λ in condition (4) with respect to the system parameters does not need to be calculated beforehand and can be estimated by $\hat{\mathcal{Q}}(t)$ when the cluster synchronization is achieved. Remarks 3 and 4 show that the impulsive gains or instants can be estimated adaptively through choosing proper parameters. Specially, from Remark 4, the upper bound of the impulsive intervals does not need to be calculated beforehand. That is, the conditions (7) are not conservative to some extent.

4. Numerical illustrations

In this section, we use three numerical examples to demonstrate the effectiveness of the theoretical results.

Choose the density function as $\phi_{pm}(\theta) = 1/\xi_{pm}$, the inner coupling matrix $\mathcal{R}$ as the identity matrix, and $\xi_{11} = 0.2$, $\xi_{12} = 0.1$, $\xi_{13} = 0.2$, $\xi_{21} = 0.4$, $\xi_{22} = 0.4$, $\xi_{23} = 0.1$, $\xi_{31} = 0.2$, $\xi_{32} = 0.4$, $\xi_{33} = 0.1$ in the following examples. In addition, choose the initial values of $y_i(t)$ and $\eta_{\pi_i}(t)$ randomly.

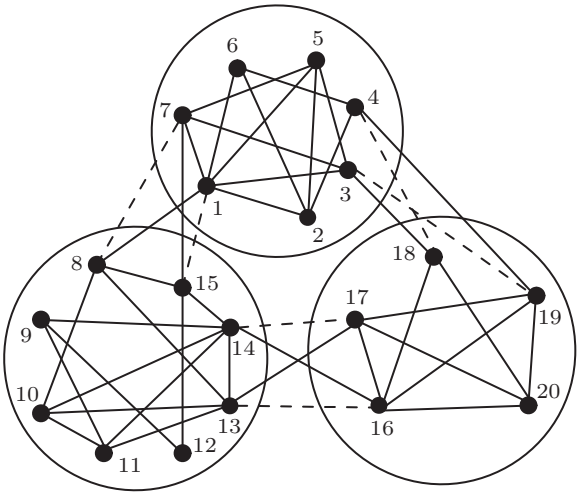


Fig. 1. A community network consisting of 20 nodes in 3 communities. The top, bottom-left, and bottom-right are the first, the second, and the third communities, respectively. The outer coupling matrix B is defined as: if nodes i and j $(j \neq i)$ are connected by a solid line, then $b_{ij} = 0.001$; if nodes i and j $(j \neq i)$ are connected by a dashed line, then $b_{ij} = -0.001$.

Example 1 Consider a community network consisting of 20 nodes in 3 communities, whose topology is shown in Fig. 1. Choose the node dynamics of the first community as the Lorenz system[32]

$$\begin{aligned}\dot{y}_1 &= 10(y_2 - y_1),\\ \dot{y}_2 &= 28y_1 - y_1y_3 - y_2,\\ \dot{y}_3 &= y_1y_2 - 8y_3/3,\end{aligned}$$

the second community as the Chen system[33]

$$\begin{aligned}\dot{y}_1 &= 35(y_2 - y_1),\\ \dot{y}_2 &= -7y_1 - y_1y_3 + 28y_2,\\ \dot{y}_3 &= y_1y_2 - 3y_3,\end{aligned}$$

and the third community as the Lü system[34]

$$\begin{aligned}\dot{y}_1 &= 36(y_2 - y_1),\\ \dot{y}_2 &= -y_1y_3 + 20y_2,\\ \dot{y}_3 &= y_1y_2 - 3y_3.\end{aligned}$$

According to the discussion in Ref. [35], we can choose $M = 53$ such that Assumption 1 holds. Choose the impulsive gains $\varphi(t_\upsilon) = -0.999$ and intervals $\rho_\upsilon = 0.12$. Then the maximum eigenvalue of $(2M+k)I + (B \otimes \mathcal{R})(B \otimes \mathcal{R})^{\mathrm{T}}$ is $\lambda = 109.0001$, and $\ln \mu(t_\upsilon) + \delta + \lambda\rho_\upsilon = -0.7345 < 0$ with $\delta = 0.001$, i.e., the condition 4 holds. Figure 2 shows the orbits of synchronization errors in 3 communities.

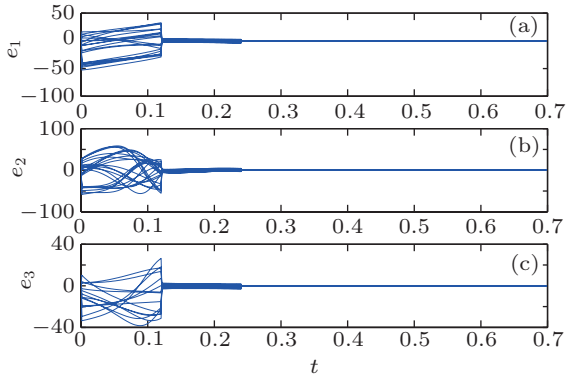


Fig. 2. (color online) The orbits of synchronization errors in 3 communities.

Example 2 Consider the same community network in Example 1 via adaptive impulsive control.

First, choose $\rho_\upsilon = 0.12$, $\delta = 0.001$, $\omega = 0.0003$, and $\hat{\mathcal{Q}}(0) = 0.05$. According to Remark 3, choose $\varphi(t_\upsilon) = \exp(-\frac{\delta + \hat{\mathcal{Q}}(t_\upsilon)\rho_\upsilon}{2}) - 1 - \alpha$ with $\alpha = 0.001$. Figure 3 shows the orbits of synchronization errors in 3 communities. Figure 4 shows the impulsive gain $\varphi(t_\upsilon)$.

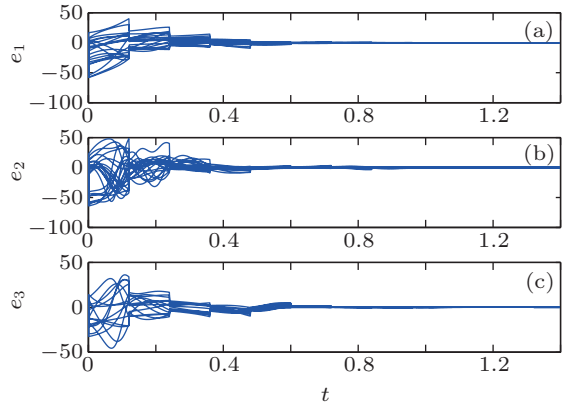


Fig. 3. (color online) The orbits of synchronization errors in 3 communities.

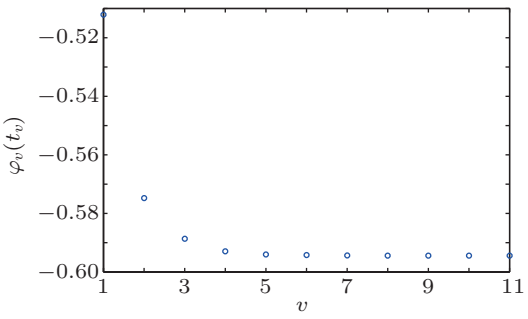


Fig. 4. (color online) The impulsive gain $\varphi(t_\upsilon)$.

Second, choose the impulsive gain $\varphi(t_\upsilon) = -0.999$, $\omega = 0.0003$, $\hat{\mathcal{Q}}(0) = 0.05$, $\delta = 0.001$. According to Remark 4, we can estimate the impulsive instants. Figure 5 shows the orbits of synchronization errors in 3 communities. Figure 6 shows the impulsive interval ρ_υ.

In Example 1, the impulsive interval is $\rho_\upsilon = 0.12$. Figure 6 shows that the network achieves synchronization when the impulsive interval converges to 0.848. Obviously, the impulsive interval with the adaptive impulsive strategy is larger than that without the strategy.

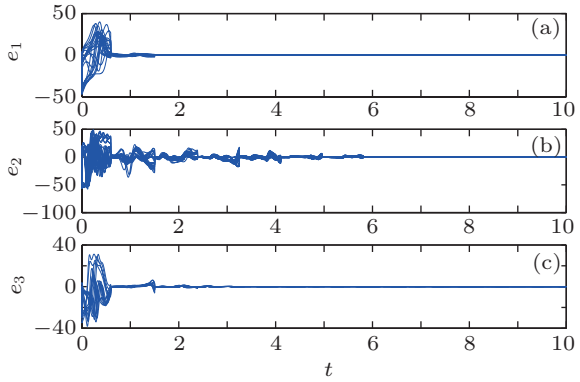


Fig. 5. (color online) The orbits of synchronization errors in 3 communities.

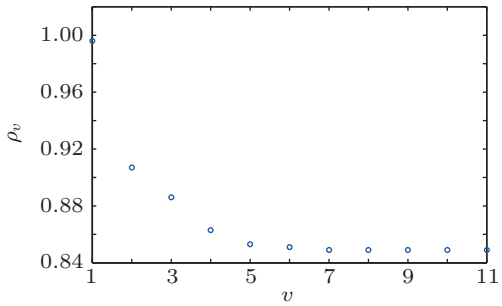


Fig. 6. (color online) The impulsive interval ρ_υ.

Example 3 We generate the first community network topology as the WS small-world network[36] with $N_1 = 50$, $K = 5$, and $p = 0.2$, the second as the WS small-world network[36] with $N_2 = 50$, $K = 6$, and $p = 0.1$, and the third as the NW small-world network[37] with $N_3 = 50$, $K = 2$, and $p = 0.1$. Further, we add a few connections among the three communities. Choose the same node dynamics as Example 1.

First, choose $\rho_\upsilon = 0.2$, $\delta = 0.001$, $\omega = 0.00005$, and $\hat{\mathcal{Q}}(0) = 0.05$. According to Remark 3, choose $\varphi(t_\upsilon) = \exp(-\frac{\delta+\hat{\mathcal{Q}}(t_\upsilon)\rho_\upsilon}{2}) - 1 - \alpha$ with $\alpha = 0.001$. Figure 7 shows the orbits of synchronization errors in 3 communities. Figure 8 shows the impulsive gain $\varphi(t_\upsilon)$.

Second, choose the impulsive gain $\varphi(t_\upsilon) = -0.9$, $\omega = 0.00005$, $\hat{\mathcal{Q}}(0) = 0.05$, $\delta = 0.001$. According to Remark 4, we can estimate the impulsive instants. Figure 9 shows the orbits of synchronization errors in 3 communities. Figure 10 shows the impulsive interval ρ_υ.

Examples 2 and 3 indicate that the cluster synchronization of networks with nonidentical node dynamics and topological structures can be achieved without calculating the constants with respect to the system parameters in advance. Thus, the adaptive impulsive controllers are unified for different community networks.

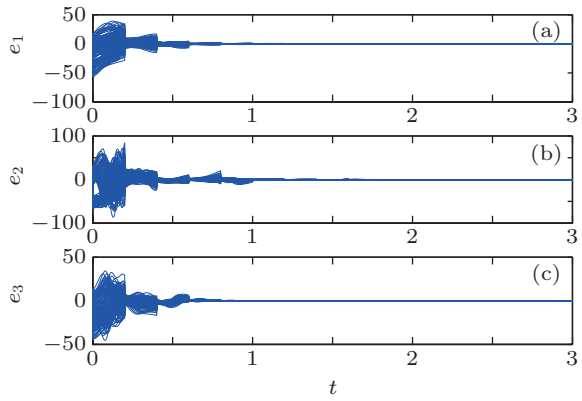


Fig. 7. (color online) The orbits of synchronization errors in 3 communities.

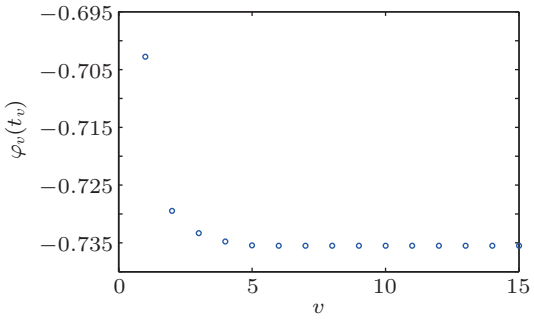


Fig. 8. (color online) The impulsive gain $\varphi(t_\upsilon)$.

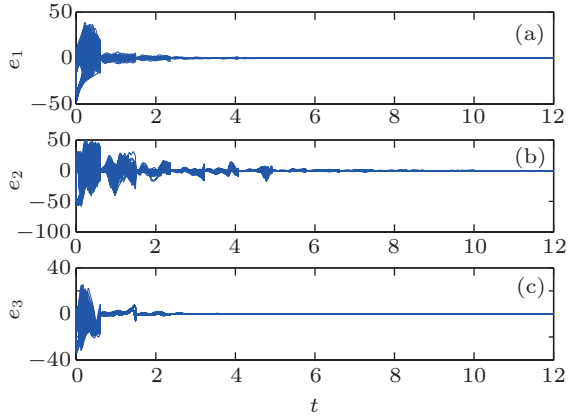


Fig. 9. (color online) The orbits of synchronization errors in 3 communities.

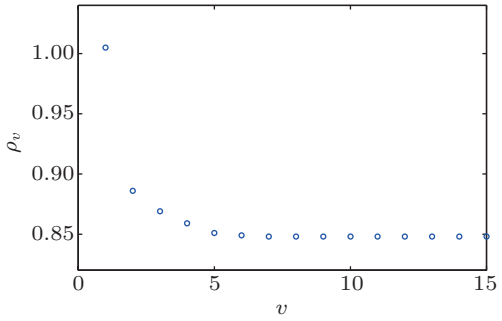


Fig. 10. (color online) The impulsive interval ρ_υ.

5. Conclusion

Community networks with distributed time delays and nonidentical node dynamics are investigated. Further, the coupling delays in different communities are also assumed to be nonidentical. Through designing proper impulsive controllers, the nodes in the different communities are synchronized onto different orbits and the nodes in the same community are synchronized with each other. That is, the cluster synchronization of the community network is achieved. With the linear matrix inequality technique and Lyapunov function method, some synchronization criteria with respect to the impulsive gains, instants, and system parameters are obtained. Noticeably, an adaptive strategy is adopted to make the impulsive controllers possess universality and the corresponding synchronization criteria are obtained as well. When the impulsive intervals are fixed, the impulsive gain can adaptively approach to the proper value according to the updating law. When the impulsive gains are fixed, the impulsive instants can be estimated by solving a sequence of maximum value problems. Finally, the effectiveness of the theoretical results are demonstrated by three numerical examples.

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