Stopping time of a one-dimensional bounded quantum walk*

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The stopping time of a one-dimensional bounded classical random walk (RW) is defined as the number of steps taken by a random walker to arrive at a fixed boundary for the first time. A quantum walk (QW) is a non-trivial generalization of RW, and has attracted a great deal of interest from researchers working in quantum physics and quantum information. In this paper, we develop a method to calculate the stopping time for a one-dimensional QW. Using our method, we further compare the properties of stopping time for QW and RW. We find that the mean value of the stopping time is the same for both of these problems. However, for short times, the probability for a walker performing a QW to arrive at the boundary is larger than that for a RW. This means that, although the mean stopping time of a quantum and classical walker are the same, the quantum walker has a greater probability of arriving at the boundary earlier than the classical walker.

Keywords: quantum walk, stopping time, cumulative probability

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1. Introduction

The random walk (RW) has been an important mathematical problem in probability theory and stochastic processes.[1] The quantum walk (QW)[2] is a nontrivial extension of the classical RW, in which the walker obeys quantum mechanical rules. Because of quantum interference effects, the properties of QW are significantly different from those of RW.[3]

A motivation for the study on QW is to develop new quantum algorithms, which provide exponential speed-ups[4–10] when compared with classical algorithms. In addition, QW is a valuable model for studying physical processes of several important quantum protocols.[11–24] QW has been realized in kinds of experiments involving linear optical systems,[25–31] cold atoms on an optical lattice,[32] trapped ions,[33,34] nuclear magnetic resonance techniques,[35,36] optical waveguides,[37] and optical fiber networks.[38] Moreover, recent researches proposed schemes to realize QW with quantum dots system.[39,40]

In previous work, the research on QW has mainly focused on the probability distribution of the position of a quantum walker after a fixed number of steps.[11–24] In RW theory, there is another important concept called stopping time.[41] The stopping time is defined as the number of steps at which a random walker arrives at a fixed boundary for the first time. For a RW with given boundaries, the stopping time is a random variable, and the corresponding mean value and probability distribution have been studied by many authors. The results are shown to be useful. For instance, they can be used to solve the Gambler's Ruin problem.[42] Nevertheless, the stopping time of a QW has not been studied before.

In this paper, we develop a method to calculate the probability distribution of the stopping time for a discrete-time one-dimensional bounded QW. With our method, we calculate the probability distribution and the mean value of the stopping time and moreover compare the properties of stopping time for QW and RW. We show that their mean values for the stopping time are the same. However, the probability distributions associated with the stopping time are quite different for these two problems. In particular, over short-time intervals, the probability density of the stopping time of a QW is larger than that for a RW. This means that a quantum walker has a greater probability of arriving at boundary earlier than a classical walker. Our result is also illustrated via the analysis of the commutative probability of a QW and a RW.

The remainder of this paper is organized as follows. In Section 1, we briefly introduce the definition and properties of the stopping time of a RW. In Section 3, we show our method for the calculation of the probability distribution of the stopping time for a QW. In Section 4, we use our method to compare the properties of the stopping time for QW and RW. There is a brief summary in Section 5.

2. Stopping time of a RW

2.1. Classical random walk

To introduce the definition and mathematical properties of a discrete-time one-dimensional RW, we consider a walker starting from the origin. In the RW, in each step the walker flips an unbiased coin and moves to the right (left) if the coin

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shows a head (tail). After t steps, the position x of the walker is a random variable, and satisfies the binomial distribution

$$p^{(t)}(x) = \left(\frac{1}{2}\right)^t \frac{t!}{\left(\frac{t+x}{2}\right)\left(\frac{t-x}{2}\right)!}. \quad (1)$$

Here, x takes discrete values $0, \pm 1, \pm 2, \ldots$

2.2. Stopping time of a RW

Now, we introduce the definition of stopping time of a RW. We consider a RW in which the walker starts from the origin $x = 0$. We assume that there is a pair of dam-boards (i.e., boundary) at positions $x = \pm X_d$, and the RW terminates immediately when the walker arrives at one of these dam-boards (Fig. 1). The time T_s at which the process terminates, i.e., the time when the walker arrives at $x = X_d$ or $x = -X_d$, is defined as the stopping time of RW.

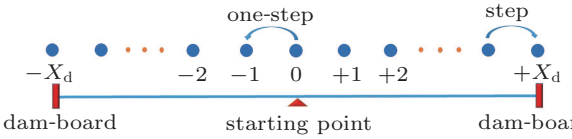


Fig. 1. (color online) Bounded discrete-time one-dimensional RW. The walker starts from position 0 and moves to the right or left in each step. The process stops the moment the walker arrives at one of two dam-boards at positions $x = \pm X_d$. In this problem, the stopping time is a random variable.

The stopping time T_s defined above is clearly a random variable; its distribution depends on the position X_d of the dam-boards, and can be denoted as $P^{(X_d)}(T_s)$. Accordingly, the mean value of the stopping time is also a function of X_d, here denoted as

$$E^{(X_d)}[T_s] \equiv \sum_{T_s=1}^{\infty} T_s P^{(X_d)}(T_s). \quad (2)$$

An analytical proof[42] showed that for a RW, we have

$$E^{(X_d)}[T_s] = X_d^2. \quad (3)$$

3. Stopping time of a QW

3.1. Quantum walk

We now introduce the QW, in which both walker and coin are quantum systems. The total Hilbert space is $\mathcal{H} = \mathcal{H}_w \otimes \mathcal{H}_c$, where $\mathcal{H}_w$ and $\mathcal{H}_c$ are the Hilbert spaces for the walker and coin, respectively. Here, $\mathcal{H}_w$ is spanned by position eigenstates $|x\rangle_w$ ($x = 0, \pm 1, \pm 2, \ldots$) of the walker, and $\mathcal{H}_c$ is spanned by "spin" states $|H\rangle_c$ and $|T\rangle_c$, representing the head and tail, respectively, of the coin.

In the beginning of a QW, the walker is at the origin and the coin is prepared in a superposition of head and tail. Specifically, the initial state is

$$|\Psi_0\rangle = |0\rangle_w \left(\frac{|H\rangle_c + \mathrm{i}|T\rangle_c}{\sqrt{2}}\right). \quad (4)$$

In the first step, the coin is first operated by a Hadamard gate

$$U_H = \frac{1}{\sqrt{2}}\left(|H\rangle\langle H| - |T\rangle\langle T| + |H\rangle\langle T| + |T\rangle\langle H|\right), \quad (5)$$

and then the system is operated by a "conditional-shift transformation"

$$U_S = \sum_x \left[|H\rangle\langle H| \otimes |x+1\rangle\langle x| + |T\rangle\langle T| \otimes |x-1\rangle\langle x|\right]. \quad (6)$$

The physical meaning of U_S is that the walker moves to the right if the coin shows a head, and moves to the left if the coin shows a tail. Thus, after the first step, the state of the total system is

$$|\Psi(1)\rangle = U_S U_H |\Psi_0\rangle. \quad (7)$$

Similarly, after t steps, the total system is in the state

$$|\Psi(t)\rangle = [U_S U_H]^t |\Psi_0\rangle. \quad (8)$$

In this QW process, the walker and coin are entangled. Because of quantum interference effects, which can be constructive or destructive, the probability distribution of the walker's position is completely different from that for RW (Eq. (1)). The properties of this distribution were discussed in Refs. [3], [11], [41] and [43].

It is pointed out that, the QW can be reduced to a classical RW under the decoherence operation in each step.[44] For our system, we can obtain the results of RW via replacing Eqs. (7) and (8) with $\rho(1) = U_S U_H \mathcal{D}[\rho(0)] U_H^\dagger U_S$ and $\rho(t) = U_S U_H \mathcal{D}[\rho(t-1)] U_H^\dagger U_S$, respectively. Here, $\rho(t)$ is the density operator of the walker and the coin, $\rho(0) = |\Psi_0\rangle\langle\Psi_0|$, and the decoherence operator $\mathcal{D}$ is defined as follows: in the bases $\{|x\rangle\}$, all the non-diagonal elements of $\mathcal{D}[\rho]$ are zero, while the diagonal elements of $\mathcal{D}[\rho]$ are the same as the ones of ρ.

3.2. Stopping time of a QW

Next, we introduce our method for the calculation of the probability distribution of the stopping time of a QW, i.e., the probability distribution for the number of steps T_s at which the walker arrives at the dam-board at $x = \pm X_d$ ($X_d > 0$) for the first time. This probability distribution can be denoted as $P^{(X_d)}(T_s)$ and can be calculated as follows.

(i) For time $t < x_d$, the walker would not have arrived at a dam-board. Therefore, we have

$$P^{(X_d)}(T_s < X_d) = 0. \quad (9)$$

(ii) Furthermore, according to Eq. (8), at step $t = X_d$ (before arrival), the density matrix of our system is

$$\rho(X_d) = [U_S U_H]^{X_d} \rho_0 \left[U_H^\dagger U_S^\dagger\right]^{X_d}, \quad (10)$$

where

$$\rho_0 = |\Psi_0\rangle\langle\Psi_0| \tag{11}$$

is the initial density matrix of our system. The probability that the walker arrives at $\pm X_\mathrm{d}$ at this step is

$$P^{(X_\mathrm{d})}(T_\mathrm{s} = X_\mathrm{d}) = \mathrm{Tr}\left[D\rho\left(X_\mathrm{d}\right)\right], \tag{12}$$

where the operator D is the projection operator for the positions $x = \pm X_\mathrm{d}$ and can be expressed as

$$D = |X_\mathrm{d}\rangle_\mathrm{w}\langle X_\mathrm{d}| + |-X_\mathrm{d}\rangle_\mathrm{w}\langle -X_\mathrm{d}|. \tag{13}$$

(iii) If X_d is an odd (even) value, then the walker can only have arrived at $x = \pm X_\mathrm{d}$ in an odd (even) number of steps. Hence, we have

$$P^{(X_\mathrm{d})}(T_\mathrm{s} = X_\mathrm{d} + 2n + 1) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{14}$$

(iv) The probability $P^{(X_\mathrm{d})}(T_\mathrm{s} = X_\mathrm{d} + 2)$ that walker arrives at the $X_\mathrm{d} + 2$ step is the product of the probability p_* that the walker had not arrived at the $t = X_\mathrm{d}$ step and the probability q_* that the walker arrives at $x = \pm X_\mathrm{d}$ at the $X_\mathrm{d} + 2$ step; i.e., we have $P^{(X_\mathrm{d})}(T_\mathrm{s} = X_\mathrm{d} + 2) = p_* q_*$ and hence $p_* = \mathrm{Tr}\left[T\rho\left(X_\mathrm{d}\right)T\right]$ where $T = 1 - D$. Furthermore, if the walker had not arrived at $t = X_\mathrm{d}$, then after arrival the density matrix of our system is

$$\rho'(X_\mathrm{d}) \equiv \frac{T\rho(X_\mathrm{d})T}{\mathrm{Tr}\left[T\rho\left(X_\mathrm{d}\right)T\right]}. \tag{15}$$

Thus, at the $t = X_\mathrm{d} + 2$ step (before arrival), the density matrix is $[U_\mathrm{S}U_\mathrm{H}]^2\rho'(X_\mathrm{d})\left[U_\mathrm{H}^\dagger U_\mathrm{S}^\dagger\right]^2$. Therefore, the probability q_* can be expressed as

$$q_* = \mathrm{Tr}\left(D\left[U_\mathrm{S}U_\mathrm{H}\right]^2\rho'(X_\mathrm{d})\left[U_\mathrm{H}^\dagger U_\mathrm{S}^\dagger\right]^2\right). \tag{16}$$

Using the relation $P^{(X_\mathrm{d})}(T_\mathrm{s} = X_\mathrm{d} + 2) = p_* q_*$, we then obtain

$$\begin{aligned} &P^{(X_\mathrm{d})}(T_\mathrm{s} = X_\mathrm{d} + 2) \\ &= \mathrm{Tr}\left(D\left[U_\mathrm{S}U_\mathrm{H}\right]^2 T\rho(X_\mathrm{d})T\left[U_\mathrm{H}^\dagger U_\mathrm{S}^\dagger\right]^2\right). \end{aligned} \tag{17}$$

(v) By repeating the above steps, we can further obtain the probability $P^{(X_\mathrm{d})}(T_\mathrm{s} = X_\mathrm{d} + 2n)$ $(n = 2, 3, 4, \ldots)$ that the walker arrives at the $X_\mathrm{d} + 2n$ step.

Given the above method, we calculated the probability $P^{(X_\mathrm{d})}(T_\mathrm{s})$ numerically. Because this QW is bounded between $-X_\mathrm{d}$ and X_d, the dimension of the Hilbert space for the walker's spatial position is $2X_\mathrm{d} + 1$; similarly, the dimension for the coin's spin space is 2. Therefore, the density matrix of the system is a matrix $(4X_\mathrm{d} + 2) \times (4X_\mathrm{d} + 2)$ for the entire process. Theoretically, the random variable t extends to infinity and therefore this evolution is capable of working out all probabilities $P(t)$ (Fig. 2). The truncation time is chosen as $t = N = 100X_\mathrm{d}^2$, which is large enough for the precision of $E^{(X_\mathrm{d})}[T_\mathrm{s}]$. Through this calculation and a sufficiently large truncation time, the mean value of the stopping time of QW can be obtained.

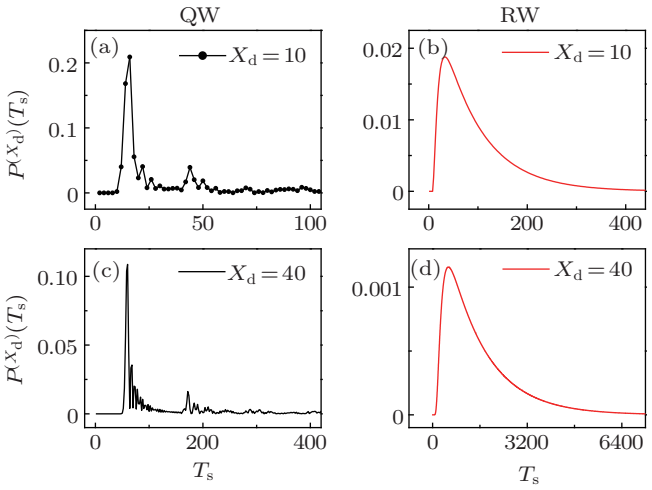


Fig. 2. (color online) $P^{(X_\mathrm{d})}(T_\mathrm{s})$ for QW ((a) and (c)) and RW ((b) and (d)) with $X_\mathrm{d} = 10$ ((a) and (b)) and $X_\mathrm{d} = 40$ ((c) and (d)). Because X_d is even in our cases, we have $P^{(X_\mathrm{d})}(T_\mathrm{s}) = 0$ for odd steps. $T_\mathrm{s} = \pm 1, \pm 3, \pm 5, \ldots$. Hence, we have only shown $P^{(X_\mathrm{d})}(T_\mathrm{s})$ for $T_\mathrm{s} = 0, \pm 2, \pm 4, \ldots$

4. Comparison of QW and RW

4.1. Mean value and probability distribution

With the method developed above, we numerically calculate the probability distribution of the stopping time T_s for many discrete-time one-dimensional QWs with different positions X_d of the dam-boards. With our result, we further derive the mean $E^{(X_\mathrm{d})}[T_\mathrm{s}]$ of T_s. We find that for fixed values of X_d, the mean value of the stopping time of a QW and a RW are same. Specifically, for a QW we also have $E^{(X_\mathrm{d})}[T_\mathrm{s}] = X_\mathrm{d}^2$. We illustrate this result in Fig. 3.

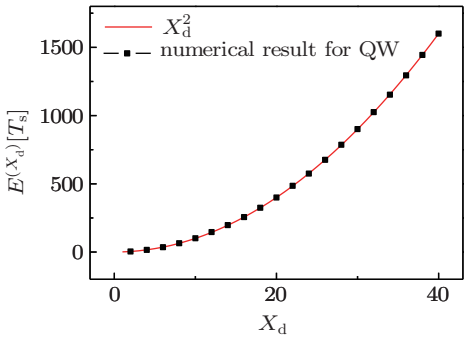


Fig. 3. (color online) Mean value $E^{(X_\mathrm{d})}[T_\mathrm{s}]$ of the stopping time of a discrete-time one-dimensional QW. The red solid line is the result $E^{(X_\mathrm{d})}[T_\mathrm{s}] = X_\mathrm{d}^2$; the black line is the numerical result for a QW, which is given by the method in Section 3.2.

Although the mean value of the stopping time T_s for both QW and RW are the same, the probability distribution of T_s for these two cases are completely different. Figure 2 illustrates the probability distribution $P^{(X_\mathrm{d})}(T_\mathrm{s})$ corresponding to $X_\mathrm{d} = 10$ and $X_\mathrm{d} = 40$, for both QW and RW. We find that for a given value of X_d, $P^{(X_\mathrm{d})}(T_\mathrm{s})$ for a QW peaks at a smaller T_s, and has a larger peak value. That is, if we define T_max as the value of T_s

at which the probability density $P^{(X_d)}(T_s)$ takes its maximum value, and define P_{max} as the corresponding peak value (i.e., $P_{max} \equiv P^{(X_d)}(T_{max})$), then QW has a smaller T_{max} and a larger P_{max}. For instance, as shown in Fig. 2, when $X_d = 40$, for QW we have $T_{max} = 60$ and $P_{max} = 0.1087$, whereas for RW we have $T_{max} = 532$ and $P_{max} = 1.157 \times 10^{-3}$.

Our numerical calculations show that this character is universal for QW for different values of X_d. From Fig. 4, when X_d is large (larger than 10), we have $T_{max} \approx \sqrt{2}X_d$ and $P_{max} \propto 1/\sqrt{X_d}$ for QW and $T_{max} \approx X_d^2/3$ and $P_{max} \propto 1/X_d^2$ for RW. Therefore, we know that as X_d increases, the disparity in stopping times for QW and RW increases.

In addition to the above features, our calculation also shows that when T_s is large, the decay of $P^{(X_d)}(T_s)$ for QW is slower than that for RW (Fig. 5). This explains the fact that although the probability distribution $P^{(X_d)}(T_s)$ peaks at a shorter time for QW, the mean of the stopping time T_s is the same for both QW and RW.

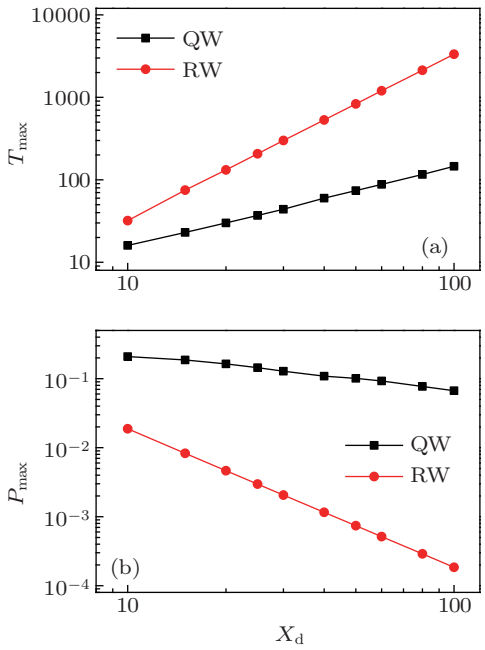


Fig. 4. (color online) (a) Time T_{max} for the maximum point in the probability distribution $P^{(X_d)}(T_s)$; (b) the peak value P_{max} of $P^{(X_d)}(T_s)$ at time T_{max}.

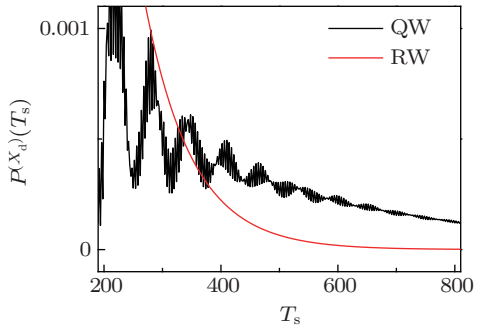


Fig. 5. (color online) Probability distribution $P^{(X_d)}(T_s)$ for QW and RW of $X_d = 10$ in the large-T_s region.

4.2. The probability that "QW is faster"

In the section above, we showed that in comparison with a RW, the probability distribution $P^{(X_d)}(T_s)$ of the stopping time for a QW peaks at smaller values of T_s, and has a larger peak value P_{max}. This property implies that for a given value of X_d, the walker of a QW has a large probability (more than 50%) of arriving at the dam-boards earlier than a walker of a RW. That is, we have a large probability for the fact that "QW is faster".

To verify this result, we numerically calculated the probability $Q(X_d)$ for the fact that "QW is faster". This probability can be reasonably defined as

$$Q(X_d) = \sum_{T_s, T_s'} P_{QW}^{(X_d)}(T_s) P_{RW}^{(X_d)}(T_s') \theta(T_s' - T_s). \tag{18}$$

Here, $P_{QW}^{(X_d)}(T_s)$ and $P_{RW}^{(X_d)}(T_s)$ are the probability distributions $P^{(X_d)}(T_s)$ of the stopping time for a QW and RW, respectively, and the θ-function is the Heaviside function, defined as $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \le 0$. In Fig. 6, we show $Q(X_d)$ for $X_d \in (10, 50)$. Clearly, in all of these cases we have $Q(X_d) > 50\%$, and $Q(X_d)$ increases with X_d. Hence, we know that because of the difference in probability distribution $P^{(X_d)}(T_s)$ of the stopping time for QW and RW, the walker in a QW has a greater probability $Q(X_d)$ of arriving at the dam-boards earlier than the walker of a RW, and this probability increases with distance X_d between the dam-boards and the starting point of the walker. This is also consistent with our previous result that the difference between $P^{(X_d)}(T_s)$ for QW and RW increases with X_d.

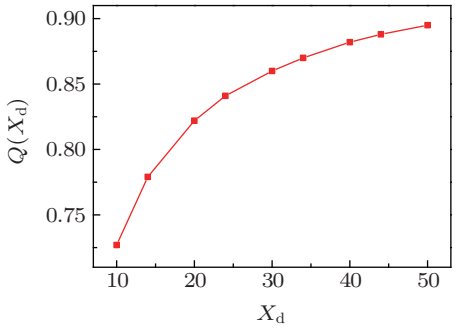


Fig. 6. (color online) Probability $Q(X_d)$ that a walker arrives faster by a QW than by a RW.

4.3. Cumulative probability

Finally, we study the cumulative probability for a QW, which is defined as the probability that the walker arrives at the dam-boards before time T. Denoted as $F^{(X_d)}(T)$, we can express it as

$$F^{(X_d)}(T) = \sum_{T_s \le T} P^{(X_d)}(T_s). \tag{19}$$

Clearly, we have

$$\lim_{T \to \infty} F^{(X_d)}(T) = 1. \tag{20}$$

In Fig. 7, we show the cumulative probability $F^{(X_d)}(T)$ for both QW and RW with $X_d = 40$. For a QW, the cumulative probability increases very fast for small T. As a result, for short times T, a walker of a QW has a larger probability of arriving at the dam-boards than a walker of the RW. The reason is that the probability distribution $P^{(X_d)}(T_s)$ peaks at a shorter time and has a larger peak value for a QW, as demonstrated in Section 4.1.

Figure 7 also shows that when time T is increased, the cumulative probability $F^{(X_d)}(T)$ of a RW can exceed that for a QW. As a result, when T is sufficiently large, the cumulative probability for a RW is larger than that for a QW. This is consistent with the fact that the mean of the stopping time is the same for both QW and RW.

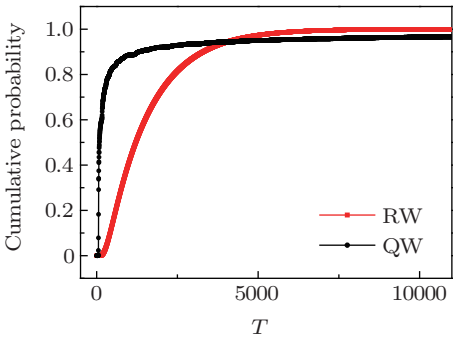


Fig. 7. (color online) Cumulative probability $F^{(X_d)}(T)$ for QW and RW with $X_d = 40$.

5. Summary

We have developed a method to calculate the probability distribution of the stopping time of a QW. With this method, we further studied the properties of this probability distribution and its mean value. We find that the mean value of the stopping time for QW and RW are the same. However, the distribution of QW and RW are quite different. For QW, the probability for short stopping times is significantly larger than that for RW, and the difference between the distribution of QW and RW increases with distance X_d between the dam-boards and the starting point of the walker. Our result implies that the walker in a QW has a large probability (more than 50%) of arriving at the boundary earlier than the walker of a RW. We also illustrated our result with the analysis for the cumulative probability for a QW and a RW. Our results, which show that a quantum walker has a very large probability to be faster than a classical walker, may be used in quantum search problems. In addition, with our result one can also further study the quantum version of classical problems related to stopping time, such as the Gambler's ruin problem.

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