Consensus for second-order multi-agent systems with position sampled data*

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In this paper, the consensus problem with position sampled data for second-order multi-agent systems is investigated. The interaction topology among the agents is depicted by a directed graph. The full-order and reduced-order observers with position sampled data are proposed, by which two kinds of sampled data-based consensus protocols are constructed. With the provided sampled protocols, the consensus convergence analysis of a continuous-time multi-agent system is equivalently transformed into that of a discrete-time system. Then, by using matrix theory and a sampled control analysis method, some sufficient and necessary consensus conditions based on the coupling parameters, spectrum of the Laplacian matrix and sampling period are obtained. While the sampling period tends to zero, our established necessary and sufficient conditions are degenerated to the continuous-time protocol case, which are consistent with the existing result for the continuous-time case. Finally, the effectiveness of our established results is illustrated by a simple simulation example.

Keywords: multi-agent systems, distributed control, consensus, observer, sampled data

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1. Introduction

In the last few decades, collective behaviors of the multi-agent systems have been attracting a great deal of researchers because of their numerous extensive applications such as sensor networks, formation control, consensus, containment, etc. (See Refs. [1]–[5]). Among the major areas of coordination control of multi-agent systems, consensus is one of the most significant and fundamental problems, which aims to design the distributed protocol such that the final states of all agents can converge to an agreement (Ref. [1]).

As all we know, the design of a consensus protocol and consensus analysis mainly rely on the agents' dynamics and their interaction topologies. In Ref. [6], the authors investigated the first-order consensus problem by the stochastic matrix approach, which provided the theoretical explanation of the Vicsek model (Ref. [7]). In Refs. [8]–[13], difference second-order multi-agent consensus problems were discussed, which have attracted increasing interest due to their numerous practical applications. Up to now, numerous interesting results have been established for the multi-agent consensus. Some other relevant research topics have also been addressed, such as event-triggered, time-delay, finite-time, input saturation, and communication constraint (See Refs. [4], [13]–[17]).

Most of the present approaches to solve the consensus problems of multi-agent systems are based on the state information. However, in many practical applications, some state variables such as velocity cannot be obtained directly because of the equipment constraints or cost saving. Thus, it may be a good choice to design the observer to estimate unavailable state variables, by which the control law is constructed. Up to now, the observer-based consensus problem becomes a significant topic in the area of multi-agent systems. In Ref. [8], Hong *et al.* proposed a distributed consensus protocol for first-order following agents based on the velocity observer to estimate the leader's unavailable velocity, which was extended to the case that the leader agent has general linear dynamics (See Refs. [18] and [19]). To track the accelerated motion leader, a distributed tracking law based on the acceleration observer was proposed for the second-order following-agents in Ref. [11]. For consensus problems with general linear dynamics, different design approaches have been provided to construct the full-order, reduced-order, and function observer-based consensus protocols in Refs. [20]–[23].

In many practical applications, due to the application of digital sensors and controllers, agents might only communicate with their neighbors periodically rather than continuously, that is, only sampled-data at discrete sampling instants are available for the synthesis of control protocols. Sampled-data control has increasingly attracted study, and some interesting and useful results have been established. By using direct discretization, the sampled-data based consensus protocols were proposed in Ref. [24], and the observer-based consensus laws with sampled-data was proposed under the fixed and Markov-switching topology in Ref. [25]. The results of Refs. [24]

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and [25] are essentially obtained for second-order discrete-time multi-agent system but not for second-order continuous-time multi-agent systems. In Ref. [26], necessary and sufficient conditions were established to solve the consensus problem with second-order dynamics via sampled control. The distributed consensus protocols only with sampled position data were proposed for second-order agents in Ref. [27], and the distributed consensus protocols only with sampled position and velocity data were proposed in Ref. [28]. Some sampled control with time-varying sampling intervals have been investigated in Refs. [29] and [30]. In Ref. [31], the authors studied the second-order consensus with sampled position and velocity data via pinning control. The sampled-data consensus problems with inherent nonlinear dynamics was investigated in Ref. [32].

Motivated by the aforementioned works, this paper investigates an observer-based consensus problem of second-order multi-agent systems with position sampled-data under directed fixed topology. Since the observer and sampled-data are both taken into account in the design of consensus protocols in this paper, our addressed problem becomes more interesting and difficult. The main contribution of this paper is that two kinds of observer-based consensus protocols with sampled data are proposed for the second-order multi-agent systems. Different to Ref. [25], the observers involved in this paper have continuous-time dynamics, the analysis and design are based on the multi-agent systems itself but not on its discretization system. By using matrix theory and the sampled control analysis method, some sufficient and necessary conditions with respect to the coupling parameters, spectrum of the Laplacian matrix and sampling period are established.

The rest of the paper is organized as follows. In Section 2, some preliminaries on algebraic graph theory and problem statement are introduced. Two kinds of distributed observer-based consensus protocols with sampled-data are provided in Section 3. Following that, a simulation example is presented to illustrate the theoretical result in Section 4. Finally, conclusions are drawn in the last section.

Now, some notations used throughout this paper are given as follows. $R^{m\times n}$, $C^{m\times n}$ are the sets of $m\times n$ real matrices and complex matrices, respectively. $\|\cdot\|$ denotes the Euclidean norm. The transpose of matrix A is represented by A^{T}. I_n and 0_n represent the identity and the zero matrices of dimension n, respectively. $\mathbf{1}_n \in R^n$ is the column vector with all elements equal to one. $\mathrm{Re}(\mu)$ and $\mathrm{Im}(\mu)$ expressed the real and imaginary part of the complex number μ, respectively. $A\otimes B$ denotes the Kronecker product of matrices A and B. A matrix is said to be Schur stable if all its eigenvalues lie inside a unit circle. A polynomial is said to be stable if all its roots have negative real parts.

2. Problem Formulation and Preliminaries

Consider a group of N identical agents, whose dynamics is modeled by second-order dynamics as follows:

$$\begin{aligned}\dot{r}_i(t) &= v_i(t),\\ \dot{v}_i(t) &= u_i(t), \quad i=1,\ldots,N,\end{aligned} \tag{1}$$

where $r_i(t)\in R^n$ and $v_i(t)\in R^n$ are the position and velocity of the i-th agent, respectively. $u_i(t)\in R^n$ is the control input.

The interaction relationships among N agents can be conveniently modeled by a simple weighted digraph. Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ be a directed weighted graph of order N, where $\mathcal{V}=\{v_1,v_2,\ldots,v_N\}$ is a group of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of edges. The adjacency matrix $\mathcal{A}=[a_{ij}]_{N\times N}$ of directed graph $\mathcal{G}$ is defined as $a_{ii}=0$, if $(v_j,v_i)\in\mathcal{E}$, then $a_{ij}>0$ and $a_{ij}=0$ otherwise. The neighbor set of node v_i is denoted by $\mathcal{N}_i=\{j|(v_i,v_j)\in\mathcal{E}\}$. The Laplacian matrix is signed by $L=[L_{ij}]_{N\times N}$ with $L_{ii}=\sum_{j\neq i}a_{ij}$ and $L_{ij}=-a_{ij}$ for $i\neq j$. For a directed graph $\mathcal{G}$, the Laplacian matrix L has at least one 0 eigenvalue and all other nonzero eigenvalues have positive real parts. Moreover, matrix L has a simple eigenvalue 0 if and only if the directed graph $\mathcal{G}$ has a directed spanning tree (See Ref. [1]). For convenience, let μ_i, $i=1,2,\ldots,N-1$ be all nonzero eigenvalues of Laplacian matrix L. The interaction topology $\mathcal{G}$ involved in this paper satisfies the following assumption.

Assumption 1 The interaction topology $\mathcal{G}$ contains a directed spanning tree.

The multi-agent system is said to achieve consensus if $\lim_{t\to\infty}(r_i(t)-r_j(t))=0$ and $\lim_{t\to\infty}(v_i(t)-v_j(t))=0$ hold for all $i,j=1,2,\ldots,N$. We say that the protocol $u_i(t)$ can solve the consensus problem, if the closed-loop system achieves consensus.

Due to the wide use of computer-based control, sampled data-based protocols are considered. Let $t_k=kT$, $k=0,1,2,\ldots$ be the sampling times, where $T>0$ is the sampling period. In many applications, the velocity information cannot be available because the velocity measurements are inaccurate or it is not equipped with constrains in space, cost, and weight. In our problem, it is assumed that only the position information $r_i(t_k)$ can be used for designing the agent's controller. The main objective of this paper is to design a distributed consensus protocol $u_i(t)$ only based on sampled position information to make the closed-loop feedback system achieve consensus.

Before establishing our main conclusions, some Lemmas are introduced, which will be used in the convergence analysis.

Lemma 1[1] The complex coefficient polynomial $s^2+as+b=0$, and all roots lie inside a unit circle if and only if $(1+a+b)\lambda^2+2(1-b)\lambda+b-a+1=0$ is stable.

Lemma 2[33] Consider a complex coefficient quadratic polynomial as follows:

$$f(s)=s^2+(\theta_1+\mathrm{i}\delta_1)s+\theta_0+\mathrm{i}\delta_0, \tag{2}$$

where $\theta_1, \theta_0, \delta_1$, and δ_0 are real constants. Then, $f(s)$ is stable if and only if $\theta_1 > 0$ and $\theta_1\delta_1\delta_0 + \theta_1^2\theta_0 - \delta_0^2 > 0$.

Lemma 3 Consider a sampled-data based system

$$\dot{x}(t) = A_1 x(t) + A_2 x(t_k), \ \ t \in [t_k, t_{k+1}). \tag{3}$$

Then, system (3) is asymptotically stable if and only if matrix

$$\mathrm{e}^{A_1 T} + \int_0^T \mathrm{e}^{A_1 t}\,\mathrm{d}t A_2$$

is Schur-stable.

Proof Integrating both sides of system (3) from t_k to t, one has

$$x(t) = \mathrm{e}^{A_1(t-t_k)}x(t_k) + \int_0^{t-t_k} \mathrm{e}^{A_1 s}\,\mathrm{d}s A_2 x(t_k), \ \ t \in [t_k, t_{k+1}), \tag{4}$$

from which we can get

$$x(t_{k+1}) = \left(\mathrm{e}^{A_1 T} + \int_0^T \mathrm{e}^{A_1 t}\,\mathrm{d}t A_2\right) x(t_k). \tag{5}$$

While system (3) is asymptotically stable, we know that $\lim\limits_{t\to\infty} x(t) = 0$. Thus, we have $\lim\limits_{k\to\infty} x(t_k) = 0$, which means discrete-time system (5) is asymptotically stable, that is,

$$\mathrm{e}^{A_1 T} + \int_0^T \mathrm{e}^{A_1 t}\,\mathrm{d}t A_2$$

is Schur-stable. On the other hand, while

$$\mathrm{e}^{A_1 T} + \int_0^T \mathrm{e}^{A_1 t}\,\mathrm{d}t A_2$$

is Schur stable, we have $\lim\limits_{k\to\infty} x(t_k) = 0$. It is easy to see

$$\| \mathrm{e}^{A_1(t-t_k)} + \int_0^{t-t_k} \mathrm{e}^{A_1 s}\,\mathrm{d}s A_2 \|, \ \ t \in [t_k, t_{k+1})$$

is bounded. Thus, we know that $\lim\limits_{t\to\infty} x(t) = 0$.

3. Main results

In this section, the sampled-data consensus problem of the second-order multi-agent system will be investigated under the fixed directed topology. Full-order and reduced-order observer-based consensus protocols are proposed to achieve consensus, respectively.

3.1. Sampled-data consensus protocol with full-order observer

Under the assumption that the interaction topology $\mathcal{G}$ is directed, consider a full-order observer of agent i with sampled data

$$\begin{aligned} \dot{\hat{r}}_i(t) &= \hat{v}_i(t) - h_1(\hat{r}_i(t_k) - r_i(t_k)), \\ \dot{\hat{v}}_i(t) &= u_i(t) - h_2(\hat{r}_i(t_k) - r_i(t_k)), \ \ t \in [t_k, t_{k+1}), \end{aligned} \tag{6}$$

where $\hat{r}_i$ and $\hat{v}_i$ are the estimation of the position r_i and velocity v_i, respectively. h_1, h_2 are the positive parameters, which will be determined later.

By adopting the zero-order holder strategy, a distributed consensus protocol is proposed as follows:

$$\begin{aligned} u_i(t) = {} & k_1 \sum_{j=1}^{N} a_{ij}(\hat{r}_j(t_k) - \hat{r}_i(t_k)) \\ & + k_2 \sum_{j=1}^{N} a_{ij}(\hat{v}_j(t_k) - \hat{v}_i(t_k)), \ \ t \in [t_k, t_{k+1}), \end{aligned} \tag{7}$$

where k_1, k_2 are designed to be positive constants.

Now, a consensus condition is established with respect to the coupling parameters, sampling period and spectrum of the Laplacian matrix.

Theorem 1 Suppose that Assumption 1 is satisfied. Then, the multi-agent system (1) can achieve consensus via control protocol (7) and the position and velocity estimation errors asymptotical converge to zero if and only if k_1, k_2, h_1, h_2, and T satisfy the following conditions.

$$\frac{k_2}{Tk_1} > \frac{1}{2}, \ \frac{h_1}{Th_2} > \frac{1}{2}, \ T < \frac{2}{h_1}, \tag{8}$$

$$\left(1 - \frac{2k_2}{Tk_1}\right)^2 \left(\frac{2\,\mathrm{Re}\,(\mu_i)}{T\|\mu_i\|^2} - k_2\right) - \frac{8\,\mathrm{Im}\,(\mu_i)^2}{T^3 k_1 \|\mu_i\|^4} > 0. \tag{9}$$

Proof Let $e_{ri}(t) = \hat{r}_j(t) - r_i(t)$ and $e_{vi}(t) = \hat{v}_j(t) - v_i(t)$ be the position and velocity estimation errors, respectively. Then, it follows from Eqs. (1) and (6) that

$$\begin{aligned} \dot{e}_{ri}(t) &= e_{vi}(t) - h_1 e_{ri}(t_k), \\ \dot{e}_{vi}(t) &= -h_2 e_{ri}(t_k), \ \ t \in [t_k, t_{k+1}). \end{aligned} \tag{10}$$

According to Lemma 3, it is not difficult to see that the position and velocity estimation errors asymptotically converge to zero if and only if the following matrix is Schur-stable

$$\begin{aligned} F_1 &\triangleq \exp\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} T\right\} + \int_0^T \exp\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t\right\} \mathrm{d}t \begin{bmatrix} -h_1 & 0 \\ -h_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 - h_1 T - \dfrac{1}{2}h_2 T^2 & T \\ -h_2 T & 1 \end{bmatrix}. \end{aligned}$$

Let $\varphi_i = [r_i^{\mathrm{T}}, v_i^{\mathrm{T}}, e_{ri}^{\mathrm{T}}, e_{vi}^{\mathrm{T}}]^{\mathrm{T}}$. By combining Eqs. (1), (7), and (10), the dynamics of the closed-loop system can be expressed as

$$\begin{aligned} \dot{\varphi}_i(t) = {} & A \otimes I_n \varphi_i(t) + B \otimes I_n \varphi_i(t_k) \\ & - \sum_{j=1}^{N} a_{ij} C \otimes I_n \varphi_i(t_k), \ \ t \in [t_k, t_{k+1}) \end{aligned} \tag{11}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -h_1 & 0 \\ 0 & 0 & -h_2 & 0 \end{bmatrix},$$

$$C=\begin{bmatrix} 0 & 0 & 0 & 0 \\ k_1 & k_2 & k_1 & k_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

By defining $\varphi=[\varphi_1^{\mathrm{T}},\ldots,\varphi_N^{\mathrm{T}}]^{\mathrm{T}}$, the overall system can be rewritten as

$$\begin{aligned}\dot{\varphi}(t)=&(I_N\otimes A\otimes I_n)\varphi(t)+(I_N\otimes B\otimes I_n)\varphi(t_k)\\&-(L\otimes C\otimes I_n)\varphi(t_k).\end{aligned} \tag{13}$$

While all systems (10), $i=1,2,\ldots,N$ are stable, it is not difficult to see that the multi-agent system (1) achieves consensus via control protocol (7) if and only if system (13) reaches consensus.

Noticing that the digraph $\mathscr{G}$ contains a directed spanning tree, thus 0 is a simple eigenvalue of matrix L, and $L\mathbf{1}_N=\mathbf{0}_N$. Moreover, there exists a nonnegative vector $r\in R^N$ such that $r^{\mathrm{T}}L=0_{1\times N}$ and $w_l^{\mathrm{T}}\mathbf{1}_N=1$. Define J to be the Jordan form associated with the Laplacian matrix L, there exist nonsingular matrices $T\in R^{N\times N}, Y\in R^{N\times(N-1)}$ such that

$$T=[\mathbf{1}_N\ \ Y],\quad T^{-1}LT=J=\begin{bmatrix}0&0\\0&\bar{J}\end{bmatrix}, \tag{14}$$

where $\bar{J}=\mathrm{diag}\,(J_1,J_2,\ldots,J_s)$, and J_l is the Jordan block corresponding to μ_l, in which μ_l are the nonzero eigenvalues of L, with multiplicity $N_l, l=1,2,\ldots,s$ and $\sum_{l=1}^{s}N_l=N-1$. It is easy to see that T^{-1} has form

$$T^{-1}=\begin{bmatrix}r^{\mathrm{T}}\\S\end{bmatrix}.$$

Due to $T^{-1}T=I$, we have $r^{\mathrm{T}}\mathbf{1}_N=1$, $r^{\mathrm{T}}Y=0$, $S\mathbf{1}=\mathbf{0}$, and $SY=I_{N-1}$. Then, we can verify directly that

$$\begin{aligned}T^{-1}(I_N-\mathbf{1}r^{\mathrm{T}})T&=T^{-1}T-T^{-1}\mathbf{1}r^{\mathrm{T}}T\\&=\begin{bmatrix}0&0\\0&I_{N-1}\end{bmatrix}.\end{aligned}$$

Let $\xi(t)=(T^{-1}\otimes I_{4n})\varphi(t)$. Then, equation (13) can be represented as the following system

$$\begin{aligned}\dot{\xi}(t)=&(I_N\otimes A\otimes I_n)\xi(t)+(I_N\otimes B\otimes I_n)\xi(t_k)\\&-(J\otimes C\otimes I_n)\xi(t_k),\end{aligned} \tag{15}$$

which can be divided into the following two subsystems: one is

$$\dot{\xi}^0(t)=A\otimes I_n\xi^0(t)+B\otimes I_n\xi^0(t_k), \tag{16}$$

and the other one is

$$\begin{aligned}\dot{\xi}^1(t)=&(I_{N-1}\otimes A\otimes I_n)\xi^1(t)+(I_{N-1}\otimes B\otimes I_n)\xi^1(t_k)\\&-(\bar{J}\otimes C\otimes I_n)\xi^1(t_k),\end{aligned} \tag{17}$$

where $\xi=[\xi^{0\mathrm{T}},\xi^{1\mathrm{T}}]^{\mathrm{T}}$.

Denote $e_{\varphi i}$ as

$$e_{\varphi i}=\varphi_i-\sum_{j=1}^{N}r_j\varphi_j,\quad i=1,\ldots,N. \tag{18}$$

Obviously, for all $e_{\varphi i}=0$ $(i=1,2,\ldots,N)$ if and only if $\varphi_i=\varphi_j$ $(i,j=1,2,\ldots,N)$, that is, system (13) reaches consensus. Let

$$e_\varphi=[e_{\varphi 1}{}^{\mathrm{T}},e_{\varphi 2}{}^{\mathrm{T}},\ldots,e_{\varphi N}{}^{\mathrm{T}}]^{\mathrm{T}}.$$

Then, we have

$$e_\varphi=((I_N-\mathbf{1}r^{\mathrm{T}})\otimes I_n)\varphi,$$

by which we can get

$$\begin{aligned}(T^{-1}\otimes I)e_\varphi&=(T^{-1}\otimes I)((I_N-\mathbf{1}r^{\mathrm{T}})\otimes I)(T\otimes I)\xi\\&=\begin{bmatrix}0&0\\0&I_{N-1}\end{bmatrix}\otimes I\xi=\begin{bmatrix}0\\\xi^1\end{bmatrix}.\end{aligned}$$

From the above analyses, we know that multi-agent system (13) can achieve consensus if and only if $\lim\limits_{t\to\infty}\dot{\xi}^1(t)=0$, that is, system (17) is stable.

According to Lemma 3, it is not difficult to see that system (17) is stable if and only if matrix

$$\mathrm{e}^{I_{N-1}\otimes A\otimes I_nT}+\int_0^T\mathrm{e}^{I_{N-1}\otimes A\otimes I_nt}\,\mathrm{d}t(I_{N-1}\otimes B\otimes I_n-\bar{J}\otimes C\otimes I_n)$$

is Schur stable. Notice that

$$\mathrm{e}^{I_{N-1}\otimes A\otimes I_nT}+\int_0^T\mathrm{e}^{I_{N-1}\otimes A\otimes I_nt}\,\mathrm{d}t(I_{N-1}\otimes B\otimes I_n-\bar{J}\otimes C\otimes I_n)$$

is a block upper triangular matrix, whose diagonal block matrices are

$$\mathrm{e}^{AT}+\int_0^T\mathrm{e}^{At}\,\mathrm{d}t(B-\mu_iC),\ \ i=1,2,\ldots,N-1.$$

Therefore, we know that the control law (7) can solve the consensus problem if and only if

$$\mathrm{e}^{AT}+\int_0^T\mathrm{e}^{At}\,\mathrm{d}t(B-\mu_iC)$$

are Schur-stable, that is, all its eigenvalues are within the unit circle. In the sequel, we will prove that

$$\mathrm{e}^{AT}+\int_0^T\mathrm{e}^{At}\,\mathrm{d}t(B-\mu_iC)$$

are Schur-stable under conditions (8) and (9).

According to Eq. (12), by a simple calculation, one has

$$\mathrm{e}^{AT}=\begin{bmatrix}1&T&0&0\\0&1&0&0\\0&0&1&T\\0&0&0&1\end{bmatrix}.$$

Then, the following matrix form can be obtained

$$\mathrm{e}^{AT}+\int_0^T \mathrm{e}^{At}\,\mathrm{d}t(B-\mu_i C)=\begin{bmatrix} 1-\frac{1}{2}k_1T^2\mu_i & T-\frac{1}{2}k_2T^2\mu_i & -\frac{1}{2}k_1T^2\mu_i & -\frac{1}{2}k_2T^2\mu_i \\ -k_1T\mu_i & 1-k_2T\mu_i & -k_1T\mu_i & -k_2T\mu_i \\ 0 & 0 & 1-h_1T-\frac{1}{2}h_2T^2 & T \\ 0 & 0 & -h_2T & 1 \end{bmatrix}. \tag{19}$$

Denote

$$F_{2i}=\begin{bmatrix} 1-\frac{1}{2}k_1T^2\mu_i & T-\frac{1}{2}k_2T^2\mu_i \\ -k_1T\mu_i & 1-k_2T\mu_i \end{bmatrix}. \tag{20}$$

If all matrices F_1 and F_{2i} $(i=1,2,\ldots,N-1)$ are Schur-stable, then

$$\mathrm{e}^{AT}+\int_0^T \mathrm{e}^{At}\,\mathrm{d}t(B-\mu_i C)$$

is Schur-stable. Thus, the multi-agent system (1) can achieve consensus via control protocol (7) and the position and velocity estimation errors asymptotically converge to zero if and only if F_1 and F_{2i} are Schur-stable.

The characteristic equations $|sI-F_{2i}|=0$ and $|sI-F_1|=0$ of matrices F_{2i} and F_1 have the following form respectively

$$s^2+\left(\frac{1}{2}k_1T^2\mu_i+k_2T\mu_i-2\right)s+\frac{1}{2}k_1T^2\mu_i-k_2T\mu_i+1=0, \tag{21}$$

$$s^2+\left(h_1T+\frac{1}{2}h_2T^2-2\right)s-h_1T+\frac{1}{2}h_2T^2+1=0. \tag{22}$$

In fact, let $s=(\lambda+1)/(\lambda-1)$. Then, equation (21) can be transformed to

$$k_1T^2\mu_i\lambda^2+(2k_2T\mu_i-k_1T^2\mu_i)\lambda+4-2k_2T\mu_i=0. \tag{23}$$

From Lemma 1, the roots of Eq. (21) are within the unit circle if and only if the roots of Eq. (23) are in the open left-half plane. Thus, F_{2i} is Schur-stable if and only if all the roots in Eq. (23) have negative real parts.

According to Lemma 2, equation (23) is stable if and only if

$$\begin{aligned} &\frac{k_2}{Tk_1}>\frac{1}{2},\\ &\left(1-\frac{2k_2}{Tk_1}\right)^2\left(\frac{2\mathrm{Re}\,(\mu_i)}{T\|\mu_i\|^2}-k_2\right)-\frac{8\,\mathrm{Im}\,(\mu_i)^2}{T^3k_1\|\mu_i\|^4}>0. \end{aligned} \tag{24}$$

Using the same way to analyze Eq. (22), we can obtain that the roots of Eq. (22) are within the unit circle if and only if $h_1/Th_2>1/2$ and $T<2/h_1$. Therefore, the second-order multi-agent system (1) can achieve consensus via protocol (7) if and only if equations (8) and (9) are satisfied. The proof is completed. ■

Remark 1 According to Eq. (10), the position and velocity estimation errors asymptotically converge to zero if and only if F_1 is Schur-stable, from which we know that while $T<\min\{2h_1/h_2,\ 2/h_1\}$ the position and velocity estimation errors asymptotically converge to zero, while $h_1=0$ or $h_2=0$, F_1 is not Schur-stable. To get large T, small h_1, h_2 should be chosen. Unfortunately, small h_1 and h_2 will slow down the convergence rate.

While $T\to 0$, the protocol (7) degenerates to the continuous-time observer-based consensus protocol

$$\begin{aligned} \dot{\hat{r}}_i(t)&=\hat{v}_i(t)-h_1(\hat{r}_i(t)-r_i(t)),\\ \dot{\hat{v}}_i(t)&=u_i(t)-h_2(\hat{r}_i(t)-r_i(t)),\\ u_i(t)&=k_1\sum_{j=1}^N a_{ij}(\hat{r}_j(t)-\hat{r}_i(t))+k_2\sum_{j=1}^N a_{ij}(\hat{v}_j(t)-\hat{v}_i(t)). \end{aligned} \tag{25}$$

According to conditions (8) and (9), it is easy to establish the consensus condition for continuous-time observer-based consensus protocol (25) as follows.

Corollary 1 Suppose that Assumption 1 is satisfied. Then, the multi-agent system (1) can achieve consensus via control protocol (25) and the position and velocity estimation errors asymptotically converge to zero if and only if all positive constants satisfy

$$\frac{k_2^2}{k_1}>\max_{i=2,\ldots,N}\left\{\frac{\mathrm{Im}^2(\mu_i)}{\mathrm{Re}\,(\mu_i)\|\mu_i\|^2}\right\}. \tag{26}$$

Proof According to Theorem 1, as $T\to 0$, condition (8) must be satisfied. Condition (9) can be expressed as

$$\left(T-\frac{2k_2}{k_1}\right)^2\left(\frac{2\mathrm{Re}\,(\mu_i)}{\|\mu_i\|^2}-k_2T\right)-\frac{8\,\mathrm{Im}\,(\mu_i)^2}{k_1\|\mu_i\|^4}>0. \tag{27}$$

As $T\to 0$, condition (27) degenerates to

$$\left(\frac{2k_2}{k_1}\right)^2\frac{2\mathrm{Re}\,(\mu_i)}{\|\mu_i\|^2}-\frac{8\,\mathrm{Im}\,(\mu_i)^2}{k_1\|\mu_i\|^4}>0,$$

from which the consensus condition (26) is obtained. ■

Condition (26) is the same as the consensus condition for state consensus protocol

$$u_i(t)=k_1\sum_{j=1}^N a_{ij}(r_j(t)-r_i(t))+k_2\sum_{j=1}^N a_{ij}(v_j(t)-v_i(t)),$$

which was established in Ref. [10]. By adopting the full-order state observer (25), it need not add an additional condition to reach consensus. Naturally, it is interesting to probe how large T can be taken under the condition (26), which is addressed in the sequel. For convergence, denote

$$f_i(s)=\left(T-\frac{2k_2}{k_1}\right)^2\left(\frac{2\mathrm{Re}\,(\mu_i)}{\|\mu_i\|^2}-k_2T\right)-\frac{8\,\mathrm{Im}\,(\mu_i)^2}{k_1\|\mu_i\|^4}. \tag{28}$$

Let s_i^* be the minimum positive zero of $f_i(s)$. Then, we can obtain the following result.

Theorem 2 Suppose that Assumption 1 is satisfied. Then, there exists a positive constant T^* such that for any $T < T^*$, the consensus problem can be solved by protocol (7) and the position and velocity estimation errors asymptotically converge to zero if and only if condition (26) holds. Moreover,

$$T^* = \min\left\{\frac{2k_2}{k_1}, \frac{2h_1}{h_2}, \frac{2}{h_1}, s_i^* \ \ (i=1,2,\ldots,N-1)\right\}. \quad (29)$$

Proof The necessity is obvious. Now, we prove the sufficiency. While condition (26) holds, it is not difficult to see that equation (29) is well-defined. Obviously, condition (8) holds while $T < T^*$. On the other hand, noticing that $0 < T < T^* \le s_i^*$, we have $f_i(T) > 0$, which implies that condition (9) holds. According to Theorem 1, the consensus problem can be solved by protocol (7) with any $T < T^*$. ■

Considering the special case that the graph $\mathcal{G}$ is undirected and connected, the following corollary can be easily obtained via Theorems 1 and 2.

Corollary 2 Suppose that interaction topology $\mathcal{G}$ is undirected and connected. Then the second-order consensus problem can be solved by protocol (7), and the position and velocity estimation errors asymptotically converge to zero if and only if k_1, k_2, h_1, h_2, and T satisfy the following condition

$$T < \min\left\{\frac{2k_2}{k_1}, \frac{2h_1}{h_2}, \frac{2}{h_1}, \frac{2}{k_2\mu_i} \ \ (i=1,2\ldots,N-1)\right\}. \quad (30)$$

Proof While the graph $\mathcal{G}$ is undirected and connected, we know that the Laplacian matrix L is symmetric and positive-semi-definite. Furthermore, we know that μ_i, $i=1,2,\ldots,N-1$ are real and positive, that is, $\mathrm{Im}(\mu_i)=0$ and $\mathrm{Re}(\mu_i)>0$. Then, the Corollary 2 can be obtained directly from Theorem 2. ■

Remark 2 While condition (26) holds, we can obtain that $f_i(s)>0$ and $f_i(+\infty)<0$ $(i=1,2,\ldots,N-1)$, from which $f_i(s)$ $(i=1,2,\ldots,N-1)$ has at least a positive zero point. Moreover, we know that

$$0 < s^* < \min\left\{\frac{2k_2}{k_1}, \frac{2\,\mathrm{Re}(\mu_i)}{k_2\|\mu_i\|^2}\right\}.$$

Thus, T^* defined by Eq. (29) is positive. According to Theorem 2, while the sampling period T is sufficiently small, the consensus problem can be solved by protocol (7) if condition (26) holds. For a given sampling period T, we can design the gain parameters in protocol (7) under undirected and connected interaction topology. Let $\mu_{\max}$ be the maximum eigenvalue of L. The gain parameters can be chosen by $0<k_2<2/T\mu_{\max}$, $0<k_1<2k_2/T$, $0<h_1<2/T$, and $0<h_2<h_1/T$.

3.2. Sampled-data control protocol with reduced-order observer

In some applications, the agent can have access to its position information continuously, but might communicate with the neighbor agents periodically. In this part, by assumption that the continuous-time position information can be used by the agent itself but only sampled-data can be used by the neighbor agents, a reduced-order observer based consensus protocol with sampled-data is proposed.

Consider a reduced-order observer for agent i

$$\begin{aligned}&\dot{z}_i(t) = -gz_i(t) - g^2 r_i(t) + u_i(t),\\ &\hat{v}_i(t) = z_i(t) + gr_i(t), \ \ i=1,\ldots,N, t\in[t_k,t_{k+1}),\end{aligned} \quad (31)$$

where $z_i(t)$ is the protocol, $\hat{v}_i(t)$ is the restructured velocity of $v_i(t)$, $g>0$ is coupling gain to be determined.

Similarly, the consensus protocol with sampled-data is proposed as follows:

$$\begin{aligned}u_i(t) = k_1\sum_{j=1}^{N} a_{ij}(r_j(t_k) - r_i(t_k)) + k_2\sum_{j=1}^{N} a_{ij}(\hat{v}_j(t_k)\\ - \hat{v}_i(t_k)), \ \ t\in[t_k,t_{k+1}),\end{aligned} \quad (32)$$

where k_1, k_2 are designed positive constants.

Then, the consensus condition can be obtained by the following theorem.

Theorem 3 Suppose that Assumption 1 is satisfied. Then, the multi-agent system (1) can achieve consensus via control protocol (32) and the velocity estimation errors asymptotically converge to zero if and only if k_1, k_2, g, and T satisfy the following conditions

$$\frac{k_2}{Tk_1} > \frac{1}{2}, \ \ T < \frac{2}{g}, \quad (33)$$

$$\left(1-\frac{2k_2}{Tk_1}\right)^2\left(\frac{2\,\mathrm{Re}(\mu_i)}{T\|\mu_i\|^2} - k_2\right) - \frac{8\,\mathrm{Im}(\mu_i)^2}{T^3k_1\|\mu_i\|^4} > 0. \quad (34)$$

Proof Let $e_i(t) = \hat{v}_i(t) - v_i(t)$. For $t\in[t_k,t_{k+1})$, it follows from Eq. (31) that the dynamical equations for $e_i(t)$ can be described as

$$\begin{aligned}\dot{e}_i(t) &= \dot{\hat{v}}_i(t) - \dot{v}_i(t)\\ &= -gz_i(t) - g^2 r_i(t_k) + u_i(t) + gv_i(t) - u_i(t)\\ &= -g(z_i(t) + gr_i(t_k) - v_i(t))\\ &= -ge_i(t),\end{aligned} \quad (35)$$

which means that the velocity estimation errors asymptotically converge to zero.

By the definition of $e_i(t)$ and the protocol (32), the closed-loop dynamics of system (1) can be expressed as

$$\begin{aligned}&\dot{r}_i(t) = v_i(t),\\ &\dot{v}_i(t) = k_1\sum_{j=1}^{N} a_{ij}(r_j(t_k) - r_i(t_k)) + k_2\sum_{j=1}^{N} a_{ij}(e_j(t_k) - e_i(t_k))\end{aligned}$$

$$+k_2\sum_{j=1}^{N}a_{ij}(v_j(t_k)-v_i(t_k)),\ \ t\in[t_k,t_{k+1}). \tag{36}$$

Then, let $\varepsilon=[r^{\mathrm{T}},v^{\mathrm{T}},e^{\mathrm{T}}]^{\mathrm{T}}$. From Eqs. (35) and (36), the following closed-loop dynamic can be rewritten as

$$\dot{\varepsilon}(t)=(I_N\otimes\bar{A}\otimes I_n)\varepsilon(t)-(L\otimes\bar{B}\otimes I_n)\varepsilon(t_k), \tag{37}$$

where

$$\bar{A}=\begin{bmatrix}0&1&0\\0&0&0\\0&0&-g\end{bmatrix},\quad \bar{B}=\begin{bmatrix}0&0&0\\k_1&k_2&k_2\\0&0&0\end{bmatrix}.$$

Following the same line of the previous subsection, it is not difficult to see that the multi-agent system (37) achieves consensus if and only if the following system is stable

$$\dot{\eta}^1(t)=(I_{N-1}\otimes\bar{A}\otimes I_n)\eta^1(t)-(\bar{J}\otimes\bar{B}\otimes I_n)\eta^1(t_k),\ \ t\in[t_k,t_{k+1}), \tag{38}$$

where $\bar{J}$ is the same as that in Eq. (17).

According to Lemma 3, system (38) is asymptotically stable if and only if matrices

$$\mathrm{e}^{\bar{A}T}-\int_0^T\mathrm{e}^{\bar{A}t}\,\mathrm{d}t\mu_i\bar{B},\ \ i=1,2,\ldots,N-1$$

are Schur-stable. Since

$$\mathrm{e}^{\bar{A}T}-\int_0^T\mathrm{e}^{\bar{A}t}\,\mathrm{d}t\mu_i\bar{B}=\begin{bmatrix}1-\frac{1}{2}\mu_ik_1T^2&T-\frac{1}{2}\mu_ik_2T^2&-\frac{1}{2}\mu_ik_2T^2\\-\mu_ik_1T&1-\mu_ik_2T&-\mu_ik_2T\\0&0&1-gT\end{bmatrix}.$$

Similarly, we can obtain that the above matrices are Schur-stable if and only if

$$F_{2i}=\begin{bmatrix}1-\frac{1}{2}\mu_ik_1T^2&T-\frac{1}{2}\mu_ik_2T^2\\-\mu_ik_1T&1-\mu_ik_2T\end{bmatrix}$$

and $1-gT$ are Schur-stable. From the proof of Theorem 1, we know that F_{2i} is Schur-stable if and only if the following condition hold

$$\frac{k_2}{Tk_1}>\frac{1}{2},$$
$$\left(1-\frac{2k_2}{Tk_1}\right)^2\left(\frac{2\mathrm{Re}\,(\mu_i)}{T\|\mu_i\|^2}-k_2\right)-\frac{8\,\mathrm{Im}\,(\mu_i)^2}{T^3k_1\|\mu_i\|^4}>0.$$

Moreover, due to $|1-gT|<1$, $T<2/g$ is easily obtained. Thus, second-order multi-agent system (1) can achieve consensus via protocol (32) if and only if equations (33) and (34) are satisfied. The proof is completed. ■

Similarly, the following result can be obtained as Theorem 2, and the proof is omitted.

Corollary 3 Suppose that Assumption 1 is satisfied. Then, there exists a positive constant $\bar{T}^*$ such that for any $T<\bar{T}^*$, the consensus problem can be solved by protocol (32) and the position and velocity estimation errors asymptotically converge to zero if and only if condition (26) holds. Moreover,

$$\bar{T}^*=\min\left\{\frac{2k_2}{k_1},\frac{2}{g},s_i^*\ \ (i=1,2,\ldots,N-1)\right\}, \tag{39}$$

where s_i^* are the same as in Theorem 2.

Remark 3 Similarly, while $T\to 0$, the protocol (32) degenerates to the continuous-time observer-based consensus protocol

$$\begin{aligned}\dot{z}_i(t)&=-gz_i(t)-g^2r_i(t)+u_i(t),\\ \hat{v}_i(t)&=z_i(t)+gr_i(t),\\ u_i(t)&=k_1\sum_{j=1}^{N}a_{ij}(\hat{r}_j(t)-\hat{r}_i(t))+k_2\sum_{j=1}^{N}a_{ij}(\hat{v}_j(t)-\hat{v}_i(t)).\end{aligned} \tag{40}$$

According to conditions (8) and (9), it is easy to obtain the consensus condition for continuous-time observer-based consensus protocol (40) as follows:

$$\frac{k_2^2}{k_1}>\max_{i=2,\ldots,N}\left\{\frac{\mathrm{Im}^2(\mu_i)}{\mathrm{Re}\,(\mu_i)\|\mu_i\|^2}\right\}. \tag{41}$$

3.3. Consensus protocol with time delay

In some applications, there exists the input time delay which will damage system performance. While time delay is considered in protocols (6) and (7), the consensus protocol has form as follows:

$$\begin{aligned}\dot{\hat{r}}_i(t)&=\hat{v}_i(t)-h_1(\hat{r}_i(t_k-\tau)-r_i(t_k-\tau)),\\ \dot{\hat{v}}_i(t)&=u_i(t)-h_2(\hat{r}_i(t_k-\tau)-r_i(t_k-\tau)),\\ u_i(t)&=k_1\sum_{j=1}^{N}a_{ij}(\hat{r}_j(t_k-\tau)-\hat{r}_i(t_k-\tau))\\&\quad+k_2\sum_{j=1}^{N}a_{ij}(\hat{v}_j(t_k-\tau)-\hat{v}_i(t_k-\tau)),\\&\quad t\in[t_k,t_{k+1}).\end{aligned} \tag{42}$$

In the sequel, we solve the consensus problem by protocol (42). The notations are the same as the proof of Theorem 1. By following a similar analysis process, the multi-agent system (13) can achieve consensus if the following system is stable

$$\dot{\xi}^1(t)=(I_{N-1}\otimes A\otimes I_n)\xi^1(t)+(I_{N-1}\otimes B\otimes I_n)\xi^1(t_k-\tau)-(\bar{J}\otimes C\otimes I_n)\xi^1(t_k-\tau), \tag{43}$$

from which we can get

$$\xi^1(t)=\mathrm{e}^{(I_{N-1}\otimes A\otimes I_n)(t-t_k)}\xi^1(t_k)+\int_0^{t-t_k}\mathrm{e}^{(I_{N-1}\otimes A\otimes I_n)s}\mathrm{d}s(I_{N-1}\otimes B-\bar{J}\otimes C)\otimes I_n\xi^1(t_k-\tau),\quad t\in[t_k,t_{k+1}),\tag{44}$$

where A, B, C are defined in Eq. (12). Suppose that $\tau<T$. From Eq. (44), one has

$$\begin{aligned}\xi^1(t_{k+1})&=\mathrm{e}^{(I_{N-1}\otimes A\otimes I_n)T}\xi^1(t_k)+\int_0^{T}\mathrm{e}^{(I_{N-1}\otimes A\otimes I_n)s}\mathrm{d}s(I_{N-1}\otimes B-\bar{J}\otimes C)\otimes I_n\xi^1(t_k-\tau),\\ \xi^1(t_{k+1}-\tau)&=\mathrm{e}^{(I_{N-1}\otimes A\otimes I_n)(T-\tau)}\xi^1(t_k)+\int_0^{T-\tau}\mathrm{e}^{(I_{N-1}\otimes A\otimes I_n)s}\mathrm{d}s(I_{N-1}\otimes B-\bar{J}\otimes C)\otimes I_n\xi^1(t_k-\tau).\end{aligned}\tag{45}$$

It is not too difficult to see that system (45) is Schur-stable if all matrices

$$G_i\triangleq\begin{bmatrix}\mathrm{e}^{AT} & \displaystyle\int_0^T\mathrm{e}^{As}\mathrm{d}s(B-\mu_iC)\\ \mathrm{e}^{A(T-\tau)} & \displaystyle\int_0^{T-\tau}\mathrm{e}^{As}\mathrm{d}s(B-\mu_iC)\end{bmatrix}$$

$(i=1,2,\ldots,N-1)$ are Schur-stable. Denote

$$\bar{F}_1\triangleq\begin{bmatrix}1 & T & -h_1T & -\dfrac{1}{2}h_2T^2\\ 0 & 1 & -h_2T & 0\\ 1 & T-\tau & -h_1(T-\tau) & -\dfrac{1}{2}h_2(T-\tau)^2\\ 0 & 1 & -h_2(T-\tau) & 0\end{bmatrix},$$

$$\bar{F}_{2i}\triangleq\begin{bmatrix}1 & T & -\dfrac{1}{2}k_1T^2\mu_i & -\dfrac{1}{2}k_2T^2\mu_i\\ 0 & 1 & -k_1T\mu_i & -k_2T\mu_i\\ 1 & T-\tau & -\dfrac{1}{2}k_1(T-\tau)^2\mu_i & -\dfrac{1}{2}k_2(T-\tau)^2\mu_i\\ 0 & 1 & -k_1(T-\tau)\mu_i & -k_2(T-\tau)\mu_i\end{bmatrix}.$$

By simple calculation, we know that G_i is Schur-stable if and only if all matrices $\bar{F}_1$ and $\bar{F}_{2i}$ $(i=1,2,\ldots,N-1)$ are Schur-stable. Similar to Lemma 3, we know that if system (45) is Schur-stable, then $\lim\limits_{t\to\infty}\xi^1(t)=0$. Thus, we can obtain the following result directly.

Theorem 4 Suppose that Assumption 1 is satisfied and $\tau<T$. Then, the multi-agent system (1) can achieve consensus via control protocol (42) and the position and velocity estimation errors asymptotically converge to zero if and only if all matrices $\bar{F}_1$ and $\bar{F}_{2i}$ $(i=1,2,\ldots,N-1)$ are Schur-stable.

Remark 4 Similar to Theorem 1, we can use the characteristic polynomial approach to discuss the Schur stability of $\bar{F}_1$ and $\bar{F}_{2i}$. Since the characteristic polynomial of $\bar{F}_{2i}$ has degree 4 and complex coefficients, the established condition will be very complicated. Thus, we do not intend to address this issue. For a reduced-observer case, we can establish a similar result as Theorem 4.

4. Numerical example

In this section, a simple simulation example is proposed to illustrate the theoretical result.

Consider a second-order multi-agent system composing of $N=6$ agents. For simplicity, let $n=1$. Its topology structure is described by a directed graph $\mathcal{G}$ as shown in Fig. 1, and its Laplacian matrix L is obtained as

$$L=\begin{bmatrix}3 & 0 & -2 & -1 & 0 & 0\\ -2.5 & 4.5 & 0 & -2 & 0 & 0\\ 0 & -2 & 3.5 & 0 & -1.5 & 0\\ 0 & 0 & -1 & 4 & -2 & -1\\ 0 & -2.5 & 0 & 0 & 2.5 & 0\\ -1 & -2 & 0 & 0 & -2 & 5\end{bmatrix}.$$

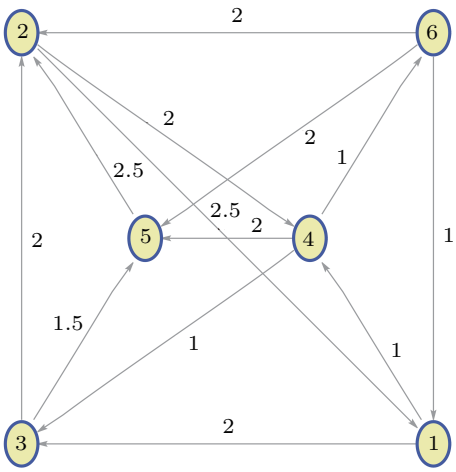


Fig. 1. (color online) Directed graph $\mathcal{G}$ for six agents.

Apparently, interaction topology $\mathcal{G}$ has a directed spanning tree. By simple calculation, the eigenvalues of L are $\mu_1=0$, $\mu_2=4.7178$, $\mu_{3,4}=4.7190\pm2.5214\mathrm{i}$, $\mu_{5,6}=4.1721\pm0.2678\mathrm{i}$.

First, we consider a full-order observer case. The initial positions and velocities are taken as

$$r_0^{\mathrm{T}}=[26,-36,41,67,-16,-37]^{\mathrm{T}}$$

and

$$v_0^{\mathrm{T}}=[35,-13,20,45,-33,27]^{\mathrm{T}}.$$

The initial positions and velocities for the observers are taken as

$$\begin{aligned}\hat{r}_0^{\mathrm{T}}&=[-16,50,-47,-26,60,-39]^{\mathrm{T}},\\ \hat{v}_0^{\mathrm{T}}&=[-8,19,-21,-18,29,21]^{\mathrm{T}}.\end{aligned}$$

Take $k_1 = 1.1$, $k_2 = 0.8$, $h_1 = 0.6$, $h_2 = 0.9$. According to Theorem 2, we can get $T^* = 0.3539s$ by a simple computation. Take $T = 0.2s$, which satisfies the conditions (8) and (9). The position, velocity, estimated position error, and estimated velocity error are depicted in Fig. 2, which shows that the second-order multi-agent system can reach consensus via the protocol (7) and the estimated errors converge to zero. Apparently, while $T = 1.4s$, which cannot satisfy the conditions in Theorem 1. The related error trajectories are depicted in Fig. 3, which shows that the multi-agent system cannot reach consensus. Thus, the simulation result is compatible with our obtained theoretical result.

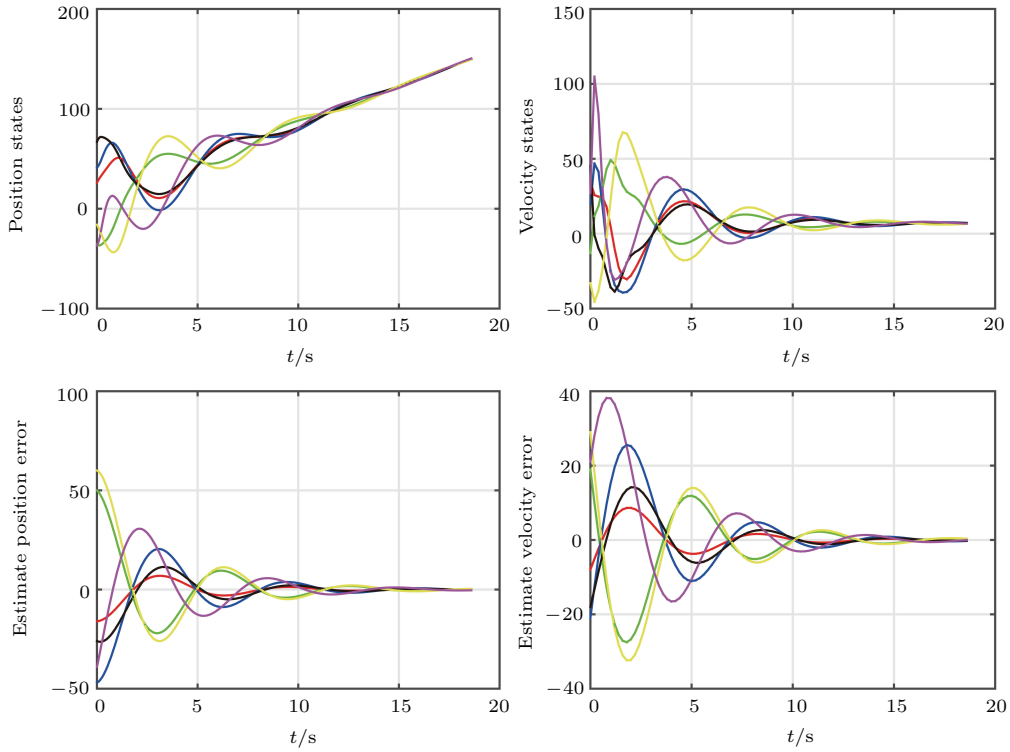


Fig. 2. (color online) Convergence results with $k_1 = 1.1$, $k_2 = 0.8$, $h_1 = 0.6$, $h_2 = 0.9$, and $T = 0.2$ s.

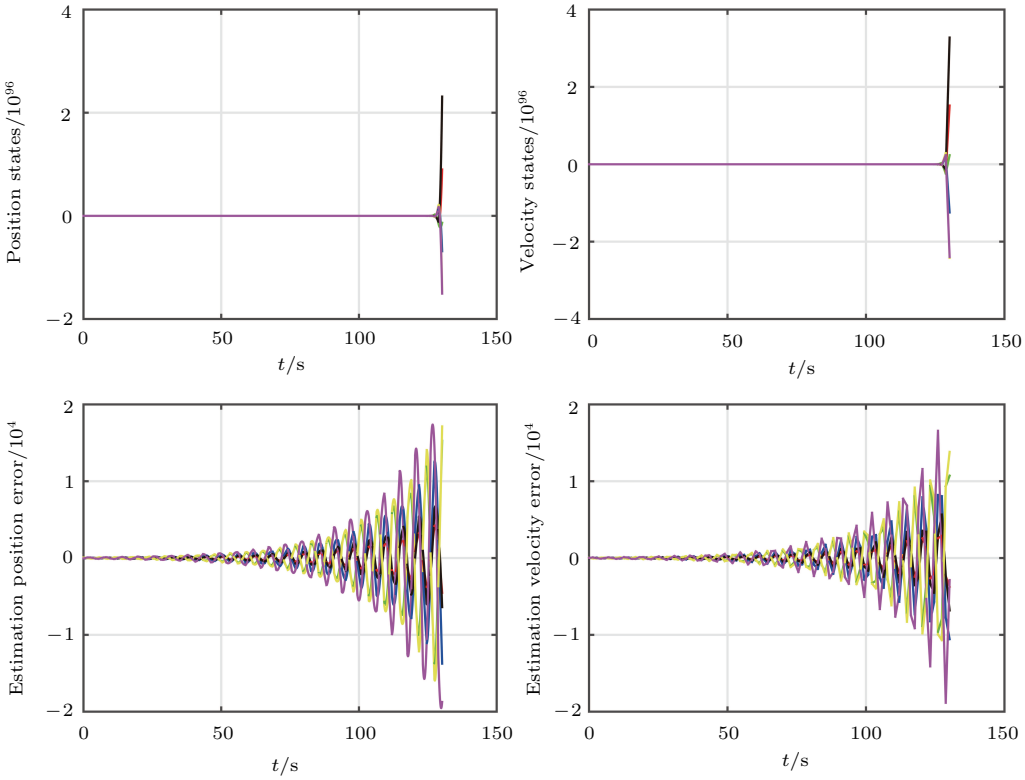


Fig. 3. (color online) Convergence results with $k_1 = 1.1$, $k_2 = 0.8$, $h_1 = 0.6$, $h_2 = 0.9$, and $T = 1.4$ s.

For the reduced-order observer case, we take

$$r_0^{\mathrm{T}} = [20, -25, 27, 17, 40, -33]^{\mathrm{T}},$$
$$v_0^{\mathrm{T}} = [5, 16, 8, 6, -9, 19]^{\mathrm{T}},$$
$$\hat{v}_0^{\mathrm{T}} = [-6, 11, 7, -16, -7, 15]^{\mathrm{T}}.$$

We choose the scalar parameters and the sampling period as $k_1 = 1.4$, $k_2 = 0.9$, $g = 4$. By Corollary 3, we can get $\bar{T}^* = 0.3141s$. Take $T = 0.26s < \bar{T}^*$. Obviously, conditions (33) and (34) hold in this case. The simulation results are depicted in Fig. 4, from which we know that the multi-agent system can achieve consensus and the estimated error of velocity converges to zero.

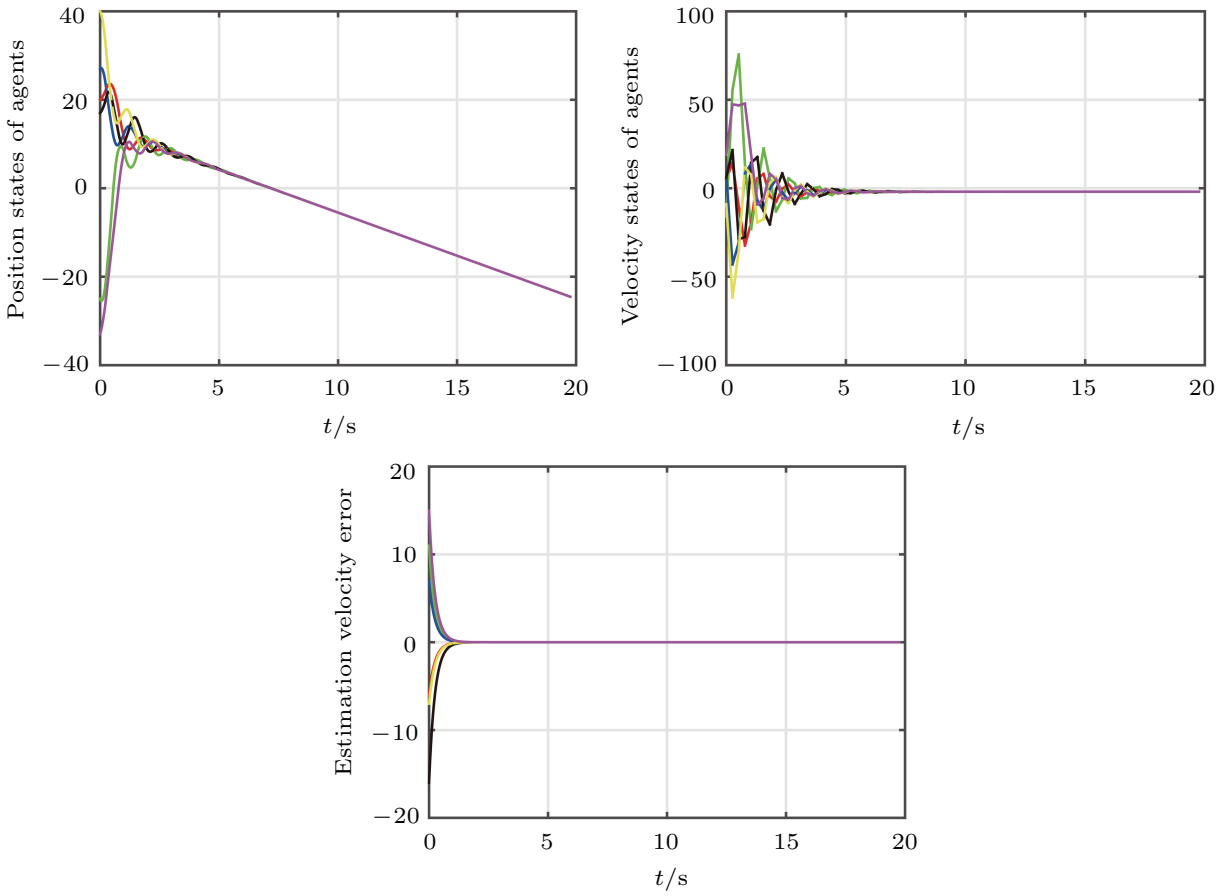


Fig. 4. (color online) Convergence results with $k_1 = 1.4$, $k_2 = 0.9$, $g = 4$, and $T = 0.26s$.

Here, we only consider the fixed interaction topology case. Certainly, while the interaction topology arbitrarily switches within a finite set of directed topologies, the multi-agent system may not achieve consensus by our proposed protocols. But, while the interaction topology switches with a large enough dwell time, the simulation results also show that the multi-agent system can achieve consensus via our proposed protocols. Our established consensus condition may be able to be weakened by using the concept of an average dwell time. The average dwell time approach to discuss the consensus problem can be found in Refs. [22] and [23].

5. Conclusions

In this paper, the second-order consensus problem with position sampled-data has been addressed. By assumption that the interaction topology is fixed and directed, the continuous-time full-order and reduced-order observers are adopted to estimate the agents' unavailable states, by which two kinds of sampled-data based consensus protocols have been constructed. By using matrix theory and sampled control analysis method, some sufficient and necessary consensus conditions about coupling parameters and sampling period have been obtained. As a special case, we can obtain the necessary and sufficient conditions for the continuous-time consensus protocols directly. Our future works will concentrate on the sampled data-based consensus issue with the time-varying sampling interval, general linear dynamics, and nonlinear dynamics.

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