An application of a combined gradient system to stabilize a mechanical system*

Xiang-Wei Chen(陈向炜)[1,†], Ye Zhang(张晔)[2], and Feng-Xiang Mei(梅凤翔)[3]

[1] Department of Physics and Information Engineering, Shangqiu Normal University, Shangqiu 476000, China
[2] School of Mathematics and Physics, Suzhou University of Science and Technology, Suzhou 215009, China
[3] School of Aerospace, Beijing Institute of Technology, Beijing 100081, China

(Received 13 February 2016; revised manuscript received 23 May 2016; published online 25 August 2016)

A gradient system and a skew-gradient system can be merged into a combined gradient system. The differential equations of the combined gradient system are established and its property is studied. If a mechanical system can be represented as a combined gradient system, the stability of the mechanical system can be studied by using the property of the combined gradient system. Some examples are given to illustrate the applications of the results.

Keywords: gradient system, combined gradient system, mechanical system, stability

PACS: 02.30.Jr, 45.20.Jj, 11.30.–j **DOI:** 10.1088/1674-1056/25/10/100201

1. Introduction

Two important systems have been studied in chapter 9 "A wide range of nonlinear skills" of Ref. [1], one is the gradient system, and the other is the Hamilton system. The gradient system is an important problem of differential equations and dynamical systems, and is especially suitable for being studied by using the Lyapunov function. The skew-gradient system is given in Ref. [2], which is of great significance for studying the integral and stability of a mechanical system. In Refs. [3]–[16] the general gradient system was applied to the mechanical system, and in Refs. [17] and [18] the skew-gradient system was applied to the mechanical system. The gradient properties of the kinetic equation, not only play an important role in revealing the internal structure of a dynamical system, but also help to explore the dynamical behavior of the system, and are widely used in thermodynamics, optics, classical and quantum field theory and many other fields.[2,19] The semi-inverse method proposed by He[20] has been proved to be a most convenient and effective way to establish generalized variational principles, and has been applied successfully to many physical and mechanical fields.[21] In the present paper, the combination of the general gradient system and the skew-gradient system is called the combined gradient system. This kind of combined gradient system can be applied to the problem of the stability of the mechanical system which cannot be solved by the general gradient system nor by the skew-gradient system. Firstly, differential equations, properties and applications of the general gradient system and the skew-gradient system are introduced. Secondly, differential equations and some important properties of the combined gradient system are studied. Finally, some examples are given to illustrate the application to stabilize the mechanical system.

2. General gradient system

2.1. Differential equation

The differential equations of the general gradient system are written as[1,2]

$$\dot{x}_i = -\frac{\partial V(\boldsymbol{x})}{\partial x_i}, \quad (i = 1, 2, \ldots, m), \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ and $V = V(\boldsymbol{x})$ are potential functions.

2.2. Properties

The general gradient system (1) has the following important properties[1]

(i) function V is a Lyapunov function of system (1) and $\dot{V} = 0$ if and only if $\boldsymbol{x}$ is an equilibrium point;

(ii) supposing that Z is an α limit point or ω limit point of the solution of a gradient system, then Z is an equilibrium point;

(iii) for a gradient system, the linearized system at an arbitrary equilibrium point only has real characteristic roots.

2.3. Applications

By the properties of the gradient system (1), we can study the stability of the solution of the system. The second order Lagrange system and the first order Lagrange system can become a gradient system under certain conditions, and the potential function V can become a Lyapunov function, then the stability of the system can be studied by using Lyapunov theorem, and the partial variable stability can be studied by using Rumyatsev theorem too.[10] At the same time, the stability

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11372169 and 11272050).
†Corresponding author. E-mail: hnchenxw@163.com
© 2016 Chinese Physical Society and IOP Publishing Ltd

can also be studied by using the third property of the gradient system. According to the third property and Lyapunov first approximation theory, the solution is asymptotically stable or unstable. In addition, we also know that this gradient system cannot be used to investigate the asymptotic stability of a damped oscillator which has plural characteristic roots.

3. Skew-gradient system

3.1. Differential equation

The differential equations of the skew-gradient system are[17]

$$\dot{x}_i = b_{ij}(\boldsymbol{x})\frac{\partial V(\boldsymbol{x})}{\partial x_j}, \quad (i,j=1,2,\ldots,m), \tag{2}$$

where $\boldsymbol{x}=(x_1,x_2,\ldots,x_m)$, the same subscript represents summation, $V=V(\boldsymbol{x})$ is an energy function[2] and matrix $(b_{ij}(\boldsymbol{x}))$ is antisymmetrical, i.e.

$$b_{ij} = -b_{ji}. \tag{3}$$

3.2. Properties

The skew-gradient system has important properties as follows.

i) The energy function $V=V(\boldsymbol{x})$ is an integral of the skew-gradient system (2). Indeed, derivative of V with respect to time t according to Eq. (2) is obtained as

$$\dot{V} = \frac{\partial V}{\partial x_i} b_{ij} \frac{\partial V}{\partial x_j} = 0;$$

ii) If V can be a Lyapunov function, then the solutions $x_i = x_{i0}$ $(i=1,2,\ldots,m)$ of the skew-gradient system (2) are stable.

3.3. Applications

The skew-gradient system can be used to investigate the integral and the stability of the system, but cannot be used to study the asymptotic stability of the solution. The steady Hamilton system, the autonomous Birkhoff system and the generalized Hamilton system all are naturally skew-gradient systems, therefore, the Hamilton function and the Birkhoff function are the integrals of the system. If the Hamilton function and the Birkhoff function can become the Lyapunov functions, the solution is stable. The Lorenz equation is one of the earliest examples of chaos, and its Robbins model is a skew-gradient system.[17]

4. Combined gradient system

4.1. Differential equation

The combined gradient system is obtained by combining the gradient system (1) and the skew-gradient system (2). The differential equations of the system are

$$\dot{x}_i = -\frac{\partial V(\boldsymbol{x})}{\partial x_i} + b_{ij}(\boldsymbol{x})\frac{\partial V(\boldsymbol{x})}{\partial x_j}, \quad (i,j=1,2,\ldots,m). \tag{4}$$

4.2. Properties

(I) The matrix of a combined gradient system is negative definite. In fact, we have

$$(x_1,x_2,\ldots,x_m)\begin{pmatrix} -1 & b_{12} & \ldots & b_{1m} \\ -b_n & -1 & \ldots & b_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ -b_{1m} & -b_{2m} & \ldots & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_m \end{pmatrix} = -\left(x_1^2+x_2^2+\ldots+x_m^2\right).$$

(II) If function V is positive definite, then the solution of the system is asymptotic stable.

4.3. Application

If the differential equations of the mechanical system can be expressed as Eq. (4) of the combined gradient system, we can use the properties of the combined gradient system to investigate the stability of the solution of the mechanical system.

Example 1 We study a damped oscillator system with a single degree of freedom, in which the Lagrange function and the generalized force are expressed respectively as

$$L = \frac{1}{2}\dot{q}^2 - q^2 - \frac{1}{2}q^4,$$
$$Q = -2\dot{q} - 3q^2\dot{q}, \tag{5}$$

where the quantities are dimensionless. Now, we merge them into a combined gradient system, and study the stability of the zero solution.

The differential equation of system (5) is

$$\ddot{q} = -2q - 2q^3 - 2\dot{q} - 3q^2\dot{q}.$$

This is a damped oscillator system.

Let

$$x_1 = -q - q^3 - \dot{q}, \quad x_2 = q.$$

Taking the derivative of x_1 and x_2, we have

$$\dot{x}_1 = -x_1 + x_2 + x_2^3, \quad \dot{x}_2 = -x_1 - x_2 - x_2^3.$$

It can be written as the form of a combined gradient system, i.e.,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\begin{pmatrix} \dfrac{\partial V}{\partial x_1} \\ \dfrac{\partial V}{\partial x_2} \end{pmatrix},$$

where $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_2^4$ is positive-definite in the neighborhood of $x_1 = x_2 = 0$. Therefore, the solution $x_1 = x_2 = 0$ is asymptotically stable.

Example 2 The Lagrange function and the generalized force of the single degree of freedom system respectively are

$$L=\frac{1}{2}\dot{q}^2-\frac{1}{2}q^2-\int q(2+\sin q)^2\mathrm{d}q,$$
$$Q=-2\dot{q}+\frac{(q+\dot{q})\dot{q}}{2+\sin q}\cos q. \tag{6}$$

The differential equation of system (6) is

$$\ddot{q}=-q-q\,(2+\sin q)^2-2\dot{q}+\frac{(q+\dot{q})\,\dot{q}}{2+\sin q}\cos q.$$

Let

$$x_1=q,\quad x_2=\frac{q+\dot{q}}{2+\sin q}$$

and take the derivative of x_1 and x_2, then we will have

$$\dot{x}_1=-x_1+x_2\,(2+\sin x_1),\quad \dot{x}_2=-x_1\,(2+\sin x_1)-x_2.$$

They can be written as the form of a combined gradient system (4), i.e.,

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix}+\begin{pmatrix}0&2+\sin x_1\\-(2+\sin x_1)&0\end{pmatrix}\right)\times\begin{pmatrix}\dfrac{\partial V}{\partial x_1}\\ \dfrac{\partial V}{\partial x_2}\end{pmatrix},$$

where $V=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2$ is positive-definite in the neighbourhood of $x_1=x_2=0$. Therefore, the solution $x_1=x_2=0$ is asymptotically stable.

Example 3 The single degree of freedom system is

$$L=\frac{1}{2}\dot{q}^2-\frac{1}{2}q^2-\frac{1}{4}\,(2q-1)\exp(2q),\tag{7}$$
$$Q=-2\dot{q}+\dot{q}^2+q\dot{q},\tag{8}$$

where the quantities are dimensionless. Now we merge them into a combined gradient system (4), and study the stability of zero solution.

The Lagrange equation is

$$\ddot{q}=-q-q\exp(2p)-2\dot{q}+\dot{q}^2+q\dot{q}.$$

Let

$$\dot{x}_1=q,\quad x_2=(q+\dot{q})\exp(-q).$$

Taking the derivative of x_1 and x_2, we have

$$\dot{x}_1=-x_1+x_2\exp x_1,\quad \dot{x}_2=-x_1\exp x_1-x_2.$$

They can be written as the form of combined gradient system (4), i.e.,

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix}+\begin{pmatrix}0&\exp x_1\\-\exp x_1&0\end{pmatrix}\right)\begin{pmatrix}\dfrac{\partial V}{\partial x_1}\\ \dfrac{\partial V}{\partial x_2}\end{pmatrix},$$

where $V=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2$ is positive-definite in the neighbourhood of $x_1=x_2=0$. Therefore, the solution $x_1=x_2=0$ is asymptotic stable.

The three examples above are all for a Lagrange system, but these properties are also applicable to other mechanical systems. The three examples cannot be represented as the general gradient system nor as the skew-gradient system.

5. Conclusions

The study of the stability of a mechanical system usually needs one to construct the Lyapunov function. It is generally difficult to construct the Lyapunov function directly according to the differential equation. Gradient systems including the general gradient system, skew-gradient system, and combined gradient system, etc., are especially suitable for being studied by using the Lyapunov function. We can use the properties of gradient systems to investigate the stability of mechanical systems, if a mechanical system can be represented as a gradient system. The stabilities of solutions of these examples above cannot be studied by the general gradient system nor by the skew-gradient system, but can be investigated by the combined gradient system. In this paper, an indirect method of studying the stability of a mechanical system is presented.

References

[1] Hirsch M W, Smale S and Devaney R L 2008 *Differential Equations, Dynamical Systems, and an Introduction to Chaos* (Singapore: Elsevier) pp. 203–206

[2] Mc Lachlan R I, Quispel G R W and Robidoux N 1999 *Phil. Trans. R. Soc. Lond. A* **357** 1021

[3] Mei F X and Li Y M 2011 *J. Shangqiu Normal University* **27** 1 (in Chinese)

[4] Chen X W, Zhao G L and Mei F X 2013 *Nonlinear Dyn.* **73** 579

[5] Luo S K, He J M and Xu Y L 2016 *Int. J. Non-Linear Mech.* **78** 105

[6] Mei F X, Cui J C and Wu H B 2012 *Trans. Beijing Inst. Tech.* **32** 1298 (in Chinese)

[7] Tomáš B, Ralph C and Eva F 2012 *Monatsh Math.* **166** 57

[8] Mei F X and Wu H B 2012 *J. Dyn. Control* **10** 289 (in Chinese)

[9] Marin A M, Ortiz R D and Rodriguez J A 2013 *Int. Math. Forum* **8** 803

[10] Mei F X 2012 *Mechanics in Engineering* **34** 89

[11] Yin X W and Li D S 2015 *Acta Math. Sci.* **35A** 464 (in Chinese)

[12] Mei F X and Wu H B 2013 *Sci. China: Phys. Mech. Astron.* **43** 538 (in Chinese)

[13] Li L and Luo S K 2013 *Acta Mechanica* **224** 1757

[14] Chen X W, Li Y M and Mei F X 2014 *Appl. Math. Mech.* **35** 1392 (in Chinese)

[15] Mei F X and Wu H B 2015 *Chin. Phys. B* **24** 104502

[16] Ge W K, Xue Y and Lou Z M 2014 *Acta Phys. Sin.* **63** 110202 (in Chinese)

[17] Mei F X 2013 *Mechanics in Engineering* **35** 79 (in Chinese)

[18] Mei F X 2013 *Analytical Mechanics II* (Beijing: Beijing Institute of Technology Press) pp. 564–581 (in Chinese)

[19] Hirsch M W and Smale S 1974 *Differential Equations, Dynamical Systems, and Linear Algebra* (New York: Academic Press) pp. 199–203

[20] He J H 2000 *Appl. Math. Mech.* **21** 797

[21] Cao X Q, Song J Q, Zhang W M, Zhu X Q and Zhao J 2011 *Acta Phys. Sin.* **60** 080401 (in Chinese)