# Two kinds of generalized gradient representations for holonomic mechanical systems*

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Two kinds of generalized gradient systems are proposed and the characteristics of the two systems are studied. The conditions under which a holonomic mechanical system can be considered as one of the two generalized gradient systems are obtained. The characteristics of the generalized gradient systems can be used to study the stability of the holonomic system. Some examples are given to illustrate the application of the results.

**Keywords:** holonomic mechanical system, generalized gradient system, Lyapunov function, stability

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## 1. Introduction

In Ref. [1], it was pointed out that the gradient system is especially suitable for study by using the Lyapunov function. Reference [2] indicated that there are the skew-gradient system, the gradient system with a symmetric negative definite matrix, the gradient system with a negative semidefinite matrix, and so on, in addition to the general gradient system. References [3] and [4] demonstrated that an autonomous ordinary differential equation has a first integral if and only if it can be written as a skew-gradient system, which provides a general way to rewrite such a system as a skew-gradient system, and from which several new integral-preserving discrete gradient methods are constructed. Reference [5] studied the skew-gradient representation for autonomous stochastic differential equations with a conserved quantity, from which direct/indirect discrete gradient approaches are constructed. Some results have been obtained in the research of the relation between constrained mechanical systems and gradient systems.[6–18] However, in those studies, the matrix or the function of the gradient system has no time $t$. When its matrix or function contains time $t$, the system can be called a generalized gradient system. There are two types of generalized gradient systems that are particularly useful for the study of stability. One is the generalized skew-gradient system, and the other is the generalized gradient system with a symmetric negative definite matrix. In this paper, we will transform the holonomic mechanical system into the two kinds of generalized gradient systems under certain conditions, then discuss the stability of solution of the mechanical system by using the property of the generalized gradient system.

## 2. Two kinds of generalized gradient systems

### 2.1. Generalized skew-gradient system

The differential equations of the generalized skew-gradient system have the form

$$\dot{x}_i = b_{ij}(t,\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_j},\quad (i,j=1,2,\ldots,m), \tag{1}$$

where $\boldsymbol{X}=(x_1,x_2,\ldots,x_m)$, $b_{ij}(t,\boldsymbol{X})=-b_{ji}(t,\boldsymbol{X})$; here and in the following, the same indexes in a term indicate to take sum with it. If $b_{ij}$ and $V$ contain no time $t$, then equations (1) become a skew-gradient system.[2]

Taking the derivative of $V$ with respect to time, according to Eq. (1), we have

$$\dot{V}=\frac{\partial V}{\partial t}. \tag{2}$$

If $V$ can be a Lyapunov function, i.e., it is positive definite and satisfying $\partial V/\partial t<0$, then the solution of system (1) is stable according to the Lyapunov theorem. This property can be used to discuss the stability of solution of the mechanical system which can be transformed into a generalized skew-gradient system.

### 2.2. Generalized gradient system with a symmetric negative definite matrix

The differential equations of the system have the form

$$\dot{x}_i = S_{ij}(t,\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_j},\quad (i,j=1,2,\ldots,m), \tag{3}$$

where the matrix $S_{ij}(t,\boldsymbol{X})$ is symmetric negative definite. If $S_{ij}$ and $V$ contain no time $t$, then equations (3) become a gradient system.[2]

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Taking the derivative of $V$ with respect to time, according to Eq. (3), we have

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i} S_{ij} \frac{\partial V}{\partial x_j}, \quad (i, j = 1, 2, \ldots, m), \tag{4}$$

where the second term on the right side is less than zero. If $V$ is positive definite and $\dot{V}$ is negative definite, then the solution of the system is asymptotically stable. This property can be used to discuss the stability of solution of the mechanical system which can be transformed into the generalized gradient system (3).

## 3. Generalized gradient representations for holonomic systems

The differential equations of a general holonomic system in the generalized coordinates have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s, \quad (s = 1, 2, \ldots, n), \tag{5}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian of the system, and $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are non potential generalized forces.

Suppose that the system is nonsingular, namely,

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0, \tag{6}$$

then one can solve all generalized accelerations from Eq. (5)

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, 2, \ldots, n). \tag{7}$$

Let

$$a^s = q_s, \quad a^{n+s} = \dot{q}_s, \tag{8}$$

then equations (7) can be written in the following first order form:

$$\dot{a}^\mu = F_\mu(t, \boldsymbol{a}), \quad (\mu = 1, 2, \ldots, 2n), \tag{9}$$

where

$$F_s = a^{n+s}, \quad F_{n+s} = \alpha_s. \tag{10}$$

By introducing the generalized momentum $p_s$ and the Hamiltonian $H$,

$$p_s = \frac{\partial L}{\partial \dot{q}_s}, \quad H = p_s \dot{q}_s - L, \tag{11}$$

equations (5) can be written in the following first order form:

$$\dot{a}^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} + \Lambda_\mu, \quad (\mu, \nu = 1, 2, \ldots, 2n), \tag{12}$$

where

$$a^s = q_s, \quad a^{n+s} = p_s,$$

$$(\omega^{\mu\nu}) = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix},$$

$$\Lambda_s = 0, \quad \Lambda_{n+s} = \widetilde{Q}_s(t, \boldsymbol{a}). \tag{13}$$

Here $\widetilde{Q}_s$ are the generalized forces $Q_s$ written in canonical variables.

Generally speaking, equations (9) and (12) are not a generalized gradient system. For Eq. (9), if there exist an antisymmetric matrix $b_{\mu\rho}(t, \boldsymbol{a})$ and a function $V(t, \boldsymbol{a})$ such that

$$F_\mu = b_{\mu\rho} \frac{\partial V}{\partial a^\rho}, \quad (\mu, \rho = 1, 2, \ldots, 2n), \tag{14}$$

then the system can be written in the form of the generalized skew-gradient system (1). If there exist a symmetric negative definite matrix $S_{\mu\nu}(t, \boldsymbol{a})$ and a function $V(t, \boldsymbol{a})$ such that

$$F_\mu = S_{\mu\nu} \frac{\partial V}{\partial a^\nu}, \quad (\mu, \nu = 1, 2, \ldots, 2n), \tag{15}$$

then the system can be written in the form of the generalized gradient system (3).

For Eq. (12), if there exist an antisymmetric matrix $b_{\mu\rho}(t, \boldsymbol{a})$ and a function $V(t, \boldsymbol{a})$ such that

$$\omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} + \Lambda_\mu = b_{\mu\rho} \frac{\partial V}{\partial a^\rho}, \quad (\mu, \nu, \rho = 1, 2, \ldots, 2n), \tag{16}$$

then the system can be written in the form of the generalized skew-gradient system (1). If there exist a symmetric negative definite matrix $S_{\mu\rho}(t, \boldsymbol{a})$ and a function $V(t, \boldsymbol{a})$ such that

$$\omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} + \Lambda_\mu = S_{\mu\rho} \frac{\partial V}{\partial a^\rho}, \quad (\mu, \nu, \rho = 1, 2, \ldots, 2n), \tag{17}$$

then the system can be written in the form of the generalized gradient system (3).

It is worth noting that, if conditions (14)–(17) are not satisfied, it is not determined whether the system is a generalized gradient system, because it is related to the first order form of the equations. In order to transform a mechanical system into a gradient system, one can do the opposite, that is to rewrite a given gradient system as a mechanical system.

For the general holonomic system in the generalized coordinates, if it can be transformed into the generalized gradient system (1) or (3), and the function $V$ can be a Lyapunov function, then one can determine the stability of solution of the mechanical system by using expression (2) or (4).

## 4. Examples

**Example 1** A holonomic system with a single degree of freedom is given as

$$L = \frac{1}{2}\dot{q}^2 - 2q^2\left(1 + \frac{1}{1+t}\right)(2 + \sin t)^2, \; Q = \frac{\dot{q}\cos t}{2 + \sin t}. \tag{18}$$

We transform it into a generalized gradient system and discuss the stability of zero solution.

Equations (5) give

$$\ddot{q} = -4q\left(1+\frac{1}{1+t}\right)(2+\sin t)^2 + \dot{q}\frac{\cos t}{2+\sin t}.$$

If let

$$a^1 = q,\ \ a^2 = \dot{q},$$

it cannot be transformed into a generalized gradient system. Now let

$$a^1 = q,\ \ a^2 = \frac{\dot{q}}{2(2+\sin t)},$$

then it can be rewritten in the following first order form:

$$\dot{a}^1 = 2a^2(2+\sin t),\ \ \dot{a}^2 = -2a^1\left(1+\frac{1}{1+t}\right)(2+\sin t).$$

It also has the following matrix form:

$$\begin{pmatrix}\dot{a}^1\\ \dot{a}^2\end{pmatrix} = \begin{pmatrix}0 & 2+\sin t\\ -(2+\sin t) & 0\end{pmatrix}\begin{pmatrix}\partial V/\partial a^1\\ \partial V/\partial a^2\end{pmatrix},$$

where the matrix is antisymmetric, and the function $V$ is given as

$$V = \left(a^1\right)^2\left(1+\frac{1}{1+t}\right) + \left(a^2\right)^2.$$

It is a generalized skew-gradient system (1). Function $V$ is positive definite in the neighborhood of $a^1 = a^2 = 0$, and

$$\frac{\partial V}{\partial t} = -\frac{\left(a^1\right)^2}{(1+t)^2} < 0.$$

Therefore, zero solution $a^1 = a^2 = 0$ is stable.

**Example 2** The system's Lagrangian and generalized force are

$$\begin{aligned} L &= \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2\{4(1+t)\,[1+\exp(-t)]\\ &\quad - 2\exp(-t) - (1+t)\}(1+t),\\ Q &= -\dot{q}\{2(1+t)[2+\exp(-t)] - \frac{1}{1+t}\}. \end{aligned} \tag{19}$$

We transform it into a generalized gradient system and discuss the stability of zero solution.

Equations (5) give

$$\begin{aligned} \ddot{q} &= -\dot{q}\left\{2(1+t)[2+\exp(-t)] - \frac{1}{1+t}\right\}\\ &\quad - q\{4(1+t)\,[1+\exp(-t)] - 2\exp(-t) - (1+t)\}\,(1+t). \end{aligned}$$

Let

$$a^1 = q,\ \ a^2 = 2q[1+\exp(-t)] + \frac{\dot{q}}{1+t},$$

then the equation can be rewritten in the following first order form:

$$\dot{a}^1 = -\left\{2a^1[1+\exp(-t)] - a^2\right\}(1+t),$$

$$\dot{a}^2 = -(2a^2 - a^1)(1+t).$$

It also has the following matrix form:

$$\begin{pmatrix}\dot{a}^1\\ \dot{a}^2\end{pmatrix} = \begin{pmatrix}-(1+t) & 0\\ 0 & -(1+t)\end{pmatrix}\begin{pmatrix}\partial V/\partial a^1\\ \partial V/\partial a^2\end{pmatrix},$$

where the matrix is symmetric negative definite, and the function $V$ is given as

$$V = \left(a^1\right)^2[1+\exp(-t)] + \left(a^2\right)^2 - a^1a^2.$$

It is a generalized gradient system (3). Function $V$ is positive definite in the neighborhood of $a^1 = a^2 = 0$. Taking $\dot{V}$ according to the above equation leads to

$$\begin{aligned} \dot{V} &= -\left\{2a^1[1+\exp(-t)] - a^2\right\}^2(1+t)\\ &\quad - (2a^2 - a^1)^2(1+t) - \left(a^1\right)^2\exp(-t), \end{aligned}$$

which is negative definite. Therefore, zero solution $a^1 = a^2 = 0$ is asymptotically stable.

**Example 3** We study a single degree of freedom system whose Lagrangian and generalized force are

$$\begin{aligned} L &= \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2(1+t^2)\cos t - [4(2+\sin t) - 1]\\ &\quad \times (1+t^2)\left(\frac{1}{2}q^2 + \frac{1}{3}q^3\right),\\ Q &= -2\dot{q}\left\{(2+\sin t)(1+t^2) + (1+q^2) - \frac{t}{1+t^2}\right\}. \end{aligned} \tag{20}$$

We transform it into a generalized gradient system and discuss the stability of zero solution.

Equations (5) give

$$\begin{aligned} \ddot{q} &= -q\left\{[4(2+\sin t) - 1](1+q^2)(1+t^2) + 2(1+t^2)\cos t\right\}\\ &\quad - 2\dot{q}\left\{(2+\sin t)(1+t^2) + (1+q^2) - \frac{t}{1+t^2}\right\}. \end{aligned}$$

Let

$$a^1 = q,\ \ a^2 = 2q(2+\sin t) + \frac{\dot{q}}{1+t^2},$$

then the equation can be rewritten in the following first order form:

$$\begin{aligned} \dot{a}^1 &= -\left[2a^1(2+\sin t) - a^2\right](1+t^2),\\ \dot{a}^2 &= -(2a^2 - a^1)\left[1+\left(a^1\right)^2\right]. \end{aligned}$$

It also has the following matrix form:

$$\begin{pmatrix}\dot{a}^1\\ \dot{a}^2\end{pmatrix} = \begin{pmatrix}-(1+t^2) & 0\\ 0 & -\left[1+\left(a^1\right)^2\right]\end{pmatrix}\begin{pmatrix}\partial V/\partial a^1\\ \partial V/\partial a^2\end{pmatrix},$$

where the matrix is symmetric negative definite, and the function $V$ is given as

$$V = \left(a^1\right)^2(2+\sin t) + \left(a^2\right)^2 - a^1a^2.$$

It is a generalized gradient system (3). Function $V$ is positive definite in the neighborhood of $a^1 = a^2 = 0$. Taking $\dot{V}$ according to the above equation leads to

$$\begin{aligned}\dot{V} = & -\left[2a^1(2+\sin t)-a^2\right]^2(1+t^2)\\ & -(2a^2-a^1)^2\left[1+\left(a^1\right)^2\right]+\left(a^1\right)^2\cos t,\end{aligned}$$

which is negative definite. Therefore, zero solution $a^1 = a^2 = 0$ is asymptotically stable.

## 5. Conclusion

The stability problem of non-constant mechanical systems is important and difficult. It is often not easy to construct the Lyapunov function directly from the differential equations. In this paper, we have studied two kinds of generalized gradient representations for general holonomic mechanical systems. When a system has been transformed into a generalized gradient system, one can discuss the stability of solution of the mechanical system by using the property of the generalized gradient system. If the system is rewritten in the generalized skew-gradient system (1), it is convenient to study the stability of solution; and if the system is rewritten in the generalized gradient system (3), it is convenient to study the asymptotic stability of solution.

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