Non-Noether symmetries of Hamiltonian systems with conformable fractional derivatives*

Lin-Li Wang (王琳莉) and Jing-Li Fu(傅景礼)†

Institute of Mathematical Physics, Zhejiang Sci-Tech University, Hangzhou 310018, *China*

(Received 26 June 2015; revised manuscript received 28 August 2015; published online 8 December 2015)

In this paper, we present the fractional Hamilton's canonical equations and the fractional non-Noether symmetry of Hamilton systems by the conformable fractional derivative. Firstly, the exchanging relationship between isochronous variation and fractional derivatives, and the fractional Hamilton principle of the system under this fractional derivative are proposed. Secondly, the fractional Hamilton's canonical equations of Hamilton systems based on the Hamilton principle are established. Thirdly, the fractional non-Noether symmetries, non-Noether theorem and non-Noether conserved quantities for the Hamilton systems with the conformable fractional derivatives are obtained. Finally, an example is given to illustrate the results.

Keywords: conformable fractional derivative, Hamilton's canonical equation, non-Noether conserved quantity

PACS: 45.10.Hj, 02.30.Xx, 11.10.Ef **DOI:** 10.1088/1674-1056/25/1/014501

1. Introduction

The subject of symmetries plays a prominent role in physics, life sciences, and engineering. Using the symmetry theories to study the conservation laws is a recent development direction in mathematical physics. The symmetry theory mainly covers Noether symmetry, Lie symmetry, and Mei symmetry. Noether symmetry is an invariance of the Hamiltonian's action under the infinitesimal transformations of groups.[1] Lie symmetry is an invariance of differential equations under the infinitesimal transformations of groups.[2] Mei symmetry means that the form of the equations of motion remains invariant under the infinitesimal transformations of groups for the state parameter.[3,4] In 1992, a theorem regarding the conserved quantities for second-order dynamical systems has recently been given by Hojman,[5] and generalized by Gonzalez-Gascon,[6] using geometric techniques. Subsequently, the generalized theorem of a velocity-dependent symmetry is presented by Lutzky.[7] Lately, Mei has studied the non-Noether conserved quantities for the Hamiltonian system in phase space.[8,9] Fu *et al*. developed a new non-Noether conserved quantity for Hamilton canonical equations.[10] Fang *et al*. gave a directive method to find the non-Noether conserved quantity.[11] In recent years, the symmetry theories with fractional calculus are carried out. Zhou *et al*. studied the symmetry theories of Hamiltonian systems with Riemann–Liouville fractional derivatives.[12] Zhang *et al*. investigated the Hamilton formalism and Noether symmetry for mechanico-electrical systems with fractional derivatives.[13] Zhou *et al*. gave Noether's theorems of a fractional Birkhoffian system within Riemann–Liouville derivatives.[14] Sun *et al*. presented Lie symmetry theorem of fractional nonholonomic systems.[15]

For fractional systems, the Riemann–Liouville (RL) and Caputo fractional derivatives are popular among mathematicians and physicists.[16–31] For example, Riewe has developed a new approach to mechanics, and obtained a version of the Euler–Lagrange equations to problems of the calculus of variations with fractional derivatives for the dynamical systems.[32,33] Agrawal further presented the fractional variational problems with right and left RL fractional derivatives.[34,35] Zhang gave the fractional differential equations in terms of combined Riemann–Liouville derivatives.[36] Wang *et al*. studied the cyclic integrals and Routh equations of fractional Lagrange system.[37] However, these fractional derivatives are in integral form, thus it is extremely difficult to work with these definitions. Recently, Khalil *et al*. introduced a new simple well-behaved definition of the fractional derivative called conformable fractional derivative.[38] Subsequently, Abdeljawad developed the definitions there and set the basic concepts in this new simple interesting fractional calculus.[39] The definition and properties of the conformable fractional derivative are coincident with the usual derivative, which is in limit form. In this paper, we use the conformable fractional derivatives to study the fractional non-Noether symmetries of Hamilton systems. We firstly investigate the fractional Hamilton's canonical equations on the conformable fractional derivative, then, the corresponding fractional non-Noether conserved quantities are presented based on the motion equations.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11272287 and 11472247), the Program for Changjiang Scholars and Innovative Research Team in University, China (Grant No. IRT13097), and the Key Science and Technology Innovation Team Project of Zhejiang Province, China (Grant No. 2013TD18).

†Corresponding author. E-mail: sqfujingli@163.com

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2. Definitions and properties of conformable fractional derivative and integral

In this section, we present some fundamental definitions and properties of conformable fractional derivative and integral.[38,39] Furthermore, we also study the multivariable differential calculus of conformable fractional derivative, which will be used in the following sections.

2.1. Definition and properties of conformable fractional derivative and integral

Definition 1 Conformable fractional derivative Given a function f: $[0, \infty) \to R$. Then the "conformable fractional derivative" of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} \tag{1}$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t). \tag{2}$$

We can write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

Theorem 1 Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then

$$T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g), \quad \text{for all } a, b \in R, \tag{3}$$

$$T_\alpha(\lambda) = 0, \quad \text{for all constant functions } f(t) = \lambda, \tag{4}$$

$$T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f), \tag{5}$$

$$T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha f - fT_\alpha g}{g^2}. \tag{6}$$

If f is differentiable, then

$$T_\alpha(f)(t) = t^{1-\alpha} \frac{\mathrm{d}f}{\mathrm{d}t}(t). \tag{7}$$

Definition 2 Conformable fractional integral

$$I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} \mathrm{d}x,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

Theorem 2 Let f: $(a, b) \to R$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have

$$I_\alpha^a T_\alpha^a f(t) = f(t) - f(a). \tag{8}$$

Definition 3 Chain rule Assume f, g: $(a, \infty) \to R$ be α-differentiable functions, where $0 < \alpha \leq 1$. Let $h(t) = f(g(t))$. Then $h(t)$ is α-differentiable and for all t with $t \neq a$ and $g(t) \neq 0$ we have

$$T_\alpha h(t) = T_\alpha f(g(t)) T_\alpha g(t) g(t)^{\alpha-1} = \frac{\mathrm{d}f}{\mathrm{d}g} T_\alpha g(t). \tag{9}$$

Then, we can easily prove the following multivariable differential calculus.

Definition 4 Let $x = x(t)$ and $y = y(t)$ be differentiable at t, and let $z = f(x, y)$ be differentiable at $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and

$$T_\alpha z(t) = \frac{\partial f}{\partial x} T_\alpha x(t) + \frac{\partial f}{\partial y} T_\alpha y(t). \tag{10}$$

Equation (10) can be explained by the following formula:

$$\begin{aligned} T_\alpha z(t) &= t^{1-\alpha} \frac{\mathrm{d}z}{\mathrm{d}t} = t^{1-\alpha} \left(\frac{\partial f}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t} \right) \\ &= \frac{\partial f}{\partial x} T_\alpha x(t) + \frac{\partial f}{\partial y} T_\alpha y(t). \end{aligned} \tag{11}$$

3. The Hamilton principle with conformable fractional derivative

Firstly, let us study the exchanging relationship with respect to the isochronous variation and the fractional derivatives. Considering two infinitely closed orbits: γ and $\gamma + \mathrm{d}\gamma$, we denote the generalized coordinates by $q = q(t, \gamma)$ and $q' = q(t, \gamma + \mathrm{d}\gamma)$, corresponding to the two infinitely closed orbits individually in giving time, where $q = \{q_1, q_2, \ldots, q_n\}$. We define the isochronous variation as

$$\delta q = q(t, \gamma + \mathrm{d}\gamma) - q(t, \gamma). \tag{12}$$

Extending $q(t, \gamma + \mathrm{d}\gamma)$ to the linear terms of $\mathrm{d}\gamma$, then we obtain

$$q(t, \gamma + \mathrm{d}\gamma) = q(t, \gamma) + \frac{\partial q(t, \gamma)}{\partial \gamma} \mathrm{d}\gamma. \tag{13}$$

Substituting Eq. (13) into Eq. (12), we have

$$\delta q = \frac{\partial q(t, \gamma)}{\partial \gamma} \mathrm{d}\gamma, \tag{14}$$

then we have

$$T_\alpha \delta q = T_\alpha \frac{\partial q(t, \gamma)}{\partial \gamma} \mathrm{d}\gamma. \tag{15}$$

Using Eq. (12), we obtain

$$\delta T_\alpha q = T_\alpha q(t, \gamma + \mathrm{d}\gamma) - T_\alpha q(t, \gamma) = T_\alpha \frac{\partial q(t, \gamma)}{\partial \gamma} \mathrm{d}\gamma. \tag{16}$$

Comparing Eqs. (15) and (16), we obtain the exchanging relationship

$$T_\alpha \delta q = \delta T_\alpha q. \tag{17}$$

Now we consider a mechanical system of n generalized coordinates q_s $(s = 1, 2, \ldots, n)$, the Lagrangian of the system is $L\ (t, q_s, T_\alpha q_s)$ $(s = 1, 2, \ldots, n)$, and we define the fractional Hamilton action as

$$S = I_\alpha^{ab} L(t, q_s, T_\alpha q_s), \tag{18}$$

where all the variations of the variables are isochronous, and the terminal conditions are given by

$$\delta q_s|_{t=a} = \delta q_s|_{t=b} = 0. \tag{19}$$

Then, the fractional Hamilton principle can be written as

$$\delta S = \delta I_\alpha^{ab} L(t, q_s, T_\alpha q_s) = 0. \tag{20}$$

4. Hamilton's canonical equations with conformable fractional derivative

In this section, we study the fractional Hamilton's canonical equation based on the Hamilton principle. First, let us introduce the generalized momenta as

$$p_{\alpha s} = \frac{\partial L}{\partial T_\alpha q_s}, \quad 0 < \alpha < 1, \quad s = (1, 2, \ldots, n), \tag{21}$$

where the Lagrangian depends on the fractional time derivatives of coordinates in the form $L\ (t, q_s, D_\alpha q_s)$. Then, the Hamiltonian depending on the fractional time derivatives can be expressed as

$$H = -L + \sum_{s=1}^{n} p_{\alpha s} T_\alpha q_s. \tag{22}$$

Substituting Eq. (22) into Eq. (20), we have

$$I_\alpha^{ab} \delta \left(\sum_{s=1}^{n} p_{\alpha s} D_\alpha q_s - H \right) = 0. \tag{23}$$

Then, we obtain

$$I_\alpha^{ab} \sum_{s=1}^{n} \left(T_\alpha q_s \delta p_{\alpha s} + p_{\alpha s} \delta T_\alpha q_s - \frac{\partial H}{\partial q_s} \delta q_s - \frac{\partial H}{\partial p_{\alpha s}} \delta p_{\alpha s} \right) = 0. \tag{24}$$

Taking the commutable relation (17), we have

$$I_\alpha^{ab} \sum_{s=1}^{n} \left(T_\alpha q_s \delta p_{\alpha s} + p_{\alpha s} T_\alpha \delta q_s - \frac{\partial H}{\partial q_s} \delta q_s - \frac{\partial H}{\partial p_{\alpha s}} \delta p_{\alpha s} \right) = 0. \tag{25}$$

According to Eq. (5), we obtain

$$T_\alpha (p_{\alpha s} \delta q_s) = p_{\alpha s} T_\alpha \delta q_s + \delta q_s T_\alpha p_{\alpha s}. \tag{26}$$

Substituting Eq. (26) into Eq. (25), we derive

$$I_\alpha^{ab} \sum_{s=1}^{n} \left[T_\alpha q_s \delta p_{\alpha s} + T_\alpha (p_{\alpha s} \delta q_s) - \delta q_s T_\alpha p_{\alpha s} - \frac{\partial H}{\partial q_s} \delta q_s - \frac{\partial H}{\partial p_{\alpha s}} \delta p_{\alpha s} \right] = 0. \tag{27}$$

From Eq. (8), we obtain

$$\sum_{s=1}^{n} p_{\alpha s} \delta q_s|_a^b + I_\alpha^{ab} \sum_{s=1}^{n} \left[\left(T_\alpha q_s - \frac{\partial H}{\partial p_{\alpha s}} \right) \delta p_{\alpha s} - \left(T_\alpha p_{\alpha s} + \frac{\partial H}{\partial q_s} \right) \delta q_s \right] = 0. \tag{28}$$

Using the terminal condition (19), we obtain

$$p_{\alpha s} \delta q_s|_a^b = 0. \tag{29}$$

Then, we have

$$I_\alpha^{ab} \sum_{s=1}^{n} \left[\left(T_\alpha q_s - \frac{\partial H}{\partial p_{\alpha s}} \right) \delta p_{\alpha s} - \left(T_\alpha p_{\alpha s} + \frac{\partial H}{\partial q_s} \right) \delta q_s \right] = 0. \tag{30}$$

Calculating the derivative of Eq. (22) with respect to $p_{\alpha s}$, we acquire

$$\frac{\partial H}{\partial p_{\alpha s}} = T_\alpha q_s. \tag{31}$$

Substituting Eq. (31) into Eq. (30) and considering that δq_s $(s = 1, 2, \ldots, n)$ are independent, we obtain

$$\frac{\partial H}{\partial q_s} = -T_\alpha p_{\alpha s}. \tag{32}$$

According to Eq. (7), we can change Eqs. (31) and (32) into the following form:

$$\frac{\partial H}{\partial p_{\alpha s}} = t^{1-\alpha} \dot{q}_s, \tag{33}$$

and

$$\frac{\partial H}{\partial q_s} = -t^{1-\alpha} \dot{p}_{\alpha s}. \tag{34}$$

Equations (31) and (32), as well as equations (33) and (34), are all called Hamilton's canonical equations with conformable fractional derivative. From Eqs. (33) and (34), we can observe that the forms of fractional Hamilton's canonical equations are in accordance with the usual Hamilton's canonical equations, which just add the fractional gene $t^{1-\alpha}$. The fractional Hamilton's canonical equations are equivalent to the usual Hamilton's canonical equations by taking $\alpha = 1$.

5. Non-Noether conserved quantities with conformable fractional derivative

Now we consider a mechanical system of n generalized coordinates q_s $(s = 1, 2, \ldots, n)$ and n generalized momentums $p_{\alpha s}$ $(s = 1, 2, \ldots, n)$, and the Hamilton's canonical equations are in the form

$$\frac{\partial H}{\partial p_{\alpha s}} = T_\alpha q_s, \tag{35}$$

$$\frac{\partial H}{\partial q_s} = -T_\alpha p_{\alpha s}. \tag{36}$$

Equations (35) and (36) can be written as

$$T_\alpha q_s = g_s(t, q_k, p_{\alpha k}), \tag{37}$$

$$T_\alpha p_{\alpha s} = h_s(t, q_k, p_{\alpha k}), \quad s,k = 1,2,\ldots,n, \tag{38}$$

or

$$t^{1-\alpha}\dot{q}_s = g_s(t, q_k, p_{\alpha k}), \tag{39}$$

$$t^{1-\alpha}\dot{p}_{\alpha s} = h_s(t, q_k, p_{\alpha k}), \quad s,k = 1,2,\ldots,n. \tag{40}$$

Introducing the general infinitesimal transformations

$$q^*(t) = q(t) + \varepsilon\xi_s(t, q_s, p_{\alpha s}), \tag{41}$$

$$p^*_{\alpha s}(t) = p_{\alpha s}(t) + \varepsilon\eta_s(t, q_s, p_{\alpha s}), \tag{42}$$

where ε is a small parameter, ε_s and η_s are the generators of the infinitesimal transformations.

According to the theory of invariance of the differential equations under the infinitesimal transformations, if equations (37) and (38) keep invariant under the infinitesimal transformations, then the generators satisfy the following equations:

$$X^{(1)}(g_s - T_\alpha q_s) = 0, \tag{43}$$

$$X^{(1)}(h_s - T_\alpha p_{\alpha s}) = 0, \tag{44}$$

that is,

$$X^{(1)}(g_s - t^{1-\alpha}\dot{q}_s) = 0, \tag{45}$$

$$X^{(1)}(h_s - t^{1-\alpha}\dot{p}_{\alpha s}) = 0, \tag{46}$$

where

$$X^{(0)} = \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_{\alpha s}} \tag{47}$$

is the operator of infinitesimal generators, and its extensions

$$X^{(1)} = X^{(0)} + \dot{\xi}_s\frac{\partial}{\partial \dot{q}_s} + \dot{\eta}_s\frac{\partial}{\partial \dot{p}_{\alpha s}}. \tag{48}$$

Substituting Eqs. (45) and (46) into Eqs. (47) and (48), we obtain the following determining equations:

$$X^{(0)}(g_s) = t^{1-\alpha}\dot{\xi}_s, \tag{49}$$

$$X^{(0)}(h_s) = t^{1-\alpha}\dot{\eta}_s. \tag{50}$$

According to Eq. (7), it can also be written in the following form:

$$X^{(0)}(g_s) = T_\alpha\xi_s, \tag{51}$$

$$X^{(0)}(h_s) = T_\alpha\eta_s. \tag{52}$$

Here we point out that the determining equations with the fractional gene are in accordance with the usual determining equations when taking $\alpha = 1$. Equations (49)–(52) are applicable to the general case. Then, we have the following theorem.

Theorem 3 For the Hamiltonian system (37) and (38), if there exists a gauge function $G(t, q_s, p_{\alpha s})$ such that

$$\frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_{\alpha s}} + T_\alpha G = \psi, \tag{53}$$

where ψ is an invariant of the general infinitesimal transformations, then the following quantity can be found:

$$\phi = X^{(0)}(G) + \frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_{\alpha s}}. \tag{54}$$

Proof Firstly, we point out the fact that $X^{(0)}(\psi) = X^{(1)}(\psi) = 0$. Then, we demonstrate the theorem by taking α order derivative of ϕ

$$T_\alpha\phi = T_\alpha[X^{(0)}(G)] + T_\alpha\left(\frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_{\alpha s}}\right). \tag{55}$$

We have the following easily proven identities ($f(t, q_s, p_{\alpha s})$ is arbitrary):

$$T_\alpha\frac{\partial f}{\partial q_s} = \frac{\partial}{\partial q_s}T_\alpha f - \frac{\partial f}{\partial q_k}\frac{\partial g_k}{\partial q_s} - \frac{\partial f}{\partial p_{\alpha k}}\frac{\partial h_k}{\partial q_s}, \tag{56}$$

$$T_\alpha\frac{\partial f}{\partial p_{\alpha s}} = \frac{\partial}{\partial p_{\alpha s}}T_\alpha f - \frac{\partial f}{\partial q_k}\frac{\partial g_k}{\partial p_{\alpha s}} - \frac{\partial f}{\partial p_{\alpha k}}\frac{\partial h_k}{\partial p_{\alpha s}}, \tag{57}$$

$$X^{(0)}\left(\frac{\partial f}{\partial q_s}\right) = \frac{\partial}{\partial q_s}X^{(0)}(f) - \frac{\partial f}{\partial q_k}\frac{\partial \xi_k}{\partial q_s} - \frac{\partial f}{\partial p_{\alpha k}}\frac{\partial \eta_k}{\partial q_s}, \tag{58}$$

$$X^{(0)}\left(\frac{\partial f}{\partial p_{\alpha s}}\right) = \frac{\partial}{\partial p_{\alpha s}}X^{(0)}(f) - \frac{\partial f}{\partial q_k}\frac{\partial \xi_k}{\partial p_{\alpha s}} - \frac{\partial f}{\partial p_{\alpha k}}\frac{\partial \eta_k}{\partial p_{\alpha s}}, \tag{59}$$

$$T_\alpha[X^{(0)}(f)] = X^{(0)}(T_\alpha f). \tag{60}$$

Using Eqs. (56), (57), and (60), we obtain

$$\begin{aligned} T_\alpha\phi \;=\; & X^{(0)}(T_\alpha G) + \frac{\partial}{\partial q_s}T_\alpha\xi_s - \frac{\partial \xi_s}{\partial q_k}\frac{\partial g_k}{\partial q_s} - \frac{\partial \xi_s}{\partial p_{\alpha k}}\frac{\partial h_k}{\partial q_s} \\ & + \frac{\partial}{\partial p_{\alpha s}}T_\alpha\eta_s - \frac{\partial \eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_{\alpha s}} - \frac{\partial \eta_s}{\partial p_{\alpha k}}\frac{\partial h_k}{\partial p_{\alpha s}}. \end{aligned} \tag{61}$$

According to Eqs. (58) and (59), we have

$$\begin{aligned} & X^{(0)}\left(\frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_{\alpha s}}\right) \\ = \; & \frac{\partial}{\partial q_s}X^{(0)}(g_s) - \frac{\partial \xi_s}{\partial q_k}\frac{\partial g_k}{\partial q_s} - \frac{\partial \xi_s}{\partial p_{\alpha k}}\frac{\partial h_k}{\partial q_s} \\ & + \frac{\partial}{\partial p_{\alpha s}}X^{(0)}(h_s) - \frac{\partial \eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_{\alpha s}} - \frac{\partial \eta_s}{\partial p_{\alpha k}}\frac{\partial h_k}{\partial p_{\alpha s}}. \end{aligned} \tag{62}$$

Considering the determining equations (51) and (52), we obtain

$$\begin{aligned} & X^{(0)}\left(\frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_{\alpha s}}\right) \\ = \; & \frac{\partial}{\partial q_s}T_\alpha\xi_s - \frac{\partial \xi_s}{\partial q_k}\frac{\partial g_k}{\partial q_s} - \frac{\partial \xi_s}{\partial p_{\alpha k}}\frac{\partial h_k}{\partial q_s} \\ & + \frac{\partial}{\partial p_{\alpha s}}T_\alpha\eta_s - \frac{\partial \eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_{\alpha s}} - \frac{\partial \eta_s}{\partial p_{\alpha k}}\frac{\partial h_k}{\partial p_{\alpha s}}. \end{aligned} \tag{63}$$

Comparing Eq. (61) with Eq. (63), we derive

$$\begin{aligned} T_\alpha\phi \;=\; & X^{(0)}(T_\alpha G) + X^{(0)}\left(\frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_{\alpha s}}\right) \\ =\; & X^{(0)}\left(T_\alpha G + \frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_{\alpha s}}\right). \end{aligned} \tag{64}$$

From the gauge function (53), we have

$$T_\alpha \phi = X^{(0)}(\psi) = 0, \tag{65}$$

which clarifies the theorem.

We have shown that the form of conserved quantity with fractional calculus is in accordance with the usual one of the systems. It is common and when we take $\alpha = 1$, the conserved quantity (54) is equal to the general case.

6. An illustrated example

Let us consider the following fractional mechanical system, the Lagrangian is

$$L = \frac{1}{2}(T_\alpha^2 q_1 + T_\alpha^2 q_2 + T_\alpha^2 q_3) - q_1, \tag{66}$$

and study its non-Noether conserved quantities. Obviously, we have

$$p_{\alpha 1} = \frac{\partial L}{\partial T_\alpha q_1} = T_\alpha q_1, \tag{67}$$

$$p_{\alpha 2} = \frac{\partial L}{\partial T_\alpha q_2} = T_\alpha q_2, \tag{68}$$

$$p_{\alpha 3} = \frac{\partial L}{\partial T_\alpha q_3} = T_\alpha q_3. \tag{69}$$

The Hamilton function of the system is given by

$$\begin{aligned} H &= -L + p_{\alpha 1} T_\alpha q_1 + p_{\alpha 2} T_\alpha q_2 + p_{\alpha 3} T_\alpha q_3 \\ &= \frac{1}{2}(p_{\alpha 1}^2 + p_{\alpha 2}^2 + p_{\alpha 3}^2) + q_1. \end{aligned} \tag{70}$$

According to the Hamilton's canonical equations (31) and (32), we obtain

$$T_\alpha q_1 = p_{\alpha 1} = g_1, \tag{71}$$

$$T_\alpha q_2 = p_{\alpha 2} = g_2, \tag{72}$$

$$T_\alpha q_3 = p_{\alpha 3} = g_3, \tag{73}$$

$$T_\alpha p_{\alpha 1} = -1 = h_1, \tag{74}$$

$$T_\alpha p_{\alpha 2} = 0 = h_2, \tag{75}$$

$$T_\alpha p_{\alpha 3} = 0 = h_3. \tag{76}$$

From the determining equations (49) and (50), we have

$$\begin{aligned} &T_\alpha \xi_1 = \eta_1, \quad T_\alpha \xi_2 = \eta_2, \quad T_\alpha \xi_3 = \eta_3, \\ &T_\alpha \eta_1 = T_\alpha \eta_2 = T_\alpha \eta_3 = 0. \end{aligned} \tag{77}$$

One solution of Eq. (77) is

$$\begin{aligned} &\xi_1 = \frac{c_1}{\alpha} t^\alpha, \quad \xi_2 = \frac{c_2}{\alpha} t^\alpha, \quad \xi_3 = \frac{c_3}{\alpha} t^\alpha, \\ &\eta_1 = c_4, \quad \eta_2 = c_5, \quad \eta_3 = c_6, \end{aligned} \tag{78}$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are constants.

Then, the gauge equation gives

$$T_\alpha G = \psi. \tag{79}$$

Let $\psi = T_\alpha(q_2 p_{\alpha 3}) - T_\alpha(q_3 p_{\alpha 2})$, then $X^{(0)}(\psi) = X^{(1)}(\psi) = 0$, ψ is an invariant.

Therefore, we can find a gauge function

$$G = q_2 p_{\alpha 3} - q_3 p_{\alpha 2}. \tag{80}$$

Substituting Eqs. (78) and (80) into Eq. (54), we obtain the following non-Noether conserved quantity:

$$\phi = \frac{t^\alpha}{\alpha}(c_2 p_{\alpha 3} - c_3 p_{\alpha 2}) - c_5 q_3 + c_6 q_2. \tag{81}$$

7. Conclusions

In this paper, we propose a new method to study the complex Hamilton systems with fractional derivatives. The motion equations and non-Noether symmetry theories of fractional Hamilton systems are established by the conformable fractional derivative. The forms of the results are identical to the general ones, and they may change into the general cases when making $\alpha = 1$. Using this conformable fractional derivative, all the fractional differential equations can be translated into general differential equations and we can easily study the fractional nonholonomic system, the fractional electromechanical coupling system, and the problems of fractional anomalous diffusion.

References

[1] Noether A E 1918 *Math. Phys.* **KI II** 235
[2] Li Z P 1981 *Acta Phys. Sin.* **30** 1659 (in Chinese)
[3] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120
[4] Mei F X 2001 *Chin. Phys.* **10** 177
[5] Hojman S A 1992 *J. Phys. A: Math. Gen.* **25** L291
[6] Gonzaler-Cascon F 1994 *J. Phys. A: Math. Gen.* **27** L59
[7] Lutzky M 1995 *J. Phys. A: Math. Gen.* **28** L637
[8] Mei F X 2002 *Chin. Sci. Bull.* **47** 1544 (in Chinese)
[9] Mei F X 2003 *Acta Phys. Sin.* **52** 1048 (in Chinese)
[10] Fu J L, Chen L Q and Xie F P 2004 *Chin. Phys. Soc.* **13** 1611
[11] Fang J H, Liao Y P and Peng Y 2004 *Chin. Phys. Soc.* **13** 1620
[12] Zhou S, Fu H and Fu J L 2011 *Sci. Chin. Phys. Mech. Astron.* **54** 1847
[13] Zhang S H, Chen B Y and Fu J L 2012 *Chin. Phys. B* **21** 100202
[14] Zhou Y and Zhang Y 2014 *Chin. Phys. B* **23** 124502
[15] Sun Y, Chen B Y and Fu J L 2014 *Chin. Phys. B* **23** 110201
[16] Zaslavsky G M 2005 *Hamiltonian Chaos and Fractional Dynamics* (Oxford: Oxford University Press)
[17] Tarasov V E 2010 *Fractional Dynamics* (Beijing: Higher Education Press)
[18] Oldham K B and Spanier J 1974 *The Fractional Calculus* (New York: Academic Press)
[19] Podlubny I 1999 *Fractional Differential Equations* (San Diego CA: Academic Press)
[20] Miller K S and Ross B 1993 *An Introduction to the Fractional Calculus and Fractional Differential Equations* (New York: A Wiley-Interscience Publication)
[21] Hilf R 2000 *Applications of Fractional Calculus in Physics* (Singapore: World Scientific Publishing Company)
[22] Samko S G, Kilbas A A and Marichev O I 1993 *Fractional Integrals and Derivatives–Theory and Applications* (Linghorne: Gordon and Breach Science Publishers)

[23] Kilbas A A, Srivastava H M and Trujillo J J 2006 *Theory and Applications of Fractional Differential Equations* (Amsterdam: Elsevier B V)
[24] Zaslavsky G M 2002 *Phys. Rep.* **371** 461
[25] Zaslavsky G M and Edelman M A 2004 *Physica D* **193** 128
[26] Yang J H 2012 *Chin. Phys. Lett.* **29** 104501
[27] Tan C, Liang Z S and Zhang J Q 2014 *Acta Phys. Sin.* **63** 200502 (in Chinese)
[28] Wang F, Xie T T, Deng C and Luo M K 2014 *Acta Phys. Sin.* **63** 160502 (in Chinese)
[29] Zhou X W, Lin L F, Ma H and Luo M K 2014 *Acta Phys. Sin.* **63** 160503 (in Chinese)
[30] Tu Z, Lai L and Luo M K 2014 *Acta Phys. Sin.* **63** 120503 (in Chinese)
[31] Wang F Q and Ma X K 2013 *Chin. Phys. B* **22** 030506
[32] Riewe F 1996 *Phys. Rev. E* **53** 1890
[33] Riewe F 1997 *Phys. Rev. E* **55** 3581
[34] Agrawal O P 2002 *J. Math. Anal. Appl.* **272** 368
[35] Agrawal O P 2004 *Nonlinear Dyn.* **38** 323
[36] Zhang Y 2012 *Chin. Phys. B* **21** 084502
[37] Wang L L and Fu J L 2014 *Chin. Phys. B* **23** 124501
[38] Khalil R, Al Horani M, Yousef A and Sababheh M 2014 *J. Comput. Appl. Math.* **264** 65
[39] Abdeljawad T 2015 *J. Comput. Appl. Math.* **279** 57